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Perfect wrpp semigroups

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Abstract. In this paper, we introduce a class of wrpp semigroups called perfect wrpp semigroups which contain both perfect rpp semigroups and $C-$wrpp semigroups as subclasses. After giving some properties and characterizations of such semigroups, using $C-$wrpp semigroups and normal bands we establish their structures. Finally, we give an example of such semigroup which is neither a perfect rpp semigroup nor a $C-$wrpp semigroup.

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1 Introduction

In the last decades, generalizations of the class of Clifford semigroups have been extensively investigated by many authors and fruitful results have been obtained (see [1]–[7], [12]–[16]). Fountain in [2] introduced a class of rpp monoids with central idempotents, briefly called $C-$rpp semigroups, which is one of significant generalizations of Clifford semigroups. He showed that a semigroup is $C-$rpp if and only if it is a strong semilattice of left cancellative monoids. The class of $C-$rpp semigroups includes the class of Clifford semigroups but are not regular semigroups. Guo, Shum and Guo in [4] defined perfect rpp semigroups and showed that a perfect rpp semigroup is a strong semilattice of the direct products of left cancellative monoids and rectangular bands. On the other hand, in [16], a new generalization of $C-$rpp semigroups, the $C-$wrpp semigroups was introduced and described. It was proved that a semigroup is $C-$wrpp if and only if it is a strong semilattice of left$-\mathcal{R}$ cancellative monoids.

In this paper, we introduce a class of wrpp semigroups called perfect wrpp semigroups which contain both perfect rpp semigroups and $C-$wrpp semigroups, and then explore some basic properties and establish a construction of such semigroups. We generalize the results of both perfect rpp semigroups in [4] and $C-$wrpp semigroups in [16]. Finally, we give an example of perfect wrpp semigroup which is neither a perfect rpp semigroup nor a $C-$wrpp semigroup. Some methods in [4] are adopted.

For notations and terminologies not mentioned in this paper, readers are referred to [1], [4], [8]–[11].

2 Preliminaries

The notation $E(S)$ denotes the set of all idempotent elements of a semigroup S. If U is a subset of S, we write $E(S)\cap U$ as $E(U)$.

Let S be a semigroup. Recall in [16] that the $\mathcal{L}^{**}-$relation in S was defined by, for $a,b\in S$, $a\mathcal{L}^{**}b$ if and only if $(ax,ay)\in\mathcal{R}\Leftrightarrow(bx,by)\in\mathcal{R}$ for $x,y\in S^1$, where $\mathcal{R}$ is the usual Green's $\mathcal{R}-$relation on S.

For $a\in S$, the equivalence relation $\mathcal{L}^{**}-$class containing the element a is denoted by L_a^{**}.

Definition 2.1 ([1]) A semigroup S is called wrpp semigroup if the following conditions are satisfied:

(1) each $\mathcal{L}^{**}-$class of S contains at least one idempotent of S;

(2) for all $e\in E(L_a^{**})$, $a=ae$.

Definition 2.2 ([1]) A wrpp semigroup S is called an adequate wrpp semigroup if for each $a\in S$, there exists a unique idempotent e satisfying $a\mathcal{L}^{**}e$ and $a=ea$.

Hereafter, we denote the unique idempotent e in Definition 2.2 by a^+.

It is well–known that a band B is a rectangular band if and only if $efg=eg$ for all $e,f,g\in B$ [8]. We say a band B is normal if $efge=egfe$ for all $e,f,g\in B$ [8].

In the proof of following proposition we need the facts: if $a\mathcal{L}^{**}a^+$ for any $a\in S$, then $aa^+=a$ and $a^+a=a$, and for $e,f\in E(S)$, $e\mathcal{L}^{**}f$ if and only if $e\mathcal{L}f$.

The following proposition is crucial because it shows how we can define a nice congruence γ on an adequate wrpp semigroup such that S/γ is a $C-$wrpp semigroup.

Proposition 2.3 *Let S be an adequate wrpp semigroup whose idempotents form a normal band. Define a relation γ on S given by*

$$a\ \gamma\ b \text{ if and only if } a=ebf$$

where $a,b\in S$ and $e,f\in E(b^+)$. Then γ is a congruence on S.

Proof. Firstly, we show that if $a\ \gamma\ b$ then $E(a^+)=E(b^+)$, where a^+ and b^+ are the idempotents satisfying $a\mathcal{L}^{**}a^+$, $a^+a=a$ and $b\mathcal{L}^{**}b^+$, $b^+b=b$ respectively. In fact, if $a\ \gamma\ b$, by definition of γ, we have $a=ebf$ for some $e,f\in E(b^+)$. Clearly, $af=a$. Since $a^+\mathcal{L}^{**}a$ and $\mathcal{L}^{**}$ is a right congruence on S, we have $a^+f\mathcal{L}^{**}af=a$. Further, $a^+fa=a^+febf=a^+feb^+bf=a^+efb^+bf=a^+eb^+bf=a^+ebf=a^+a=a$, that is, $a^+f=a^+$ by the definition of adequate wrpp semigroups. This leads to $E(a^+)E(b^+)=E(b^+)E(a^+)\subseteq E(a^+)$, and then $a^+b^+,b^+a^+\in E(a^+)$. Again, by $a=ebf$, we have $(b^+a^+)a(a^+b^+)=b^+ab^+=b^+ebfb^+=(b^+eb^+)b(b^+fb^+)=b^+bb^+=b$. This means that $b\ \gamma\ a$. Hence, by using the above argument, we have $E(a^+)E(b^+) = E(b^+)E(a^+) \subseteq E(b^+)$. Therefore, $E(a^+) \cap E(b^+) \neq \emptyset$, and so $E(a^+) = E(b^+)$.

