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Catalan-like numbers and succession rules

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"[...] many an interesting combinatorial problem can be formulated algebraically as that of transforming a basis into another basis with more desirable properties."

S. A. Joni, G.-C. Rota, *Coalgebras and bialgebras in combinatorics.*

Abstract. The ECO method and the theory of Catalan-like numbers introduced by Aigner seems two completely unrelated combinatorial settings. In this work we try to establish a bridge between them, aiming at starting a (hopefully) fruitful study on their interactions. We show that, in a linear algebra context (more precisely, using infinite matrices), a succession rule can be translated into a (generalized) Aigner matrix by means of a suitable change of basis in the vector space of one-variable polynomials. We provide some examples to illustrate this fact and apply it to the study of two particular classes of succession rules.

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1 Introduction

The ECO method was founded in the 90's by a group of researchers, including Pinzani, Barcucci, Del Lungo and Pergola [6, 7]. It consists of a purely combinatorial way of constructing the objects of a given class in such a way that, if the construction is sufficiently regular and recursive, enumeration follows by more or less standard methods of combinatorial analysis. More precisely, one starts by partitioning a class of objects according to their size (suitably defined). The goal is then to perform a sort of local expansion on each object of a given size, thus producing all the objects of the successive size exactly once. Therefore a single object produces a set of new objects according to some parameter. Typically, if such a construction is regular enough, one can encode it using a *succession rule* [13, 14], which is a purely formal system, generally expressed as follows:

$$\begin{cases} (a) \\ (k) \rightsquigarrow (e_1(k)) \cdots (e_k(k)) \end{cases} . \qquad (1)$$

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Here the letters denote positive integers, (a) is called the *axiom* and $(k) \rightsquigarrow (e_1(k)) \cdots (e_k(k))$ is the *production* of (k). A succession rule can be represented by means of its *generating tree*, which is, by definition, the infinite, rooted, labelled tree whose root is labelled a (like the axiom) and such that every node labelled k produces k sons, labelled respectively $e_1(k), \ldots, e_k(k)$. One of the main enumerative information provided by a succession rule is the numerical sequence of the cardinalities of the levels of the generating tree associated with the rule: we will refer to such a sequence as the numerical sequence *determined by* the succession rule.

The basic reference for the ECO method is [6], in which many examples can also be found.

The importance of succession rules as a tool for the ECO method has lead to several investigations to get a better mathematical insight on them. In [12] the authors define the concept of *rule operator*, thus translating a succession rule into a linear operator on one-variable polynomials. In [10] every succession rule is associated with at least two infinite matrices: the *production matrix*, which is essentially the matrix of the related rule operator with respect to the canonical basis of polynomials $(x^n)_{n\in\mathbf{N}}$, and the *ECO matrix*, whose (n,k)-entry is, by definition, the number of nodes labelled k at level n in the corresponding generating tree. We point out that a few instances of the notion of production matrix appeared for the first time in [13] under the name of "transfer matrices".

Another combinatorial theory dealing with infinite matrices is Aigner's theory of Catalan-like numbers [2, 3, 4, 15]. The basic idea of [2] is to characterize those sequences for which the determinants of the Hankel matrices of order 0 are all equal to 1. It is shown that such sequences appear as the first column of certain infinite matrices, called *admissible matrices*. These numbers are referred to with the name of *Catalan-like numbers*. The reason for this name lies in the fact that Catalan numbers are the unique sequence whose Hankel determinants of orders 0 and 1 equal 1.

In [3, 4] Aigner extends this theory by considering a more general kind of matrices, which we will rename *Aigner matrices* (instead of the infelicitous name "recursive matrices" used in [4]). Generalizing the previous definition, we will call *Catalan-like numbers* every sequence appearing as the first column of an Aigner matrix.

The aim of our work is to provide a "vocabulary" to translate the ECO method into Aigner's theory, and vice versa. Such a vocabulary turns out to be based on linear algebra tools, consisting of a suitable change of basis in the vector space of one-variable polynomials. What we hope to show in this paper is that the two theories under consideration are, in some algebraic sense, the two sides of the same medal, which is quite surprising if we think of the very different starting points, and combinatorial meanings, of such theories.

After a brief survey of the notions we need from the ECO method and Aigner's theory, we provide the main linear algebra tools to be used in the sequel. In particular, we define the Aigner basis in the vector space of one-variable polynomials and prove some of its properties. Next we introduce what we call the *fundamental change of basis*, which is the key ingredient to accomplish our project, and propose some examples to illustrate our approach. The final part of the work is devoted to the study of two particular cases, for which we are able to fully describe how to switch from one theory to the other. In the last section we give some hints for possible, future works.

Before starting, a few words concerning our notations. We have chosen to use $\mathbf{N}$ to denote natural numbers (zero included), whereas $\mathbf{N}^*$ is the set of natural numbers without zero. The symbol $\mathbf{x}$ is used to denote the operator of multiplication by x, in order to distinguish it from the symbol x, used as a polynomial variable. The symbol $^\top$ denotes the transpose of a matrix. The last remark concerns the way we have chosen to index the lines of our matrices. Classically, the lines of an Aigner matrix are indexed by $\mathbf{N}$, whereas, in an ECO matrix, the first column is usually column 1 (so that columns are indexed by $\mathbf{N}^*$). There are clear combinatorial and algebraic reasons for this: the n-th column of an ECO matrix gives the distribution of label (n) in the generating tree, whereas the scalar product of the n-th and the m-th rows of an admissible matrix gives the first element of its $(n+m)$-th row. Unfortunately, keeping both these conventions would result in a purely formal, but greatly inelegant, variations of our results: namely, the fundamental change of basis (which is degree-preserving in our theory) would translate x^n into a polynomial of degree $n-1$ (which is $p_{n-1}(x)$, according to the notations of section 3). To avoid this we have preferred to use $\mathbf{N}^*$ as the set of indices for the lines of Aigner matrices. In this way the (nice) characteristic algebraic property of admissible matrices becomes a little bit difficult to read, but our theory can be described much more elegantly.

