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On a tomographic equivalence between $(0, 1)$ matrices

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Abstract. Tilings of the plane by translation show interesting regularity in the disposition of the tiles: in particular, we point out that in the case of a single tile, the obtained configurations are invariant by one translation. Furthermore, in the case of rectangular tiles, such a translation is the horizontal or the vertical one. Then we restrict our analysis on invariant matrices and we furnish a characterization result which links together invariance and homogeneity.

Mathematics Subject Classifications (2000). 05B45, 52C20

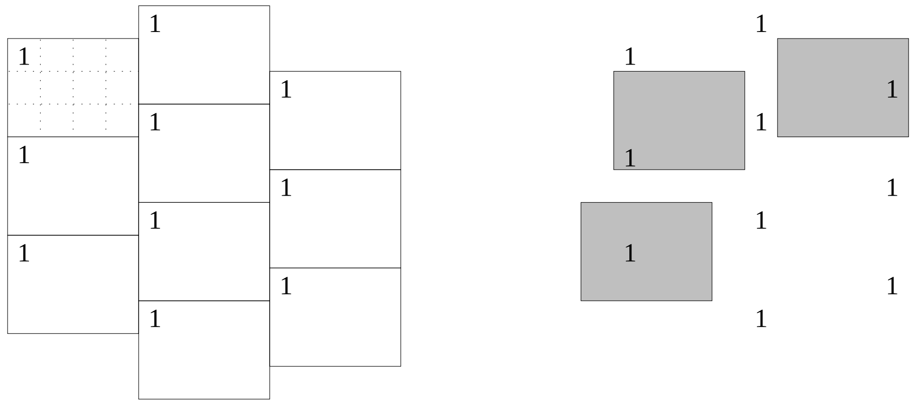
1 Introduction

The present paper relies on the basic fact that every tiling of the plane by translations of a given $m \times n$ rectangle is invariant by a single translation (either the horizontal translation of length m or the vertical translation of length n or both, in the particular case of a regular tiling).

This result integrates the studies pursued in [3] on the connections between k-homogeneous mappings in $\mathbb{Z}^2$ and the tilings of the plane with rectangles by translation.

In particular, if we mark the same point in each rectangular tile used in the planar tiling, then the obtained configuration of marked points has the property that if we look at it through a rectangular window of the same dimensions of the tile, there appears one and only one marked point in each position of the window. Figure 1 (a) shows a tiling of the plane with 4×3 rectangles which is invariant with respect to the vertical translation. Each tile has the upper-left element marked with 1. In Fig. 1 (b) the distribution of the 1's in the tiling is the same as the previous one, and some rectangular 4×3 windows are shown; a quick check reveals the presence of exactly one element 1 inside each of them.

This last fact is not characteristic of rectangles but is valid for any kind of pieces P one can use to tile the plane by translation. Therefore the following theorem holds:



(a) (b)

Figure 1: A tiling of $\mathbb{Z}^2$ with rectangles by translation, (a). Some positions of a 4×3 window moving around the obtained configuration of 1's in $\mathbb{Z}^2$, (b).

THEOREM 1 *Let $\mathcal{U}$ be a mapping of $\mathbb{Z}^2$ into $\{0,1\}$ and P be a mapping of a finite subset F of $\mathbb{Z}^2$ into $\{0,1\}$ such that $|\{f \in F : P(f) = 1\}| = 1$. The following are equivalent:*

(1) $\forall z \in \mathbb{Z}^2$, $|\{f \in F : \mathcal{U}(z+f) = 1\}| = 1$;

(2) $\mathbb{Z}^2 = \mathcal{U}^{-1}(1) \oplus F$.

This symbol $\oplus$ denotes the unambiguous Minkowski sum, i.e. $C = A \oplus B$ if and only if:

- $\forall c \in C$ $\exists a \in A$, $\exists b \in B$ such that $c = a + b$;

- $\forall a_1, a_2 \in A$, $\forall b_1, b_2 \in B$, it is $a_1 + b_1 = a_2 + b_2 \Rightarrow a_1 = a_2$ and $b_1 = b_2$.