Secondly, we show that γ is an equivalence relation. By the above proof, γ is symmetric. Also, γ is clearly reflexive. We only need to prove that γ is transitive. Suppose now $a\ \gamma\ b$ and $b\ \gamma\ c$, then we have $E(a^+) = E(b^+) = E(c^+)$. Then there exist $e, f, e_1, f_1 \in E(a^+)$ such that $ebf = c$ and $e_1af_1 = b$. Hence, $a^+b^+cb^+a^+ = a^+b^+ebfb^+a^+ = a^+ba^+ = a^+e_1af_1a^+ = a$ and $a^+b^+ \in E(c^+)$, $b^+a^+ \in E(c^+)$, that is, $a\ \gamma\ c$.

Thirdly, we show that γ is right compatible. Assume $a\ \gamma\ b$ for $a, b \in S$. Then $a = ebf$ for some $e, f \in E(b^+)$. Then, $b^+eb^+ = b^+$and $b^+fb^+ = b^+$. Hence, $ac = ebfc = ebb^+b^+fc^+c = ebb^+fb^+c^+c = ebc$. Further, $ac = (ac)^+ac = (ac)^+ebc = (ac)^+eb^+b^+bc = (ac)^+b^+eb^+bc = (ac)^+(bc)^+bc$. Since $(ac)^+\mathcal{L}^{**}ac$, we have $(ac)^+(bc)^+\mathcal{L}^{**}ac(bc)^+ = ac$ and $(ac)^+(bc)^+ac = (ac)^+(bc)^+(ac)^+(bc)^+bc = (ac)^+(ac)^+(bc)^+(bc)^+bc = (ac)^+(bc)^+bc = ac$. We obtain that $(ac)^+ = (ac)^+ (bc)^+$ since S is an adequate wrpp semigroup. Symmetrically, we can prove that $(bc)^+ = (bc)^+(ac)^+$. Therefore, $E((ac)^+) \cap E((bc)^+) \neq \emptyset$ and so $E((ac)^+) = E((bc)^+)$. Also, since $ac = (ac)^+bc = (ac)^+bc(bc)^+$ and $(ac)^+, (bc)^+ \in E((ac)^+)$, we have $ac\ \gamma\ bc$, as required.

Finally, to prove that γ is left compatible, let $a\ \gamma\ b$. Then, $a = ebf$ for some $e, f \in E(b^+)$. For any $c \in S$, $ca = c(ebf) = cc^+eb^+b^+bf = cc^+(b^+eb^+)b^+bf = cbf$. Further, $cab^+ = (cbf)b^+ = cbb^+fb^+ = cbb^+ = cb$. This leads to $(ca)^+(ca) (ca)^+b^+ = cb$. We need only to prove that $(ca)^+b^+ \in E((ca)^+)$. In fact, by $ca = cbf$, we have $ca = caf$. Notice that $(ca)^+\mathcal{L}^{**}ca$ and $(ca)^+f\mathcal{L}^{**}caf = ca\mathcal{L}^{**}(ca)^+$. Hence, $(ca)^+f\mathcal{L}(ca)^+$ and so $(ca)^+((ca)^+f)(ca)^+ = (ca)^+$. Therefore, $(ca)^+fca = (ca)^+(ca)^+f(ca)^+ca = (ca)^+ca = ca$. Since S is an adequate wrpp semigroup, $(ca)^+f = (ca)^+$. Hence, by $f \in E(b^+)$, we have $E((ca)^+) = E((ca)^+f) = E((ca)^+b^+)$, i.e., $(ca)^+b^+ \in E((ca)^+)$. Consequently, $ca\ \gamma\ cb$. □

COROLLARY 2.4 *In the above Proposition, if $a\mathcal{L}^{**}b$ for all $a, b \in S$, then $a\gamma\mathcal{L}^{**} b\gamma$ where $\mathcal{L}^{**}$ is the equivalence relation $\mathcal{L}^{**}$ on S/γ.*