2 Preliminaries on ECO and Aigner matrices

In this section we report the main facts of the two combinatorial theories we are going to compare.

Consider a succession rule as in (1). Instead of representing it by means of a generating tree, one can choose linear algebra notations. In the vector space of one-variable polynomials, define the linear operator $L = L_\Omega$ on the canonical basis $(x^n)_{n\in\mathbf{N}}$ as follows:

$$\begin{aligned} L(\mathbf{1}) &= x^a, \\ L(x^k) &= x^{e_1(k)} + \cdots + x^{e_k(k)}, \\ L(x^h) &= hx^h, \quad \text{if } (h) \text{ is not a label of } \Omega. \end{aligned}$$

L is called the *rule operator* associated with Ω (see [12]): it bears all the enumerative properties of a succession rule and allows to express such properties using algebraic notations. For example, if $(f_n)_{n\in\mathbf{N}}$ is the numerical sequence determined by Ω, then we can find f_n using the rule operator L of Ω as follows:

$$f_n = [L^{n+1}(\mathbf{1})]_{x=1}.$$

Throughout the present work, we will always deal with a special case, namely we assume that $\deg L^n(\mathbf{1}) = n$. From a combinatorial point of view, this means that the set of the labels of Ω is $\mathbf{N}^*$ and the maximum label among those produced by (k) is $(k+1)$.

The infinite matrix $P = P_\Omega$ representing L with respect to the canonical basis $(x^n)_{n\in\mathbf{N}}$ is called the *production matrix* of Ω. Such matrices are extensively studied in [10]; here we only recall some of their properties.

Let A_P be the infinite matrix whose n-th row vector is given by $u^\top P^{n-1}$ (where $u^\top = (1\ 0\ 0\ \ldots\ 0\ \ldots)$). Then A_P describes the statistic given by the distribution of the labels at the various levels of the generating tree related to Ω. Namely, the (n,k) entry of A_P is the number of nodes labelled k at level n of the generating tree of Ω. A_P is called the *ECO matrix* associated with P (or with Ω), and is also characterized by the matrix equality $DA_P = A_P P$, where

$$D = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

In terms of the production matrix P, the sequence $(f_n)_{n\in\mathbf{N}}$ is nothing else than the sequence of the row sums of the associated ECO matrix. The ordinary and exponential generating functions of Ω are given, respectively, by $f_P(t) = u^\top(I - tP)^{-1}e$ and $F_P(t) = u^\top \exp(tP)e$, where e is the column vector $(1\ 1\ 1\ \ldots\ 1\ \ldots)^\top$ and $\exp(X)$ denotes the usual matrix exponential of the (infinite) matrix X. In section 5 we deal with some examples of the theory we are going to develop; in describing such examples we also consider the production and ECO matrices of some classical succession rules. The reader is invited to have a look to those example in order to be introduced to these concepts.

In a recent series of nice and well-written papers [2, 3, 4], Martin Aigner has developed a new theory to deal with numerical sequences somehow linked to the sequence of Catalan numbers. One of the main tools of this theory is a particular class of infinite (triangular) matrices, called *admissible matrices* in [2] and renamed with the infelicitous term *recursive matrices* after [4]. In the sequel we will use the following terminology.

Consider a lower triangular matrix $A = (a_{n,k})_{n,k\in\mathbf{N}^*}$ with main diagonal equal to 1. A is called an *admissible matrix* whenever, for every $n, m \in \mathbf{N}^*$, the (ordinary) scalar product of the n-th and the m-th rows of A gives the first element of the $(n+m-1)$-th row; in symbols:

$$\sum_{k\geq 1} a_{n,k} a_{m,k} = a_{n+m-1,1}.$$

More generally, we define A to be an *Aigner matrix* when there exists a sequence of nonnegative integers $(T_n)_{n\in\mathbf{N}^*}$ such that $T_1 = 1$ and $T_n | T_{n+1}$ for which we have

$$\sum_{k\geq 1} a_{n,k}a_{m,k}T_k = a_{n+m-1,1}. \tag{2}$$

Obviously admissible matrices correspond to Aigner matrices for which $T_n \equiv 1$.

If A is an Aigner matrix, the sequence $(a_{n,1})_{n\in\mathbf{N}^*}$ is called the sequence of *Catalan-like numbers* associated with A. For an extensive study of the algebraic and enumerative properties of Aigner matrices the reader is referred to [2, 4] and to the further items cited in the references of the two papers. Here we recall only those results which we need for our purposes.

PROPOSITION 2.1 ([2, 4]) *An Aigner matrix $A = (a_{n,k})_{n,k\in\mathbf{N}^*}$ is uniquely determined by the two sequences $(a_{n+1,n})_{n\in\mathbf{N}^*}$ and $(T_n)_{n\in\mathbf{N}^*}$. Conversely, to every pair of sequences $(b_n)_{n\in\mathbf{N}^*}$ and $(T_n)_{n\in\mathbf{N}^*}$ of real numbers there exists an (and therefore precisely one) Aigner matrix $A = (a_{n,k})_{n,k\in\mathbf{N}^*}$ associated with $(T_n)_{n\in\mathbf{N}^*}$ and such that $a_{n+1,n} = b_n$ for all n.*

PROPOSITION 2.2 ([2, 4]) *Let $A = (a_{n,k})_{n,k\in\mathbf{N}^*}$ be an Aigner matrix associated with $(T_n)_{n\in\mathbf{N}^*}$. Set $s_1 = a_{2,1}, s_n = a_{n+1,n} - a_{n,n-1}$ and $t_n = \frac{T_n}{T_{n-1}}$, for $n \geq 2$. Then we have*

$$\begin{aligned} &a_{1,1} = 1, \qquad a_{1,k} = 0 \quad (k > 1) \\ &a_{n,k} = a_{n-1,k-1} + s_k a_{n-1,k} + t_{k+1}a_{n-1,k+1} \quad (n \geq 2). \end{aligned} \tag{3}$$

Conversely, if $a_{n,k}$ is given by the recursion (3), then $A = (a_{n,k})_{n,k\in\mathbf{N}^*}$ is an Aigner matrix with $T_n = t_2 \cdot \ldots \cdot t_n$ and $a_{n+1,n} = s_1 + \cdots + s_n$.