Property (1) says that $\mathcal{U}$ contains one and exactly one 1 in each position of the window F, and property (2) says that $\mathbb{Z}^2$ is tiled by the translation of F (even more precisely that if we surround each 1 in $\mathcal{U}$ by a copy of F such that the 1 is always in the same position in F we obtain a tiling of $\mathbb{Z}^2$). This leads to the definition of homogeneous bidimensional sequence given in [3]:

a mapping $\mathcal{U} : \mathbb{Z}^2 \to \{0,1\}$ is *homogeneous bidimensional sequence of degree* k with respect to a finite window F if and only if

$$\forall z \in \mathbb{Z}^2, \ |\{f \in F : \mathcal{U}(z+f) = 1\}| = k.$$

In [3] it is proved the following rather surprising result:

Theorem 2 *A bidimensional sequence $\mathcal{U} : \mathbb{Z}^2 \to \{0,1\}$ is homogeneous of degree k with respect to a rectangle R if and only of there exist k disjoint homogeneous sequences of degree 1 (with respect to the same R) such that:*

$$\mathcal{U} = \mathcal{U}_1 + \mathcal{U}_2 + \ldots + \mathcal{U}_k.$$

This last can be nicely rephrased:

If a bidimensional sequence is homogeneous of degree k with respect to F, then one can color the $1's$ with k colors in such a way that in each position of the window there is one and only one 1 of each color, as shown in Fig. 2.

Example 1 A bidimensional sequence $\mathcal{U}$ which is 4-homogeneous with respect to a 4×3 window is represented in Fig. 2, (a) (from now on, we omit the symbols 0). The symbols 1 which belong to each of the four 1 homogeneous sequences $\mathcal{U}_1$, $\mathcal{U}_2$, $\mathcal{U}_3$, and $\mathcal{U}_4$ into which $\mathcal{U}$ decomposes, are labelled with a, b, c, and d, respectively (see Fig. 2, (b)).

1				1				1	
1	1				1			1	1
			1	1			1		
	1				1				1
1		1				1		1	
	1			1	1				1
		1				1			

(a)

a				a				a	
b	c				c			b	c
			d	b			d		
	a				a				a
b		c				c		b	
	d			b	d				d
		a				a			

(b)

Figure 2: The invariance correspondence in a $(0,1)$ homogeneous matrix.

We claim that the three symbols a, c, and d are invariant by the translation $(4,0)$ and the fourth one, corresponding to b, is invariant by the translation $(0,3)$.

We conjecture that Theorem 2 is valid for all *exact* windows i.e., those windows which can be used to tile the plane by translation. In fact it can be shown that there exists a sequence $\mathcal{U}$ which is homogeneous of degree 1 with respect to an window F if and only if F is exact (by Theorem 1), and thus Theorem 2 can hold only for exact windows. For the time being we are unable to furnish a proof in this more general case.

In the next section we focus our attention on homogeneous matrices with *integer* coefficients, and we prove an analogous of Theorem 2. Section 3 deals with R-null matrices, and it contains the following main result: each R-null matrix is decomposable as sum of 1 or -1 homogeneous matrices which are invariant by the horizontal or the vertical translation.

The generalization to bidimensional sequences (i.e. infinite matrices) of the results obtained in the next two sections is straightforward.

2 A decomposition theorem for homogeneous matrices with integer coefficients

From now on, we will start dealing with matrices rather than sequences. We define a matrix M of size $p \times q$ as a bidimensional sequence of restricted domain, i.e. a mapping of $\{0,1,\ldots,p-1\} \times \{0,1,\ldots,q-1\}$ into a set of coefficients which will be either $\{0,1\}$ or $\{-1,0,1\}$ or $\mathbb{N}$ or $\mathbb{Z}$ (note that such a definition of matrix allows an indexing of its elements which is different from the standard one).

We set $[p] = \{0,1,\ldots,p-1\}$. Also we slightly modify the definition of homogeneity: Let R be the rectangle $[m] \times [n]$; the matrix $M : [p] \times [q] \rightarrow \mathbb{Z}$ is *homogeneous of degree* k with respect to R if and only if $\forall (x,y) \in [p-m+1] \times [q-n+1]$ it holds

$$\sum \{M(x+i, y+j) : (i,j) \in [m] \times [n]\} = k.$$

Notice that, if the set of coefficients is $\{0,1\}$, this definition coincides with the previous one.

We call $R-$*projection* of a matrix M with coefficients in $\mathbb{Z}$, the matrix $R(M)$ of size $(p-m+1) \times (q-n+1)$ whose coefficients are

$$R(M)(x,y) = \sum \{M(x+i, y+j) : (i,j) \in [m] \times [n]\}.$$

A matrix is homogeneous with respect to R if and only if its R-projection is constant. Furthermore, a matrix is called $R-$*null* if and only if its $R-$*projection* is the matrix having all the coefficients equal to 0.

THEOREM 3 *A matrix M with coefficients in $\mathbb{N}$ (the set of non negative integers) is homogeneous of degree k with respect to R if and only if it is the sum of k matrices $M_1, M_2, \ldots, M_k$ with coefficients in $\{0,1\}$ which are homogeneous of degree* 1.

The proof is very similar to that of Theorem 2 given in [3], but it is slightly more difficult.