Proof. We need to prove that if $a\mathcal{L}^{**}b$ for all $a, b \in S$, then $((ax)\gamma, (ay)\gamma) \in \mathcal{R}$ implies $((bx)\gamma, (by)\gamma) \in \mathcal{R}$, where $x, y \in S^1$. In fact, since $((ax)\gamma, (ay)\gamma) \in \mathcal{R}$, there exists $u\gamma, v\gamma \in (S/\gamma)^1$ such that $(ax)\gamma u\gamma = (ay)\gamma$ and $(ay)\gamma v\gamma = (ax)\gamma$, that is, $(axu, ay) \in \gamma$ and $(ayv, ax) \in \gamma$. By $(axu, ay) \in \gamma$, we have $ay = g(axu)h$ for some $g, h \in E((axu)^+)$. By using the facts that $E(S)$ is a normal band and $g \in E((axu)^+)$, we have $ay = a^+ay = a^+g(axu)^+(axu)^+(axu)h = a^+(axu)^+g(axu)^+(axu)h = a^+(axu)^+(axu)h = a^+axuh = axuh$. Similarly, by $(ayv, ax) \in \gamma$, we can deduce that $ax = ayvh'$ for some $h' \in E((ayv)^+)$. Hence, by the above argument, we have $(ax, ay) \in \mathcal{R}$. Since $a\mathcal{L}^{**}b$, $(bx, by) \in \mathcal{R}$. Thus, there exists $m, n \in S^1$, such that $bx = bym = (bym)^+bym(bym)^+$, $by = bxn = (bxn)^+bxn(bxn)^+$, that is, $(bx, bym) \in \gamma$ and $(by, bxn) \in \gamma$. Consequently, $((bx)\gamma, (by)\gamma) \in \mathcal{R}$. And then $a\gamma\mathcal{L}^{**}b\gamma$. □

3 Perfect wrpp semigroups

In this section, we will firstly introduce the concept of perfect wrpp semigroups, and then give some characterizations and a structure theorem for this kind of semigroups, finally, we give an example of such semigroups.

Definition 3.1 A wrpp semigroup S is called a perfect wrpp semigroup if the following conditions are satisfied:

(1) S is an adequate wrpp semigroup;

(2) $E(S)$ forms a normal band under the multiplication of S;

(3) $\mathcal{L}^{**}$ is a congruence on S.

Before establishing a construction of perfect wrpp semigroups, we give a characterization for perfect wrpp semigroups as follows:

Theorem 3.2 *Let S be an adequate wrpp semigroup whose set of idempotents $E(S)$ is a normal band. Then the following conditions are equivalent:*

(i) *S is a perfect wrpp semigroup;*

(ii) *$(ab)^+ = a^+b^+$ for all $a, b \in S$;*

(iii) *S/γ is a $C-$wrpp semigroup, where γ is the equivalence relation on S defined in Proposition 2.3.*

Proof. (i)$\Rightarrow$(iii)

First we prove that, for all $a, b \in S$, $E((ab)^+) = E(a^+b^+) = E(b^+a^+) = E((ba)^+)$. By the hypothesis, $\mathcal{L}^{**}$ is a congruence on S and $E(S)$ is a normal band. According to $a\mathcal{L}^{**}a^+$ and $b\mathcal{L}^{**}b^+$ we have $ab\mathcal{L}^{**}a^+b^+$ and $ba\mathcal{L}^{**}b^+a^+$. Thus, $(ab)^+\mathcal{L}a^+b^+$, $(ba)^+\mathcal{L}b^+a^+$, and so $E((ab)^+) = E(a^+b^+)$ and $E((ba)^+) = E(b^+a^+)$. Also, since $E(S)$ is a normal band, it can be expressed as a strong semilattice $[Y; E_\alpha, \phi_{\alpha,\beta}]$ of rectangular bands E_α. Then there exists $\alpha, \beta \in Y$ such that $a^+ \in E_\alpha = E(a^+)$ and $b^+ \in E_\beta = E(b^+)$. Further, notice that $E(S)$ is a strong semilattice of rectangular bands E_α, we also have $E((ab)^+) = E((ba)^+)$.

Since $a\mathcal{L}^{**}a^+$ for $a \in S$, by Corollary 2.4, $a\gamma\mathcal{L}^{**}a^+\gamma$, that is, each $\mathcal{L}^{**}$-class of S/γ contains an idempotent.

On the other hand, if $a\gamma \in E(S/\gamma)$, then $a\gamma a^2$ and $a = ea^2f$ for some $e, f \in E((a^2)^+)$. Hence, $ea = eea^2f = a$ and $af = ea^2ff = a$. Consequently, $a = a^2$.

We now show that the idempotents of S/γ are central. Consider $a\gamma \in S/\gamma$ and $e\gamma \in E(S/\gamma)$, where $e \in E(S)$. Since $E((ae)^+) = E(a^+e) = E(ea^+) = E((ea)^+)$, we have $(ea)^+, ea^+, a^+e, (ae)^+ \in E((ea)^+)$, and so $(ae)^+(ea)a^+e = (ae)^+(ea)e = (ae)^+(ea^+)(ae)^+ae = (ae)^+ae = ae$. Hence, $ae\gamma ea$, that is, $E(S/\gamma)$ are central in S/γ. Consequently, S/γ is a $C-$wrpp semigroup.