Setting $\sigma = (s_n)_{n\in\mathbf{N}^*}$, $\tau = (t_n)_{n\geq 2}$, we say that $A = A^{\sigma,\tau}$ is the Aigner matrix *of type* (σ,τ) when recursion (3) holds for its entries.

It is possible to write (2) in a compact matrix form. Define the diagonal matrix T and the infinite Hankel matrix of the sequence $(a_{n,1})_{n\in\mathbf{N}^*}$ as follows:

$$T = \begin{pmatrix} T_1 & 0 & 0 & \cdots \\ 0 & T_2 & 0 & \cdots \\ 0 & 0 & T_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad H = \begin{pmatrix} a_{1,1} & a_{2,1} & a_{3,1} & \cdots \\ a_{2,1} & a_{3,1} & a_{4,1} & \cdots \\ a_{3,1} & a_{4,1} & a_{5,1} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then (2) can be written as $ATA^\top = H$. More precisely, we have the following characterization.

PROPOSITION 2.3 ([4]) *A is an Aigner matrix if and only if $ATA^\top = H$ with $T_n \neq 0$ for all $n \geq 2$, $T_1 = 1$. The sequences σ and τ are then given as in proposition 2.2.*

Moreover, if we denote by H_n the n-th Hankel matrix of a sequence $(a_n)_{n\in\mathbf{N}^*}$ (which is, by definition, the submatrix of H consisting of rows and columns 1 to n), we have the following corollary.

COROLLARY 2.1 ([4]) *A sequence $(a_n)_{n\in\mathbf{N}^*}$ is Catalan-like if and only if $|H_n| \neq 0$ for all $n \geq 1$.*

3 Linear algebra tools

In this section we give some (elementary) linear algebra tools which will be necessary in the sequel to settle the stated analogy between ECO method and Aigner's theory of Catalan-like numbers.

In the vector space of one-variable polynomials over the real field (to be denoted $\mathbf{R}[x]$) we define the following polynomial sequences:

$$\begin{aligned} p_0(x) &= 1, \\ p_n(x) &= x(x-1)^{n-1}, \quad \forall n \geq 1. \end{aligned}$$

It is clear that deg $p_n(x) = n$, so that $(p_n(x))_{n\in\mathbf{N}}$ constitutes a basis for the vector space $\mathbf{R}[x]$. We call such a basis the *Aigner basis* of $\mathbf{R}[x]$.

Remark. We recall that the polynomial $p_n(x)$ has a very important combinatorial meaning: it is the chromatic polynomial of a tree having n vertices (see, for example, [9]). However, this fact will not be used in this paper.

It is well known that, for any basis of $\mathbf{R}[x]$, there exists a unique differential[1] linear operator mapping the n-th element of the basis into the $(n-1)$-th one. According to [12], we call *factorial derivative operator* the linear operator:

$$\begin{aligned} T : \mathbf{R}[x] &\longrightarrow \mathbf{R}[x] \\ : p_0(x) &\longrightarrow \mathbf{0}, \\ : p_n(x) &\longrightarrow p_{n-1}(x), \quad n \geq 1. \end{aligned}$$

The factorial derivative operator can also be nicely performed on the canonical basis of $\mathbf{R}[x]$.

PROPOSITION 3.1 *For any $n \in \mathbf{N}$, $T(x^n) = 1 + x + \cdots + x^{n-1} = \sum_{k=0}^{n-1} x^k$.*

Proof. For the first values of n, we have $T(\mathbf{1}) = \mathbf{0}, T(x) = \mathbf{1}$, $T(x^2) = T(p_1(x) + p_2(x)) = 1 + x$. By induction, suppose that $T(x^n) = 1 + \cdots + x^{n-1}$. Observe that we have the trivial equality $T((x-1)p_n(x)) = T(p_{n+1}(x)) = p_n(x)$, whence $T(xp_n(x)) = T(p_n(x)) + p_n(x)$. By linearity we then have:

$$T(xp(x)) = T(p(x)) + p(x).$$

Therefore, in the case $p(x) = x^n$, we get:

$$\begin{aligned} T(x^{n+1}) &= T(x \cdot x^n) = T(x^n) + x^n \\ &= 1 + \cdots + x^{n-1} + x^n, \end{aligned}$$

[1] A polynomial operator is called a *differential operator* when it maps a polynomial of degree n into a polynomial of degree $n-1$ (see [1]).

which is the thesis. □

Some properties of the factorial derivative operator are recorded in [12].

The Aigner basis has a nice behavior with respect to the usual multiplication operation on polynomials.

PROPOSITION 3.2 *Let* $n, m \geq 1$.

1. $$p_n(x) \cdot p_m(x) = x p_{n+m-1}(x);$$
2. $$x^k p_n(x) = \sum_{h=0}^{k} \binom{k}{h} p_{n+h}(x);$$

   in particular,

   $$x p_n(x) = p_{n+1}(x) + p_n(x);$$

3. *denoting by* $\mathbf{x}^{-1}$ *the linear operator defined by* $\mathbf{x}^{-1}(p(x)) = \frac{p(x)-p(0)}{x}$ *(so that* $\mathbf{x}^{-1}$ *is the usual difference quotient operator), it is*

   $$\mathbf{x}^{-k} p_n(x) = \sum_{h=0}^{n-k} (-1)^{n-k-h} \binom{n-1-h}{k-1} p_h(x) \quad (k \geq 1);$$

   in particular, setting $k = 1$*, we have*

   $$\mathbf{x}^{-1} p_n(x) = \sum_{h=0}^{n-1} (-1)^{n-h-1} p_h(x) = p_{n-1}(x) - p_{n-2}(x) + p_{n-3}(x) - \cdots .$$

Proof.