Let M and M' be two matrices of the same size $p \times q$ with coefficients in $\mathbb{N}$. We say that M' is smaller than M if and only if

$$\forall (x,y) \in [p] \times [q] \text{ it holds } M'(x,y) \leq M(x,y).$$

In order to prove Theorem 3, we show that if M is homogeneous of degree k with respect to R there exists a matrix M' with coefficients in $\{0,1\}$ which is homogeneous of degree 1 with respect to R, and it is smaller than M. Then we proceed in subtracting M' from M to obtain a new matrix whose coefficients are in $\mathbb{N}$, and which is homogeneous of degree $k-1$. Clearly we can repeat this process and eventually write M as a sum of $(0,1)$-matrices which are homogeneous of degree 1. Now we need a crucial lemma:

LEMMA 1 *If M is a matrix with coefficients in $\mathbb{Z}$, and it is homogeneous with respect to R, then for all $(x,y) \in [p] \times [q]$ such that $(x+m, y+n) \in [p] \times [q]$, it is*

$$M(x,y) + M(x+m, y+n) = M(x+m, y) + M(x, y+n).$$

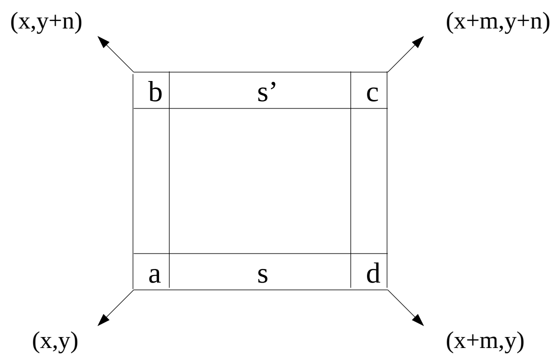
Proof. Set the values $s = \sum\{M(x+i,y) : 1 \leq i \leq m-1\}$ and $s' = \sum\{M(x+i, y+n) : 1 \leq i \leq m-1\}$.

We clearly have $M(x,y) + s = M(x,y+n) + s'$ and $s + M(x+m,y) = s' + M(x+m,y+n)$.

Consequently we get

$$s - s' = M(x,y+n) - M(x,y) = M(x+m,y+n) - M(x+m,y). \qquad \square$$

The following picture helps to visualize this property:



We just express the fact that the sum of the coefficients in the rectangle whose left inferior corner is (x,y) is equal to the sum of the coefficients in the rectangle whose left superior corner is $(x,y+n)$. The same equality holds for the sums of the coefficients in the two rectangles whose right inferior corner is $(x+m,y)$ and whose right superior corner is $(x+m,y+n)$.

Lemma 1 can be extended to get:

LEMMA 2 *If M is a matrix with coefficients in $\mathbb{Z}$, and it is homogeneous with respect to R, then for all $(x,y) \in [p] \times [q]$ and for all $\alpha, \beta \in \mathbb{Z}$, whenever $(x+\alpha m, y+\beta n)$ belongs to $[p] \times [q]$, it is*

$$M(x,y) + M(x+\alpha m, y+\beta n) = M(x, y+\beta n) + M(x+\alpha m, y).$$

The proof follows immediately by symmetry and induction.

Proof. (of Theorem 3)

First we assume that M is invariant by the translation $(m,0)$, and $\alpha, \beta \in \mathbb{Z}$. By definition, for all $(x,y) \in [p] \times [q]$, if $x + \alpha m \in [p]$, then $M(x+\alpha m, y) = M(x,y)$. Then one can find a $(0,1)$-matrix which is homogeneous of degree 1 with respect to R and smaller than M as follows: take any non zero element $M(x,y)$, with $(x,y) \in [m] \times [n]$, and consider all the values of β such that $y+\beta n \in [q]$. For each of them, there exists a strictly positive coefficient $M(x_\beta, y+\beta n)$ with $x_\beta \in x+[m]$. This is obvious since

$$\sum\{M(x_\beta, y+\beta n) : x_\beta \in x+[m]\} = \sum\{M(x_0, y) : x_0 \in x+[m]\}$$

and $\sum\{M(x_0,y) : x_0 \in x+[m]\} \geq M(x,y) > 0$.

Now all the coefficients $M(x_\beta + \alpha m, y+\beta n)$, with α, β such that$(x_\beta + \alpha m, y+\beta n) \in [p] \times [q]$, are strictly positive, and the $(0,1)$-matrix M' having all the elements equal to 0 except for those of the form $M'(x_\beta + \alpha m, y+\beta n)$ which are equal to 1, is smaller than M, and can be subtracted from it. The matrix $M - M'$ is homogeneous of degree $k-1$ and is also invariant by the translation $(m,0)$, thus a homogeneous matrix with coefficients in M which is invariant by $(m,0)$ is the sum of $(0,1)$-matrices which are homogeneous of degree 1 and are also invariant by $(m,0)$.