(iii)⇒(ii)

Suppose that S/γ is a $C-$wrpp semigroup, then $E(S/\gamma)$ is in the center of S/γ. By Corollary 2.4, $a\gamma\mathcal{L}^{**}a^+\gamma$ and $b\gamma\mathcal{L}^{**}b^+\gamma$. Since $\mathcal{L}^{**}$ is a right congruence, $(ab)^+\gamma\mathcal{L}^{**}(ab)\gamma = a\gamma b\gamma\mathcal{L}^{**}a^+\gamma b\gamma = b\gamma a^+\gamma\mathcal{L}^{**}b^+\gamma a^+\gamma = (b^+a^+)\gamma$. Thus, $E((ab)^+) = E(a^+b^+)$. By $E(a^+)E(a^+b^+) \subseteq E(a^+b^+) = E((ab)^+)$, we deduce that $(ab)^+a^+(ab)^+ = (ab)^+(a^+(ab)^+)(ab)^+ = (ab)^+$, also, we know $a^+(ab)^+(ab)^+ = a^+(ab)^+$, then we have $a^+(ab)^+\mathcal{L}(ab)^+$. Further, from $a^+(ab)^+ ab = ab$, we obtain that $a^+(ab)^+ = (ab)^+$. Since $(ab)^+b^+\mathcal{L}^{**}abb^+ = ab$ and $(ab)^+b^+ab = (ab)^+b^+a^+(ab)^+ab = (ab)^+(a^+b^+)(ab)^+ab = (ab)^+ab = ab$. This means that $(ab)^+ = (ab)^+b^+$. Thus, $(ab)^+ = a^+(ab)^+b^+ = a^+a^+(ab)^+b^+b^+ = a^+a^+b^+(ab)^+b^+ = a^+b^+(ab)^+a^+b^+ = a^+b^+$.

(ii)⇒(i)

Suppose that (ii) holds. Then, for $a, b, x, y \in S$,

$$\begin{aligned} a\mathcal{L}^{**}x \text{ and } b\mathcal{L}^{**}y &\Rightarrow a^+\mathcal{L}^{**}x^+ \text{ and } b^+\mathcal{L}^{**}y^+ \\ &\Rightarrow a^+b^+\mathcal{L}^{**}x^+b^+ = x^+b^+y^+ = x^+b^+y^+y^+ \\ &= x^+y^+b^+y^+ = x^+y^+. \end{aligned}$$

Thus, by (ii), it follows that $(ab)^+\mathcal{L}(xy)^+$. Therefore, we have $ab\ \mathcal{L}^{**}(ab)^+ \mathcal{L}(xy)^+\mathcal{L}^{**}xy$, that is, $ab\ \mathcal{L}^{**}xy$. Consequently, $\mathcal{L}^{**}$ is a congruence on S, as required. □

To describe the perfect wrpp semigroups, an analogue of locally $C-$rpp semigroups in [4], we define a class of semigroups called locally $C-$wrpp semigroups.

DEFINITION 3.3 A semigroup S is said to a locally $C-$wrpp semigroup if for all $e \in E(S)$, eSe is a $C-$wrpp semigroup. In addition, if the set of idempotents of S forms a subsemigroup of S, then we call S a locally $C-$wrpp $E-$semigroup.

For the purpose to describe the perfect wrpp semigroups with locally $C-$wrpp semigroups, we also need the following lemma whose proof is similar with Lemma 3.4 in [4].

LEMMA 3.4 *Let S be an adequate wrpp semigroup on which $\mathcal{L}^{**}$ is a congruence on S. If $E(S)$ is a semilattice, then the idempotents of S are central, that is, S is a $C-$wrpp semigroup.*

From Lemma 3.4, we can immediately know that the class of perfect wrpp semigroups is a generalization of $C-$wrpp semigroups.

THEOREM 3.5 *Let S be an adequate wrpp semigroup on which $\mathcal{L}^{**}$ is a congruence. Then S is a perfect wrpp semigroup if and only if S is a locally $C-$wrpp $E-$semigroup.*

Proof. ⇒) Suppose that S is a perfect wrpp semigroup. Let $a, b \in E(eSe)$. Then clearly we have $E(eSe)$ is a semilattice.