1. Trivial.
2. We have immediately:

   $$x^k p_n(x) = (x-1+1)^k p_n(x) = \sum_{h=0}^{k} \binom{k}{h} (x-1)^h p_n(x)$$
   $$= \sum_{h=0}^{k} \binom{k}{h} x(x-1)^{n+h-1} = \sum_{h=0}^{k} \binom{k}{h} p_{n+h}(x).$$

3. The equality can be proved by induction on $k \geq 1$. The details are left to the reader. □

We close this section by stating a technical result, useful in the computation of the powers of the factorial derivative operator T, which can be proved by induction.

PROPOSITION 3.3 *For $n, k \in \mathbf{N}$, $k \geq 1$, we have:*

$$T^k(x^n) = \sum_{h=0}^{n-k} \binom{n-h-1}{k-1} x^h.$$

In particular, specializing the above equality, we get:

1. $[T^k(x^n)]_{x=1} = \binom{n}{k}$;
2. $[T^k(x^n)]_{x=0} = \binom{n-1}{k-1}$.

4 The fundamental change of basis

For our purposes, we slightly modify the definition of ECO matrix given in [10], namely we suppose that the n-th row of F (with $n \in \mathbf{N}^*$) gives the distribution of the various labels *at level* $n-1$ of the generating tree of the rule; thus the (n, k) entry of F is the number of nodes labelled k *at level* $n-1$. Let Ω be a succession rule as in (1) and suppose that $F = (f_{n,k})_{n,k \in \mathbf{N}^*}$ is the ECO matrix associated with Ω, as explained above. We denote by $\beta_n(x)$ the polynomial canonically associated with the n-th row of F, namely:

$$\beta_n(x) = \sum_{k=1}^{n} f_{n,k} x^k \quad (n \geq 1). \tag{4}$$

As we have already remarked, we will always assume that deg $\beta_n(x) = n$. Now expand the polynomials $\beta_n(x)$ in terms of the Aigner basis, thus obtaining

$$\beta_n(x) = \sum_{k=1}^{n} a_{n,k} p_k(x). \tag{5}$$

Clearly, the coefficients $f_{n,k}$ and $a_{n,k}$ are intimately related. In particular, the following, simple result shows that this setting is the right one to achieve our project.

PROPOSITION 4.1 *The succession $(r_n)_{n \in \mathbf{N}^*} = (\sum_{k=1}^{n} f_{n,k})_{n \in \mathbf{N}^*}$ of the row sums of F is equal to the succession $(a_{n,1})_{n \in \mathbf{N}^*}$ given by the first column of the matrix $A = (a_{n,k})_{n,k \in \mathbf{N}^*}$. In symbols:*

$$a_{n,1} = \sum_{k=1}^{n} f_{n,k}.$$

Proof. It is obvious that $r_n = \beta_n(1)$, whence $\sum_{k=1}^{n} a_{n,k} p_k(1) = r_n$. By the definition of the Aigner basis, it is $p_k(1) = 0$, for $k > 1$, and $p_1(1) = 1$, and so $r_n = a_{n,1}$, as desired. □

The results obtained so far can be naturally expressed also in matrix notation. Specifically, it turns out that the fundamental change of basis described in (5) is represented as the multiplication on the right by the Pascal matrix. In other words, if $P = \left(\binom{n}{k}\right)_{n,k\in\mathbf{N}}$ is the usual Pascal matrix, and $F = (f_{n,k})_{n,k\in\mathbf{N}^*}$ a given ECO matrix, then the associated matrix $A = (a_{n,k})_{n,k\in\mathbf{N}^*}$ can be expressed as follows:

$$A = FP.$$

So the Pascal matrix P is strictly related to the matrix of the change of basis from $(x^n)_{n\in\mathbf{N}}$ to the Aigner basis $(p_n(x))_{n\in\mathbf{N}}$, which is precisely the matrix obtained by P by adding a new first row $(1\ 0\ 0\ 0\ \cdots)$ and a new first column $(1\ 0\ 0\ 0\ \cdots)^\top$.

Now cards are laid on the table: changing the canonical basis into the Aigner basis seems to be the "linear algebra" way to switch from the ECO method to Aigner's theory. At this stage, the following, very natural question can be asked:

1) *for which ECO matrices F does it happen that the matrix A is an Aigner matrix?*

Regarding things the other way round, one can start with an Aigner matrix A and perform the inverse change of basis (from $p_n(x)$ to x^n). Therefore the previous question can be inverted:

1) *for which Aigner matrices A does it happen that the matrix F is an ECO matrix?*

These two problems seems to be rather difficult to be tackled in their full generality. In the rest of the paper we will mainly focus on special examples to hopefully illustrate the interest of our approach. Finally, we will consider two particular classes of ECO matrices (namely, those arising from the so-called *factorial succession rules* and *differential succession rules*), giving for them a complete answer to question 1.

Before closing this section, we record some further notations and results which will be useful in the sequel.

If L is the rule operator associated with Ω, then for the polynomials $\beta_n(x)$ in (4) we clearly have

$$\beta_n(x) = L^n(\mathbf{1}).$$

Applying L means to shift from row n to row $n+1$ in the ECO matrix associated with Ω, whence:

$$L(\beta_n(x)) = \beta_{n+1}(x). \tag{6}$$

The coefficients $f_{n,k}$ and $a_{n,k}$ in (4) and (5) can be expressed in linear algebraic terms, as the following proposition clarifies.

Proposition 4.2 *For $n, k \in \mathbf{N}^*$, we have:*

1. $f_{n,k} = \left[\frac{D^k}{k!}(\beta_n(x))\right]_{x=0}$.

