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On finding the generalized inverse matrix for the product of matrices

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Abstract. Matrix computation is an important topic in applied mathematics and information science. There are various methods of finding the inverse and generalized inverse of a given matrix. However, for the product matrices, there does not exist a general method of finding its generalized inverse. In this note, we introduce the concept of sandwich sets of matrices. By using the new concept of sandwich sets, we are able to provide a method for finding a generalized inverse of product matrices.

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Generalized inverses of matrices and its applications have been investigated by Rao and Mitra in [3]. For some special matrices, some authors have given some interesting methods for finding their generalized inverse matrices. For example, Rakha [4] has recently given a method of finding the Moor-Penrose generalized inverse matrix. Furthermore, Werner [6] in 1994 also described the problem for finding a generalized inverse for the product of matrices. In fact he considered the problem when will B^-A^- be a generalized inverse of AB? The matrix computation for information systems was also discussed by J. Guan, Bell and Z. Guan in [1]. In this aspect, a recursive method for finding the inverse of a CSP matrix was first provided by Ramabhadra and Sharma in [2]. However, up to the present moment, except the paper by Werner [6], there does not exist a general method of finding a generalized inverse for the product of matrices. In this note, we will first introduce the concept of sandwich sets of matrices. Then by using the sandwich sets of matrices, we will provide an effective method for

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finding a generalized inverse matrix for the product of matrices. Some examples will be demonstrated how to find such a generalized inverse matrix for a product of some particular matrices.

We first denote the set of all $m \times n$ matrices over a field $\mathcal{F}$ by $\mathcal{F}^{n\times m}$. Let $A \in \mathcal{F}^{m\times n}$. If there exists some $X \in \mathcal{F}^{n\times m}$ such that

$$AXA = A \quad \text{and} \quad XAX = X \tag{1}$$

then we call this matrix X a reflexive generalized inverse of the matrix A, denoted by A^-.

For any $A \in \mathcal{F}^{m\times n}$, we can easily see that there exists $A^- \in \mathcal{F}^{n\times m}$. Now, we denote the rank of the matrix A by $\operatorname{rank} A = r$. If $r = 0$, then A is an $m \times n$ zero matrix, and so the $n \times m$ zero matrix O is a reflexive generalized inverse matrix. If $r \neq 0$, then there exists some invertible matrices $P \in \mathcal{F}^{m\times m}$ and $Q \in \mathcal{F}^{n\times n}$ such that

$$A = P^{-1} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q^{-1}, \tag{2}$$

where P^{-1} is the usual inverse matrix of the matrix P. In this case, we can verify that

$$A^- = Q \begin{bmatrix} I_r & B_1 \\ B_2 & B_2B_1 \end{bmatrix} P \tag{3}$$

for any $B_1 \in \mathcal{F}^{r\times(m-r)}$ and $B_2 \in \mathcal{F}^{(n-r)\times r}$. From the matrix A^- with the form (3) above, we can see that for any no zero matrix $A \in \mathcal{F}^{m\times n}$, A^- is unique if and only if A is invertible (see [3]).

Now, we denote the set of all the reflexive generalized inverses A^- of a matrix A by $V(A)$. Clearly, the set $V(A)$ is non-empty for any matrix A. Let $\mathbf{E}$, be the set of all $n \times n$ idempotent matrices, that is,

$$\mathbf{E} = \{E : E^2 = E,\ E \in \mathcal{F}^{n\times n}\}.$$

Then, we can easily see that for any $A \in \mathcal{F}^{m\times n}$, AA^- and A^-A are both idempotent matrices.

In order to obtain a reflexive generalized inverse for product matrices, we now introduce the following definition.

Definition 1 Suppose that $E, F \in \mathcal{F}^{n\times n} \cap \mathbf{E}$. Then we call

$$S(E, F) = \{G \in \mathbf{E} : GE = FG = G \text{ and } EGF = EF\}$$

the sandwich set of the matrices E and F.

The sandwich sets have the following properties.

Proposition 2

(i) *$S(E, F)$ defined above is non-empty.*

(ii) $|S(E,F)| = 1$ *if and only if* $GH = HG$ *for any* $G, H \in S(E,F)$.

(iii) *For any* $E \in \boldsymbol{E}$, $S(E,E)$ *contains a unique idempotent matrix* E, *i.e.* $S(E,E) = \{E\}$.

(iv) *Suppose that* I *is the usual identity matrix. Then* $S(I,I)$ *contains a unique identity matrix* I.