Now, let $a \in E(eSe)$, then $a = exe$ for some $x \in S$. Since $\mathcal{L}^{**}$ is a congruence on S, we have $a = eae\mathcal{L}^{**}ea^{+}e$. Thus, every $\mathcal{L}^{**}-$class of eSe contains at least one idempotent in $E(eSe)$. In order to prove that eSe is a wrpp semigroup, we need to show that for all $f \in E(L_a^{**}(eSe))$, $a = af$, where $a \in eSe$. In fact, for $a \in eSe$ and $f \in E(L_a^{**}(eSe))$, $f\mathcal{L}^{**}a\mathcal{L}^{**}ea^{+}e$ implies $f\mathcal{L}ea^{+}e$, $ea^{+}ef = ea^{+}e$, and so $af = aa^{+}f = eaea^{+}f = eaeea^{+}f = eaea^{+}ef = eaea^{+}e = aa^{+}e = ae = a$. Clearly, $ea^{+}ea = a$. Let $f \in E(eSe)$ such that $f\mathcal{L}^{**}(eSe)a$ and $fa = a$, then $f\mathcal{L}(eSe)ea^{+}e$. Since $E(eSe)$ is a semilattice, we have $f = ea^{+}e$. Hence, eSe is an adequate wrpp semigroup. Moreover, it is easily seen that $\mathcal{L}^{**}(eSe)$ is a congruence on eSe and hence by Lemma 3.4, eSe is a $C-$wrpp semigroup. In particular, S is a locally $C-$wrpp semigroup. Since S is a perfect wrpp semigroup, S is indeed a locally $C-$wrpp $E-$semigroup.

$\Leftarrow$) The proof is similar to the one of the sufficiency of Theorem 3.5 in [4]. □

In the following, we continue to give a characterization of perfect wrpp semigroups.

THEOREM 3.6 *Let S be an adequate wrpp semigroup. Then S is a perfect wrpp semigroup if and only if there exists a $C-$wrpp semigroup T and a surjective homomorphism ϕ preserving the $\mathcal{L}^{**}-$classes of S onto T such that for all $e, f \in E(S)$, the restriction $\phi|_{eSf}$ of ϕ to eSf is injective.*

Proof. $\Rightarrow$) The proof is similar with the one of the necessity of Theorem 3.6 in [4].

$\Leftarrow$) Assume that T and ϕ satisfy the given conditions of this Theorem. We shall show that S is a perfect wrpp semigroup by the following steps.

(i) $E(S)$ is a normal band.

The proof is analogous to the corresponding one of Theorem 3.6 in [4].

(ii) For all $a, b \in S$, $(ab)^{+} = a^{+}b^{+}$.

Since $(ab)^{+}b^{+}\mathcal{L}^{**}abb^{+} = ab\mathcal{L}^{**}(ab)^{+}$, we have $(ab)^{+}b^{+}ab = (ab)^{+}(ab)^{+}b^{+}(ab)^{+}ab = (ab)^{+}ab = ab$. Hence, $(ab)^{+}b^{+} = (ab)^{+}$. Because T is a $C-$wrpp semigroup, for $a, b \in S$ $(ab)\phi = (a^{+}(ab)b^{+})\phi = a^{+}\phi(ab)\phi b^{+}\phi = (ab)\phi a^{+}\phi b^{+}\phi = ((ab)(a^{+}b^{+}))\phi$. Notice that $\phi|_{a^{+}Sb^{+}}$ is injective. By $ab, (ab)(a^{+}b^{+}) \in a^{+}Sb^{+}$, we also have $ab = (ab)(a^{+}b^{+})$. Further, $(ab)^{+}a^{+}b^{+}\mathcal{L}^{**}(ab)a^{+}b^{+} = ab\mathcal{L}^{**}(ab)^{+}$, that is, $(ab)^{+}a^{+}b^{+}\mathcal{L}(ab)^{+}$. Also, $(ab)^{+}a^{+}b^{+}(ab) = (ab)^{+}(ab)^{+}a^{+}b^{+}(ab)^{+}ab = (ab)^{+}ab = ab$ and so $(ab)^{+} = (ab)^{+}a^{+}b^{+}$. Since $E(S)$ is normal, $a^{+}(ab)^{+}b^{+}$, $(ab)^{+}b^{+} \subseteq E((ab)^{+}a^{+}b^{+})$. Hence, $a^{+}(ab)^{+}b^{+}\mathcal{L}(ab)^{+}b^{+} = (ab)^{+}\mathcal{L}^{**}ab$. By $a^{+}(ab)^{+}b^{+}(ab) = ab$, we obtain $(ab)^{+} = a^{+}(ab)^{+}b^{+}$. It means that $(ab)^{+} \in a^{+}Sb^{+}$. Since ϕ preserves the $\mathcal{L}^{**}-$classes, $x^{+}\phi = (x\phi)^{+}$ for all $x \in S$. Thus, for all $a, b \in S$, using the fact that T itself is a $C-$wrpp semigroup, we have that

$$(ab)^{+}\phi = ((ab)\phi)^{+} = (a\phi b\phi)^{+} = (a\phi)^{+}(b\phi)^{+} = (a^{+}\phi)(b^{+}\phi) = (a^{+}b^{+})\phi.$$

Therefore, $(ab)^{+} = a^{+}b^{+}$.

By summing up (i) and (ii), and by using Theorem 3.2, we conclude that S is a perfect wrpp semigroup. □

DEFINITION 3.7 ([4],[9]) Let M and T be semigroups having a common morphic image H. Let $S = \{(a,b) \in M \times T \mid a\varphi = b\psi\}$, where $\varphi : M \to H$ and $\psi : T \to H$ are the semigroup homomorphisms which map from M and T onto H respectively. Then we call S the spined product of the semigroups M and T with respect to H, φ and ψ, denoted by $S = M \otimes_{H,\varphi,\psi} T$.