2. $a_{n,k} = [T^k(\beta_n(x))]_{x=0}$.

The easy proof is left to the reader.

COROLLARY 4.1 *For any n, $a_{n+1,n} = f_{n+1,n} + nf_{n+1,n+1}$.*

Proof. From the above proposition we have

$$a_{n+1,n} = [T^n(\beta_{n+1}(x))]_{x=0} = \sum_{k=1}^{n+1} f_{n+1,k}[T^n(x^k)]_{x=0}.$$

Now, recalling proposition 3.3, we get immediately:

$$a_{n+1,n} = \sum_{k=1}^{n+1} \binom{k-1}{n-1} f_{n+1,k} = f_{n+1,n} + nf_{n+1,n+1},$$

as desired. □

To conclude, we prove the following proposition, concerning the behavior of a rule operator when applied to the Aigner basis.

PROPOSITION 4.3 *For any rule operator L and for any $n > 2$, we have:*

$$[L(p_n(x))]_{x=1} = 0.$$

Proof. It is easy to see [12] that, for any rule operator L, it is

$$[L(p(x))]_{x=1} = [D(p(x))]_{x=1}.$$

(This is due to the fact that $[L(x^k)]_{x=1} = k = [D(x^k)]_{x=1}$). Then we get immediately:

$$[L(p_n(x))]_{x=1} = [D(p_n(x))]_{x=1} = [Dx(x-1)^{n-1}]_{x=1} = 0.$$ □

5 Detailed examples

In this section we will provide a detailed analysis of the effects of the fundamental change of basis in the case of a well-known succession rule determining Catalan numbers. Then some other examples will be dealt with; for them, we will only state the main facts (without giving proofs), however, our results can be checked out by a direct computation or by applying the theory we are going to develop in the final part of our work.

5.1 Catalan numbers

Let us consider the succession rule

$$\Omega : \begin{cases} (1) \\ (1) \rightsquigarrow (2) \\ (k) \rightsquigarrow (2)(3)(4)\cdots(k)(k+1) \end{cases},$$

defining Catalan numbers $1, 1, 2, 5, 14, 42, 132, \ldots$ (see, for example, [6]). The first lines of the ECO matrix associated with Ω looks as follows:

$$F = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 2 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 5 & 5 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 14 & 14 & 9 & 4 & 1 & 0 & 0 & \cdots \\ 0 & 42 & 42 & 28 & 14 & 5 & 1 & 0 & \cdots \\ 0 & 132 & 132 & 90 & 48 & 20 & 6 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Applying the fundamental change of basis, one gets the following matrix:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 2 & 3 & 1 & 0 & 0 & \cdots \\ 5 & 9 & 5 & 1 & 0 & \cdots \\ 14 & 28 & 20 & 7 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Is A the (unique) admissible matrix for Catalan numbers found in [2]? The first fact suggesting that it could be so follows from the application of corollary 4.1. It is well known [12] that the entries of F are the so-called *ballot numbers*, namely:

$$f_{n,k} = \frac{k-1}{n-1}\binom{2n-k-2}{n-k} \qquad (n, k > 1).$$

Therefore it follows immediately that, in A, we have:

$$\begin{aligned} a_{n+1,n} &= f_{n+1,n} + n f_{n+1,n+1} \\ &= \frac{n-1}{n}\binom{n}{1} + n \cdot \frac{n}{n}\binom{n-1}{0} = n - 1 + n = 2n - 1, \end{aligned}$$

which agrees with the $(n+1, n)$ entry of the admissible matrix for Catalan numbers. Obviously, this is not enough to conclude, however, it is in fact a strong hint. To get to the desired result we need to show, for example, that the coefficients $a_{n,k}$ obey the following recursion (deduced from [2]):

$$a_{n+1,k} = a_{n,k-1} + 2a_{n,k} + a_{n,k+1}.$$

Expressing the $a_{n,k}$'s as in proposition 4.2 we then find:

$$[T^k(\beta_{n+1}(x))]_{x=0} = [(T^{k-1} + 2T^k + T^{k+1})(\beta_n(x))]_{x=0},$$

whence, recalling proposition 3.3:

$$\sum_{h=1}^{n+1} \binom{h-1}{k-1} f_{n+1,h} = \sum_{h=1}^{n} \left(\binom{h-1}{k-2} + 2\binom{h-1}{k-1} + \binom{h-1}{k} \right) f_{n,h}. \qquad (7)$$

The sum of the binomial coefficients in the r.h.s. of (7) can be easily simplified (using well known properties of the Pascal matrix) to obtain:

$$\sum_{h=1}^{n+1} \binom{h-1}{k-1} f_{n+1,h} = \sum_{h=1}^{n} \binom{h+1}{k} f_{n,h}. \qquad (8)$$

To prove equality (8) we make use of the structural properties of the ECO matrix F, namely the recursion:

$$\begin{aligned} f_{n+1,h} &= f_{n,h-1} + f_{n,h} + \cdots + f_{n,n} \\ &= \sum_{i=h-1}^{n} f_{n,i}. \end{aligned}$$

Replacing in the l.h.s. of (8) and interchanging the order of the summations when necessary, we get:

$$\begin{aligned} \sum_{h=1}^{n} \binom{h+1}{k} f_{n,h} &= \sum_{h=1}^{n+1} \binom{h-1}{k-1} \left(\sum_{i=h-1}^{n} f_{n,i} \right) \\ &= \sum_{i=0}^{n} \left(\sum_{h=1}^{i+1} \binom{h-1}{k-1} \right) f_{n,i} = \sum_{i=0}^{n} \binom{i+1}{k} f_{n,i}, \end{aligned}$$

which is an identity ($f_{n,0} = 0$ by convention). Therefore, we have formally proved that *switching from the canonical basis to the Aigner basis translates the ECO matrix F of Catalan numbers into the (unique) admissible matrix A of Catalan numbers.*