Proof. (i) It is clear that for any idempotent matrices $E, F \in \mathcal{F}^{n\times n}$, its product EF is also an $n \times n$ matrix. Let $P \in V(EF)$, $G = FPE$. Then, we have

$$G^2 = FPE \cdot FPE = F(PEFP)E = FPE = G$$

so that G is an idempotent matrix. Also, by definition and formula (1), we can see that

$$GE = FPE \cdot E = FPE = G$$
$$FG = F \cdot FPE = FPE = G$$

and

$$EGF = E(FPE)F = (EF)P(EF) = EF.$$

This shows that $G \in S(E,F)$ and hence the proof is completed. □

(ii) The necessity is immediate since every matrix G in $S(E,F)$ is an idempotent matrix. We now prove the sufficiency. Suppose that $G, H \in S(E,F)$. Then, by definition of the sandwich set, it is evident that $G, H \in \mathbf{E}$ such that

$$GE = FG = G \quad \text{and} \quad EHF = EF.$$

This leads to

$$GHG = (GE)H(FG) = G(EHF)G = GEFG = G^2 = G.$$

By a similar argument, we can also deduce that $HGH = H$. Thus, by our hypothesis, it follows that

$$G = GHG = G^2H = GH = GH^2 = HGH = H.$$

This shows that $|S(E,F)| = 1$.

(iii) Suppose that $G \in S(E,E)$. Then, by definition of the sandwich set, we have $E \cdot G \cdot E = E^2 = E$ and $EG = GE = G$. Hence, $E = (EG)E = GE = G$.

(iv) Part (iv) follows immediately from (iii).

We are now ready to provide a method of finding a reflexive generalized inverse for the product of some particular matrices. We give the following theorem.

Theorem 3 *Suppose that* $A \in \mathcal{F}^{m\times n}$ *and* $B \in \mathcal{F}^{n\times p}$ *such that* $A^- \in V(A)$ *and* $B^- \in V(B)$. *Then* $B^-GA^- \in V(AB)$ *for any* $G \in S(A^-A, BB^-)$.

Proof. It is easy to see that A^-A and BB^- are both $n\times n$ idempotent matrices. Now, we write $A^-A = E$ and $BB^- = F$. Then by using the definition of sandwich set of the matrices E and F, for any $G \in S(E,F)$, we have

$$\begin{aligned}(AB)(B^-GA^-)(AB) &= A(BB^-)G(A^-A)B\\ &= AFGEB = AGB\\ &= AA^-AGBB^-B = A(EGF)B\\ &= AEFB = AA^-ABB^-B = AB.\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}(B^-GA^-)(AB)(B^-GA^-) &= B^-GEFGA^- = B^-G^2A^-\\ &= B^-GA^-.\end{aligned}$$

Hence, by the definition of the reflexive generalized inverses matrix, we can see immediately that $B^-GA^- \in V(AB)$.

The following corollaries are consequences of Theorem 3 and Proposition 2 (ii).

Corollary 4 *Suppose that $A \in \mathcal{F}^{m\times n}$ and $B \in \mathcal{F}^{n\times p}$. If $A^- \in V(A)$ and $B^- \in V(B)$ such that $A^-A = BB^- = E \in \mathbf{E}$, then $B^-A^- \in V(AB)$.*

Corollary 5 *Suppose that $A \in \mathcal{F}^{m\times n}$ and $B \in \mathcal{F}^{n\times p}$. If $A^- \in V(A)$ and $B^- \in V(B)$ such that $A^-A = BB^- = I$, then B^-A^- is a reflexive generalized inverse matrix for the product AB of the matrices A and B.*

Corollary 6

(i) *If A is an $n\times n$ invertible matrix with the inverse A^{-1} and B is an $n\times p$ matrix, then for any $B^- \in V(B)$, the product $B^-A^{-1} \in V(AB)$.*

(ii) *If A is an $m\times n$ matrix and B is an $n\times n$ invertible matrix with the inverse matrix B^{-1}, then for any $A^- \in V(A)$, the product $B^{-1}A^- \in V(AB)$.*

Proof. We only need to prove part (i) because the proof of part (ii) is similar. By our hypothesis, we see that an $n \times n$ matrix A is invertible and so $A^{-1}A$ is clearly the identity matrix I. Hence, we only need to consider the sandwich set $S(I, BB^-)$, for any $B^- \in V(B)$. In this cases, it can be verified that the idempotent matrix BB^- is in $S(I, BB^-)$. Consequence, by Theorem 3, we immediately see that $B^-A^{-1} \in V(AB)$. Thus, the proof is completed. □

Corollary 7 *Suppose that $A \in \mathcal{F}^{m\times 1}$ and $B \in \mathcal{F}^{1\times p}$. Then for any $A^- \in V(A)$ and $B^- \in V(B)$, the product matrix B^-A^- is a reflexive generalize inverse matrix for the product AB of the matrices A and B.*

Proof. The conclusion is obvious because the sandwich set $S(A^-A, BB^-)$ contains a unique element 1. □

We now give some examples below to demonstrate how to apply our theorem to find the generalized inverse matrix for some product of particular matrices.