Recall in [4] that a semigroup is called a left cancellative plank if it is a direct product of a left cancellative monoid and a rectangular band. Now, we say that a semigroup S is a left$-\mathcal{R}$ cancellative plank if it is a direct product of a left$-\mathcal{R}$ cancellative monoid and a rectangular band, where a semigroup is called left$-\mathcal{R}$ cancellative if for all $a, b, c \in S$, $(ca, cb) \in \mathcal{R}$ implies $(a, b) \in \mathcal{R}$.

Now, we turn to the spined product structures of perfect wrpp semigroups.

THEOREM 3.8 *The following conditions on a semigroup S are equivalent:*

(1) *S is a perfect wrpp semigroup;*

(2) *S is a spined product of a $C-$wrpp semigroup and a normal band;*

(3) *S is a strong semilattice of left$-\mathcal{R}$ cancellative planks.*

Proof (1)⇒(2)

Assume that S is a perfect wrpp semigroup and $E(S)$ is a normal band. Then $E(S)$ is a strong semilattice of rectangular bands E_α, say $E(S) = [Y; E_\alpha, \varphi_{\alpha,\beta}]$. It can be easily seen that $\gamma|_{E(S)} = \mathcal{D}^{E(S)}$, where γ is the congruence on S defined in Proposition 2.3. However, since $E(S)/\gamma|_{E(S)} = Y$, we can simply identity $E(S)/\gamma|_{E(S)}$ by Y. Thus, by Theorem 3.2 (iii), S/γ is a $C-$wrpp semigroup, in notation, $S/\gamma = [Y; M_\alpha, \psi_{\alpha,\beta}]$, where all M_α's (the $\mathcal{L}^{**}-$classes of S/γ) are the left$-\mathcal{R}$ cancellative monoids and $\psi_{\alpha,\beta}$ is the structure homomorphism of the strong semilattice $[Y; M_\alpha, \psi_{\alpha,\beta}]$. Hence, the spined product of S/γ and the band $E(S)$ with respect to the semilattice Y is $M = \cup_{\alpha\in Y}(M_\alpha \times E_\alpha)$, where the multiplication on M is defined by $(m, i)(n, j) = (mn, ij)$, and mn and ij are the semigroup products of $m, n \in S/\gamma$, $i, j \in E(S)$. To finish the proof, we still need to verify that the mapping $\theta : S \to M$ defined by $s\theta = (s\gamma, s^+)$ for all $s \in S$ is an isomorphism.

(i) θ is injective. Suppose that $(s\gamma, s^+) = (t\gamma, t^+)$ for some $(s\gamma, s^+)$ and $(t\gamma, t^+) \in M$. Then $s\gamma = t\gamma$ and $s^+ = t^+$. By the definition of γ, there exists $e, f \in E(t^+)$ such that $s = etf$. Recall that $E(S)$ is a normal band, so $E(t^+)$ is a rectangular band, we have $s = s^+ss^+ = t^+etft^+ = t^+et^+tt^+ft^+ = t^+tt^+ = t$.

(ii) θ is surjective. Let $(a, i) \in M$. Since $a \in S/\gamma$, there exists $x \in S$ such that $x\gamma = a$. According to the definition of M, we have $x^+ \in E(i)$. Next, we shall prove that $(ixi)\theta = (a, i)$. To see this, we need to show that $(ixi)^+ = i$ and $(ixi)\gamma = a$. In fact, by using the argument above, we have $x^+ \in E(i)$ and so $(ixi)\gamma = x\gamma$. It follows that $(ixi)\gamma = a$. On the other hand, since $\mathcal{L}^{**}$ is a congruence, $ixi\mathcal{L}^{**}ix^+i$. By $x^+ \in E(i)$ and $ix^+i = i$, $ixi\mathcal{L}^{**}i$. However, by $i(ixi) = ixi$, we obtain $(ixi)^+ = i$. Thus, $(ixi)\theta = (a, i)$.

(iii) θ is a isomorphism. By Theorem 3.2, $(st)\theta = ((st)\gamma, (st)^+) = (s\gamma t\gamma, s^+t^+) = (s\gamma, s^+)(t\gamma, t^+) = s\theta t\theta$.

Thus, by (i), (ii) and (iii), we have proved that θ is an isomorphism. That is, S is isomorphic to a spined product $M \otimes_{Y,\varphi,\psi} E(S)$ of a $C-$wrpp semigroup and a normal band.

(2)⇒(3)

The proof is analogous with the corresponding one of Theorem 4.3 in [4].

(3)⇒(1)

The verification can be done via the following steps:

(i) $E(S)$ is a normal band.

The proof is analogous with the corresponding one of Theorem 4.3 in [4].

(ii) S is an adequate wrpp semigroup.