5.2 Motzkin numbers

We can use the same approach to deal with Motzkin numbers. Consider the succession rule

$$\Omega : \begin{cases} (1) \\ (1) \rightsquigarrow (2) \\ (k) \rightsquigarrow (1)(2)(3)\cdots(k-1)(k+1) \end{cases}.$$

Ω determines the sequence of Motzkin numbers ([6]), and its ECO matrix is the following:

$$F = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 0 & 1 & 0 & 0 & \cdots \\ 3 & 2 & 3 & 0 & 1 & 0 & \cdots \\ 6 & 7 & 3 & 4 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Applying the fundamental change of basis (or, equivalently, multiplying on the right by the Pascal matrix P) we get to the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 2 & 1 & 0 & 0 & 0 & \cdots \\ 4 & 5 & 3 & 1 & 0 & 0 & \cdots \\ 9 & 12 & 9 & 4 & 1 & 0 & \cdots \\ 21 & 30 & 25 & 14 & 5 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

It can be proved that A is the unique admissible matrix associated with Motzkin numbers. However, it will be an immediate consequence of the results of section 6.

5.3 Bell numbers

The most popular succession rule giving rise to Bell numbers is the following:

$$\Omega : \begin{cases} (1) \\ (1) \rightsquigarrow (2) \\ (k) \rightsquigarrow (k)^{k-1}(k+1) \end{cases}.$$

Such a rule describes the usual construction of set partitions, and its ECO matrix is the following:

$$F = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 7 & 6 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 15 & 25 & 10 & 1 & 0 & \cdots \\ 0 & 1 & 31 & 89 & 65 & 15 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The matrix F is the well-known matrix of the Stirling numbers of the second kind. Applying the fundamental change of basis leads to the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 2 & 3 & 1 & 0 & 0 & \cdots \\ 5 & 10 & 6 & 1 & 0 & \cdots \\ 15 & 37 & 31 & 10 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

It is immediately seen that A is not an admissible matrix. Nevertheless, the elements of the main diagonal are all equal to 1, so A may be an Aigner matrix. Indeed, it can be shown that A is the Aigner matrix of type (σ, τ), where $\sigma = (k)_{k \in \mathbf{N}^*}$ and $\tau = (k-1)_{k \geq 2}$ [4]. This is our first example of an Aigner matrix which is not admissible and is linked to an ECO matrix by the fundamental change of basis.

5.4 Factorial numbers

The case of factorial numbers, which is extremely simple from the point of view of succession rules, turns out to be rather curious when the fundamental change of basis is applied. Indeed, the trivial rule

$$\Omega : \begin{cases} (1) \\ (k) \rightsquigarrow (k+1)^k \end{cases}$$

for the factorial numbers leads to the diagonal ECO matrix

$$F = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 6 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 24 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where, clearly, $f_{n,k} = (n-1)!\delta_{n,k}$ (δ is the usual Kronecker delta). Thus we have $\beta_n(x) = (n-1)!x^n = \sum_{k=1}^{n} (n-1)!\binom{n-1}{k-1} p_k(x)$, whence

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 2 & 4 & 2 & 0 & 0 & \cdots \\ 6 & 18 & 18 & 6 & 0 & \cdots \\ 24 & 96 & 144 & 96 & 24 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

It is clear that A cannot be an Aigner matrix, since the elements on the main diagonal are not equal to 1. However, if we consider the scalar multiplication of the $(n+1)$-th and $(m+1)$-th rows of A, supposing that $n \le m$, we get:

$$\sum_{k=1}^{n+1} a_{n+1,k} a_{m+1,k} = \sum_{k=0}^{n} n!m! \binom{n}{k} \binom{m}{k}$$
$$=^{(*)} (n+m)! \binom{n+m}{n} = a_{n+m+1,1}.$$

(Equality $^{(*)}$ is an application of Vandermonde's convolution). Thus A possesses a typical property of admissible matrices, without being neither admissible nor Aigner.

5.5 Involutions

Involutions are considered (from a succession rule point of view) in [12]. They are generated by the following succession rule:

$$\Omega : \begin{cases} (1) \\ (1) \rightsquigarrow (2) \\ (k) \rightsquigarrow (k-1)^{k-1}(k+1) \end{cases},$$

giving rise to the ECO matrix:

$$F = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 3 & 0 & 1 & 0 & \cdots \\ 3 & 0 & 6 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

In this case, the fundamental change of basis leads to the matrix:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 2 & 2 & 1 & 0 & 0 & \cdots \\ 4 & 6 & 3 & 1 & 0 & \cdots \\ 10 & 16 & 12 & 4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

which is the Aigner matrix of type (σ, τ), for $\sigma = (1)_{k \in \mathbf{N}^*}$ and $\tau = (k-1)_{k \ge 2}$ [4]. This case has some analogies with that of Bell numbers (for example, A is Aigner but not admissible).

6 Factorial succession rules

Referring to [5, 12], we recall the definition of a factorial succession rule and a factorial rule operator.

A *factorial succession rule* is a rule of the form:

$$\Omega : \begin{cases} (a) \\ (k) \rightsquigarrow (r_0)(r_0+1)\cdots(r_0+k-m-1)(k+d_1)\cdots(k+d_m) \end{cases},$$

for $k \geq r_0 \geq 1$. A *factorial rule operator* is the rule operator of a factorial rule. In [5] it is shown that factorial rules have an algebraic generating function. In [12] it is stated the following result, concerning the form of a factorial rule operator.

PROPOSITION 6.1 ([12]) *A rule operator L is factorial if and only if $L = p(\mathbf{x}, \mathbf{x}^{-1}, T)$, where $p(a, b, c)$ is a polynomial of degree 1 in c having the form:*

$$p(a, b, c) = u_0(a) + v_0(b) + u_1(a)c,$$

and T is the factorial derivative operator, as usual.