EXAMPLE 8 Let $A = (a_1, a_2, \ldots, a_m)^T$ and $B = (b_1, b_2, \ldots, b_p)$, where $a_1, b_1 \neq 0$. We now find a reflexive generalized inverse matrix for AB. According to our formula (3), we can easily see that

$$A^- = \left(\frac{1}{a_1} + CA_1, c_1, c_2, \ldots, c_{m-1}\right),$$

where

$$A_1 = \left(-\frac{a_2}{a_1}, \ldots, -\frac{a_m}{a_1}\right)^T$$
$$C = (c_1, c_2, \ldots, c_{m-1}),$$

which is an arbitrary $1 \times (m-1)$ matrix.

Similarly, we have

$$B^- = \left(\frac{1}{b_1} + B_1 D\,, d_1\, \ldots, d_{p-1}\right)^T,$$

where

$$B_1 = \left(-\frac{b_2}{b_1}, \ldots, -\frac{b_p}{b_1}\right)$$

and

$$D = (d_1\,, \ldots, d_{p-1})^T$$

which is an arbitrary $(p-1) \times 1$ matrix.

By using our Corollary 4, we have

$$(AB)^- = B^- A^- = \begin{pmatrix} \frac{1}{b_1} + B_1 D \\ d_1 \\ \vdots \\ d_{p-1} \end{pmatrix} \left(\frac{1}{a_1} + CA_1, c_1, c_2, \ldots, c_{m-1}\right).$$

In particular, if we take

$$A^- = \left(\frac{1}{a_1}, 0, 0, \ldots, 0\right)$$

and

$$B^- = \left(\frac{1}{b_1}, 0, \ldots, 0\right)^T,$$

then we immediately obtain a generalized inverse of AB as follows

$$(AB)^- = \begin{pmatrix} \frac{1}{a_1 b_1} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

EXAMPLE 9 Suppose that

$$A = \begin{pmatrix} 1 & -2 & 1 & 1 \\ 2 & -3 & 1 & 3 \end{pmatrix}^T$$

and

$$B = \begin{pmatrix} 1 & -2 & 3 & -1 & 2 \\ -1 & 1 & -2 & 1 & -1 \end{pmatrix}.$$

In order to find a reflexive generalized inverse matrix for the product of matrices AB, we first find the set $V(A)$ by using our formula (3). In fact, we can easily verify that

$$V(A) = \left\{ \begin{pmatrix} -3+a-3b & -2+a-b & a & b \\ 2+c-3d & 1+c-3d & c & d \end{pmatrix} \middle| \; a,b,c,d \in \mathcal{F} \right\}$$

and

$$V(B) = \left\{ B^- \mid B^- = \begin{pmatrix} B_{11} & B_{12} & B_{13} & B_{14} & B_{15} \\ B_{21} & B_{22} & B_{23} & B_{24} & B_{25} \end{pmatrix} \right\},$$

where

$$\begin{array}{ll} B_{11} = -1+e-g+f-h & B_{21} = -2+2e-2g+f-h \\ B_{12} = -1-e-f-i-j & B_{22} = -3-2e-f-2i-j \\ B_{13} = -e-f & B_{23} = -2e-f \\ B_{14} = -g-h & B_{24} = -2g-h \\ B_{15} = -j-i & B_{25} = -2i-j \end{array}$$

for $e,f,g,h,i,j \in \mathcal{F}$.

Now, we can find a reflexive generalized inverse matrix for the product of the matrices A and B, that is, the matrix AB. If we choose

$$A^- = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad B^- = \begin{pmatrix} -1 & 1 \\ 0 & 0 \\ 1 & 0 \\ 1 & 3 \\ 0 & 1 \end{pmatrix}$$

then we have

$$A^-A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad BB^- = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

By Corollary 5, we immediately obtain that

$$(AB)^- = B^-A^- = \begin{pmatrix} -1 & 1 \\ 0 & 0 \\ 1 & 0 \\ 1 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 \\ 1 & 3 & 4 & 2 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

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