Firstly, we show that each $\mathcal{L}^{**}-$class of S contains at least one idempotent of S. That is, for any $x = (m_\alpha, i_\alpha) \in N_\alpha \times E_\alpha = S_\alpha$, we have $(e_\alpha, i_\alpha) = e_x \in E(S_\alpha)$ and $(m_\alpha, i_\alpha) = x\mathcal{L}^{**}e_x = (e_\alpha, i_\alpha)$, where e_α is the identity of N_α. In fact, for any $y \in S_\beta$ and $z \in S_\gamma$, if $xy\mathcal{R}xz$, that is, $xg_{\alpha,\alpha\beta}yg_{\beta,\alpha\beta}\mathcal{R}xg_{\alpha,\alpha\gamma}zg_{\gamma,\alpha\gamma}$, then we have $\alpha\beta = \alpha\gamma$. Notice that N_α is a left$-\mathcal{R}$ cancellative monoid and E_α is a rectangular band, it is not hard to prove that $xg_{\alpha,\alpha\beta}yg_{\beta,\alpha\beta}\mathcal{R}xg_{\alpha,\alpha\gamma}zg_{\gamma,\alpha\gamma}$ if and only if $e_xg_{\alpha,\alpha\beta}yg_{\beta,\alpha\beta}\mathcal{R}\ e_xg_{\alpha,\alpha\gamma}zg_{\gamma,\alpha\gamma}$, that is, $xy\mathcal{R}xz$ if and only if $e_xy\mathcal{R}e_xz$, and so $x\mathcal{L}^{**}e_x$.

Secondly, we show that for all $e \in E(L_a^{**})$, $a = ae$. For $a = (m_\alpha, i_\alpha) \in S_\alpha$, by the above proof we know that $a\mathcal{L}^{**}e_a = (e_\alpha, i_\alpha)$. Since $e\mathcal{L}^{**}a$ for all $e = (e_\beta, j_\beta) \in E(L_a^{**})$, $e_a\mathcal{L}^{**}a\mathcal{L}^{**}e$, that is, $e_a\mathcal{L}e$. So we can get $\alpha = \beta$ and $e_ae = e_a$, that is, $(e_\alpha, i_\alpha)(e_\alpha, j_\alpha) = (e_\alpha e_\alpha, i_\alpha j_\alpha) = (e_\alpha, i_\alpha j_\alpha) = (e_\alpha, i_\alpha)$. Thus, $i_\alpha j_\alpha = i_\alpha$. And so $ae = (m_\alpha, i_\alpha)(e_\alpha, j_\alpha) = (m_\alpha e_\alpha, i_\alpha j_\alpha) = (m_\alpha, i_\alpha) = a$.

From the argument above, S is a wrpp semigroup. Finally, we show that S is an adequate wrpp semigroup. In fact, for any $a = (m_\alpha, i_\alpha) \in N_\alpha \times E_\alpha = S_\alpha$, by the proof above there exists $(e_\alpha, i_\alpha) = e_a \in E(N_\alpha \times E_\alpha)$ such that $a\mathcal{L}^{**}e_a$ and $e_aa = ae_a = a$, where e_α is the identity of N_α. Now assume there exists another idempotent $a^+ \in N_\alpha \times E_\alpha$ satisfying $a\mathcal{L}^{**}a^+$ and $a = a^+a$, say $a^+ = (e_\beta, j_\beta) \in S_\beta$. Then

$$\begin{aligned}
e_a\mathcal{L}^{**}a\mathcal{L}^{**}a^+ &\Rightarrow e_a\mathcal{L}a^+ \\
&\Rightarrow \alpha = \beta \text{ and } a^+ = (e_\alpha, j_\alpha) \\
&\Rightarrow (m_\alpha, i_\alpha) = (e_\alpha, j_\alpha)(m_\alpha, i_\alpha) = (e_\alpha m_\alpha, j_\alpha i_\alpha) = (m_\alpha, j_\alpha i_\alpha) \text{ and} \\
&\quad\ (m_\alpha, i_\alpha) = (m_\alpha, i_\alpha)(e_\alpha, j_\alpha) = (m_\alpha e_\alpha, i_\alpha j_\alpha) = (m_\alpha, i_\alpha j_\alpha) \\
&\Rightarrow i_\alpha = j_\alpha i_\alpha = i_\alpha j_\alpha \\
&\Rightarrow i_\alpha = j_\alpha i_\alpha j_\alpha = j_\alpha \quad (\text{since } E(N_\alpha \times E_\alpha) \text{ is a rectangular band}) \\
&\Rightarrow a^+ = (e_\alpha, j_\alpha) = (e_\alpha, i_\alpha) = e_a.
\end{aligned}$$

we have shown that S is an adequate wrpp semigroup.

(iii) $(xy)^+ = x^+y^+$ for all $x, y \in S$.