In the present section we give a complete answer to the first of the problems stated in section 4 for the class of ECO matrices arising from factorial rules. In order to accomplish our result, we need to slightly generalize Aigner's original setting.

Consider an infinite lower triangular matrix $A = (a_{n,k})_{n,k\in\mathbf{N}^*}$, with $a_{1,1} = 1$, and denote by L the linear operator associated with its rows, as in (6). We say that A is a *generalized Aigner matrix* when there exist three nonnegative integer sequences $(r_n)_{n\in\mathbf{N}}$, $(s_n)_{n\in\mathbf{N}}$, $(t_n)_{n\in\mathbf{N}}$ such that, for every $n \in \mathbf{N}$:

$$L(p_n(x)) = t_n p_{n-1}(x) + s_n p_n(x) + r_n p_{n+1}(x). \tag{9}$$

The following fact is an immediate consequence of the above definition.

PROPOSITION 6.2 *If A is a generalized Aigner matrix, then its entries obey the following recursion:*

$$\begin{cases} a_{1,1} = 1 \\ a_{n+1,k} = r_{k-1}a_{n,k-1} + s_k a_{n,k} + t_{k+1}a_{n,k+1} \end{cases}. \tag{10}$$

Proof. Since $L(\beta_n(x)) = \beta_{n+1}(x)$, using (9) we have:

$$\begin{aligned}\sum_{k=1}^{n+1} a_{n+1,k}p_k(x) &= \sum_{k=1}^{n} a_{n,k}L(p_k(x)) \\ &= \sum_{k=1}^{n} a_{n,k}(t_k p_{k-1}(x) + s_k p_k(x) + r_k p_{k+1}(x)) \\ &= \sum_{k=1}^{n+1}(r_{k-1}a_{n,k-1} + s_k a_{n,k} + t_{k+1}a_{n,k+1})p_k(x),\end{aligned}$$

whence the thesis follows. □

In particular, it is easily seen that a generalized Aigner matrix is not forced to have the elements on the main diagonal equal to 1.

Question: *does the converse of proposition 6.2 hold?*

Now we can focus on the case of factorial rules. Suppose that L is a factorial rule operator of the form

$$L = a(\mathbf{x}) + b(\mathbf{x}^{-1}) + c(\mathbf{x})T, \tag{11}$$

where $a(x) = \sum_{k\geq 0} a_k x^k$, $b(x) = \sum_{k\geq 1} b_k x^k$, $c(x) = \sum_{k\geq 0} c_k x^k$ are fixed polynomials. We can immediately find a sufficient condition for L to induce a generalized Aigner matrix.

PROPOSITION 6.3 *If $b(x) = 0$, deg $a(x) \leq 1$ and deg $c(x) \leq 2$, then A is a generalized Aigner matrix.*

Proof. We have to show that L acts as in (9). Indeed, a simple computation shows that

$$\begin{aligned} L(p_n(x)) &= ((a_0 + a_1 x) + (c_0 + c_1 x + c_2 x^2)T)(p_n(x)) \\ &= (c_0 + c_1 + c_2)p_{n-1}(x) + (a_0 + a_1 + c_1 + 2c_2)p_n(x) \\ &+ (a_1 + c_2)p_{n+1}(x), \end{aligned}$$

which is enough to conclude. □

Examples.

i) The succession rules for Catalan and Motzkin numbers described above are associated with rule operators for which, respectively, $a(x) = b(x) = 0$, $c(x) = x^2$ and $a(x) = x - 1$, $b(x) = 0$, $c(x) = x$.

ii) Consider the following succession rule, inducing Schröder numbers:

$$\begin{cases} (1) \\ (1) \rightsquigarrow (2) \\ (k) \rightsquigarrow (3)(4)(5)\cdots(k-1)(k)(k+1)^2 \end{cases} . \tag{12}$$

In this case, the rule operator L has the form:

$$L = \mathbf{x} - \mathbf{x}^2 + \mathbf{x}^3 T,$$

so it does not satisfy the hypotheses of the above proposition. Nevertheless, this rule is related to a generalized Aigner matrix, as we will see in the next pages.

THEOREM 6.1 *A factorial rule operator L as in* (11) *is associated with a generalized Aigner matrix if and only if the following conditions hold:*

i) $$\sum_{k\geq 0}\binom{k}{h}a_k+\sum_{k\geq 0}\binom{k}{h+1}c_k=0, \qquad \forall h\geq 2;$$

ii) $$\sum_{k\geq 1}(-1)^{n-k-h}\binom{n-1-h}{k-1}b_k=0, \qquad \forall h<n-1.$$

Proof. The rule operator L acts on the Aigner basis as follows:

$$\begin{aligned}L(p_n(x)) &= \sum_{k\geq 0}a_kx^kp_n(x)+\sum_{k\geq 1}b_kx^{-k}p_n(x)+\sum_{k\geq 0}c_kx^kp_{n-1}(x)\\ &= \sum_{k\geq 0}a_k\left(\sum_{h=0}^{k}\binom{k}{h}p_{n+h}(x)\right)\\ &\quad +\sum_{k\geq 1}b_k\left(\sum_{h=0}^{n-k}(-1)^{n-k-h}\binom{n-1-h}{k-1}p_h(x)\right)\\ &\quad +\sum_{k\geq 0}c_k\left(\sum_{h=0}^{k}\binom{k}{h}p_{n-1+h}(x)\right).\end{aligned}$$

Now, L must satisfy condition (9), which means that, in the above expansion, all the coefficients of the polynomials $p_k(x)$, for $k\notin\{n-1,n,n+1\}$, must be zero. This translates into conditions i) and ii) above, so the proof is complete. □