Let $S = [Y^{'}; S_\alpha, g_{\alpha,\beta}]$, where for each $\alpha \in Y^{'}$, $S_\alpha = N_\alpha \times E_\alpha$, N_α is a left–$\mathcal{R}$ cancellative monoid and E_α is a rectangular band, also, $g_{\alpha,\beta}$ is the structure homomorphism. It is not hard to prove that for any $\alpha \geq \beta$, we have the homomorphisms $\varphi : N_\alpha \to N_\beta, \psi : E_\alpha \to E_\beta$ such that $(a, i)g_{\alpha,\beta} = (a\varphi, i\psi)$ $(\forall (a, i) \in S_\alpha)$ with all the conditions needed to define the strong semilattices of semigroups $N = [Y^{'}; N_\alpha, \varphi_{\alpha,\beta}]$, $E = [Y^{'}; E_\alpha, \psi_{\alpha,\beta}]$. And then we obtain that N is a $C-$wrpp semigroup and E is a normal band. Now, for any $x, y \in S$ with $x = (a_\alpha, i_\alpha) \in S_\alpha$ and $y = (b_\beta, j_\beta) \in S_\beta$, by (ii) we know that $x^+ = (e_\alpha, i_\alpha)$ and $y^+ = (e_\beta, j_\beta)$. Then, $xy = (a_\alpha, i_\alpha)(b_\beta, j_\beta) = (a_\alpha b_\beta, i_\alpha j_\beta) = (a_\alpha\varphi_{\alpha,\alpha\beta} b_\beta \varphi_{\beta,\alpha\beta}, i_\alpha\psi_{\alpha,\alpha\beta} j_\beta\psi_{\beta,\alpha\beta})$ and

$$\begin{aligned} x^+y^+ &= (e_\alpha, i_\alpha)(e_\beta, j_\beta) \\ &= (e_\alpha e_\beta, i_\alpha j_\beta) \\ &= (e_\alpha\varphi_{\alpha,\alpha\beta} e_\beta\varphi_{\beta,\alpha\beta}, i_\alpha\psi_{\alpha,\alpha\beta} j_\beta\psi_{\beta,\alpha\beta}) \\ &= (e_{\alpha\beta} e_{\alpha\beta}, i_\alpha\psi_{\alpha,\alpha\beta} j_\beta\psi_{\beta,\alpha\beta}) \\ &= (e_{\alpha\beta}, i_\alpha\psi_{\alpha,\alpha\beta} j_\beta\psi_{\beta,\alpha\beta}) \\ &= (xy)^+. \quad \text{(by (ii))} \end{aligned}$$

According to Theorem 3.2, S is a perfect wrpp semigroup. □

In Theorem 3.8, if let the wrpp semigroup S be a rpp semigroup, then we can immediately obtain the following corollary which is Theorem 4.3 in [4].

Corollary 3.9 ([4]) *The following conditions on a semigroup S are equivalent:*

(1) *S is a perfect rpp semigroup;*

(2) *S is a spined product of a $C-$rpp semigroup and a normal band;*

(3) *S is a strong semilattice of left cancellative planks.*

On the other hand, if let the normal band $E(S)$ of S in Theorem 3.8 be trivial, then we can also obtain the following corollary which is Theorem 3.3 in [16].

Corollary 3.10 ([16]) *A semigroup S is a $C-$wrpp semigroup if and only if it is a strong semilattice of left–$\mathcal{R}$ cancellative monoids.*

Hence, by Corollary 3.9 and 3.10, we know that the class of perfect wrpp semigroups is not only a generalization of the class of perfect rpp semigroups but also the class of $C-$wrpp semigroups.

Finally, we give an example of perfect wrpp semigroup.

Example Let G be a group with $|G| > 1$. Let $K = \langle a \rangle$ be the infinite monogenic semigroup generated by a. We consider the disjoint union $M = G \dot{\cup} K$ and define a multiplication $\circ$ on M by

$$x \circ y = \begin{cases} xy, & \text{if } x, y \in G, \text{ or } x, y \in K, \\ x, & \text{if } x \in G, \text{ and } y \in K, \\ y, & \text{if } y \in G, \text{ and } x \in K. \end{cases}$$

It is easy to verify that $(M, \circ)$ is a left $\mathcal{R}-$cancellative monoid, and the $\mathcal{R}-$classes of M are the group G and the singleton subsets of K. Moreover, M is a $C-$wrpp semigroup since it contains only one idempotent (the identity). However, it is not left cancellative since for all $g, h \in G$, we always have $ag = a = ah$ for all $a \in K$. Now, let $B = I \times \Lambda$ be a rectangular band with $|I| = |\Lambda| > 1$. We now form the direct product $S = B \times M$ with the coordinatewise multiplications. Then according to Theorem 3.8, we have that $S = B \times M$ is an infinite perfect wrpp semigroup. Clearly, S is not a $C-$ wrpp semigroup because $E(S)$ is not central. On the other hand, since M is not a left cancellative monoid, S is not a perfect rpp semigroup.

Remark From the above example, we know that the class of perfect wrpp semigroups exist and also properly includes the classes of perfect rpp semigroups and $C-$ rpp semigroups.

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