Example. The previous example related to Schröder numbers can now be reconsidered. It is clear that condition ii) is trivially verified, whereas the only interesting case of condition i) occurs when $h=2$, and we have:

$$\binom{2}{2}(-1)+\binom{3}{3}1=-1+1=0.$$

So rule (12) is associated with a *generalized* Aigner matrix A. By an explicit computation for the operator L, we find for A the following expression:

$$A=\begin{pmatrix}1&0&0&0&0&\cdots\\1&1&0&0&0&\cdots\\2&4&2&0&0&\cdots\\6&16&14&4&0&\cdots\\22&68&78&40&8&\cdots\\\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix},$$

where the entries obey the following recursion:

$$\begin{cases}a_{n+1,1}=a_{n,1}+a_{n,2},\\ a_{n+1,2}=a_{n,1}+3a_{n,2}+a_{n,3},\\ a_{n+1,k}=2a_{n,k-1}+3a_{n,k}+a_{n,k+1}, & \text{for } k>2\end{cases}.$$

At this stage, it is worth noting that in [4] an Aigner matrix for Schröder numbers is taken into consideration, precisely:

$$\tilde{A} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 0 & 0 & 0 & \cdots \\ 6 & 5 & 1 & 0 & 0 & \cdots \\ 22 & 23 & 8 & 1 & 0 & \cdots \\ 90 & 107 & 49 & 11 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

If we apply the inverse of the fundamental change of basis (that is, we switch from $p_n(x)$ to x^n), we find the following matrix:

$$\tilde{F} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 2 & 3 & 1 & 0 & 0 & \cdots \\ 6 & 5 & 10 & 1 & 0 & \cdots \\ 22 & 38 & 22 & 7 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Strictly speaking, $\tilde{F}$ is not an ECO matrix, as we can immediately notice. However, if we suppose that, in $\tilde{F}$, column k represents the distribution of the label $(2k)$, we manage to find an ECO-interpretation. From a linear operator point of view, we can consider the operator $\tilde{L}$ associated with the rows of $\tilde{F}$: if we replace the variable x with x^2, we in fact obtain a rule operator L, which corresponds to a well-known ECO-interpretation of Schröder numbers [8]. The succession rule related to L is the following:

$$\begin{cases} (2) \\ (2k) \rightsquigarrow (2)(4)^2(6)^2 \cdots (2k)^2(2k+2) \end{cases}.$$

7 Differential succession rules

We call *differential succession rule* each rule such that in the production of every label (k) at least one label greater or equal than $(k-1)$ has an exponent linearly depending on k. Equivalently, a *differential rule operator* is a rule operator which can be expressed in the form

$$L = p(\mathbf{x}, \mathbf{x}^{-1}, D) = a(\mathbf{x}) + b(\mathbf{x}^{-1}) + c(\mathbf{x})D.$$

Observe that, for reasons of consistency (in a succession rule a node labelled (k) must produce exactly k sons), in the above formula we necessarily have $c(x) = x^t$, for some $t \in \mathbf{N}$, so that a differential rule operator has the following, general expansion:

$$L = p(\mathbf{x}, \mathbf{x}^{-1}, D) = a(\mathbf{x}) + b(\mathbf{x}^{-1}) + x^t D. \tag{13}$$

The rules given in section 5 for Bell numbers, factorials and the numbers of involutions of a set are examples of differential succession rules, whose associated differential rule operators are, respectively, $\mathbf{x} - 1 + \mathbf{x}D$, $\mathbf{x}^2 D$ and $\mathbf{x} - \mathbf{x}^{-1} + D$. In [12] it is shown that the sequences determined by a differential succession rule possess a transcendental (ordinary) generating function. Using arguments which are essentially analogue to those employed for factorial succession rules, we can prove the following results, concerning the relationship with Aigner's theory.

THEOREM 7.1 *A differential rule operator L as in* (13) *is associated with a generalized Aigner matrix if and only if the following conditions hold:*

1. *if $t \neq 0$, the conditions are:*
   i) $$\sum_{k \geq 0} \binom{k}{h} a_k + n\binom{t}{h+1} - \binom{t-1}{h+1} = 0, \qquad \forall h \geq 2;$$
   ii) $$\sum_{k \geq 1} (-1)^{n-k-h} \binom{n-1-h}{k-1} b_k = 0, \qquad \forall h < n-1;$$
2. *if $t = 0$, the conditions are:*
   i) $$\sum_{k \geq 0} \binom{k}{h} a_k = 0, \qquad \forall h \geq 2;$$
   ii) $$\sum_{k \geq 1} (-1)^{n-k-h} \binom{n-1-h}{k-1} b_k - (-1)^{n-h} = 0, \qquad \forall h < n-1.$$

COROLLARY 7.1 *If $b(x) = 0$, deg $a(x) \leq 1$ and $t \in \{1, 2\}$, then L is associated with a generalized Aigner matrix.*

Examples. The cases of Bell numbers and factorial numbers can be easily tackled using the corollary (see above for the rule operators involved). As far as involutions are concerned, we have to consider the rule operator $\mathbf{x} - \mathbf{x}^{-1} + D$. Applying theorem 7.1 in the case $t = 0$, we have that condition i) is trivially satisfied, whereas condition ii) becomes:

$$(-1)^{n-1-h} \binom{n-1-h}{0} (-1) - (-1)^{n-h} = (-1)^{n-h} - (-1)^{n-h} = 0,$$

which is enough to conclude.

8 Conclusions and further work

The present work is intended to be only the first step towards a more detailed investigation of the relationship between the ECO method and the theory of Catalan-like numbers.

One of the first things to be done in the next future is to provide a more complete gallery of examples to illustrate the soundness of our approach. In particular, it would be nice to find new applications of the ECO method starting from known ones of Aigner's theory, and vice versa, which is what we hope to do in a forthcoming publication.

Another line of research is provided by the last section of [4], where Aigner introduces the basics of what he calls "ballot enumeration". The analogies with the techniques of the ECO method are evident and, in fact, the combinatorial model proposed by Aigner is a particular instance of an ECO construction.

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