Now assume that M is not invariant by the translation $(m,0)$, i.e. there exists $(x,y) \in [p] \times [q]$ such that $M(x,y)$ and $M(x+m,y)$ are different, and let $M(x,y) > M(x+m,y)$ (if $M(x,y) < M(x+m,y)$, the argument is exactly the same).

As a consequence of Lemma 2, for all β such that $y+\beta n \in [q]$, it holds that $M(x, y+\beta n)$ is strictly positive.

Indeed $M(x,y+\beta n) + M(x+m,y) = M(x,y) + M(x+m, y+\beta n)$ implies that $M(x,y+\beta n) - M(x+m,y+\beta n) = M(x,y) - M(x+m,y)$ is strictly positive.

Consider now any column $x+\alpha m$, with $x+\alpha m \in [p]$, and compare the sums of n consecutive coefficients in the columns x and $x+\alpha m$, starting from the row y, i.e.

$$\sum\{M(x,y+j) : j \in [n]\} \quad \text{and} \quad \sum\{M(x+\alpha m, y+j) : j \in [n]\}.$$

Since the matrix M is homogeneous, then the two sums are equal. Thus two cases are possible

a. for all $j \in [n]$, we have $M(x+\alpha m, y-h+j) = M(x, y-h+j)$. This implies, by Lemma 2, that $M(x+\alpha m, j) = M(x,j)$ for all $j \in [q]$.

So, if we set $y_\alpha = y$, then it surely holds that $M(x+\alpha m, y_\alpha + \beta n)$ is strictly positive for all β such that $y_\alpha + \beta n \in [q]$.

b. There exists an index j such that $M(x+\alpha m, y+j) > M(x, y+j)$. Then if we set $y_\alpha = y+j$, the above argument assure that $M(x+\alpha m, y_\alpha + \beta n)$ is strictly positive for all β such that $y_\alpha + \beta n \in [q]$.

Now the matrix M' whose coefficients are all 0's except for those of the form $M'(x+\alpha m, y_\alpha + \beta n)$ which are equal to 1, is a $(0,1)$-matrix which is homogeneous of degree 1, invariant by the translation $(0,n)$ and smaller than M, so the proof is complete. □

Example 2 Consider the following matrix M which is homogeneous of degree 5 with respect to a rectangular 3×3 window:

1	2	1	1	1		1	2		1	1
				1						
		1			2			2		1
	1	3			2		1	2		
				1						
		1			2			2		1
	2	2		1	1		2	1		1
				1						

We show how to decompose it into five matrices of degree 1: first we choose a non zero element (the shadowed one in position $(0,7)$), and we check that it can only belong to a homogeneous matrix M' of degree 1 which is invariant by the horizontal translation $(3,0)$. We find one (the choice is between 2) and we subtract it from M to obtain the matrix (a) of Fig. 3, which is homogeneous of degree 4.

	2	1		1			2			1
				1						
		1			2			2		1
	1	2			1		1	1		
				1						
		1			2			2		1
	1	2			1		1	1		
				1						

(a)

	1	1					1			1
				1						
		1			2			2		
		2			1			1		
				1						
		1			2			2		
		2			1			1		
				1						

(b)

Figure 3: The first two steps of the decomposition of the matrix M in Example 2.

Then we choose a second element of $M - M'$ (that shadowed in position $(4,3)$ of Fig. 3 (a)), and we compute a second matrix M'' which is homogeneous of degree 1, invariant by the vertical translation $(0,3)$, and to which the chosen element belongs. We subtract it from the matrix (a) of Fig. 3, and we obtain the matrix in Fig. 3, (b) which is homogeneous of degree 3.

Repeating the same computations with two more elements of the matrix (b) in Fig. 3 leads to the full decomposition of M.

Theorem 3 can be easily generalized to homogeneous matrices with integer coefficient as follows:

Theorem 4 *A matrix with coefficients in $\mathbb{Z}$ is homogeneous of degree k with respect to a rectangle R if and only if it can be obtained as the difference between two sums of $(0,1)$-matrices which are homogeneous of degree 1 with respect to R. Furthermore, the number of elements in each of these two sums are bounded by:*

$$k + \sum\{-M(x,y) : M(x,y) < 0, (x,y) \in [p] \times [q]\}$$

and

$$k + \sum\{-M(x,y) : M(x,y) > 0, (x,y) \in [p] \times [q]\}.$$

Proof. Let M be a homogeneous matrix with coefficients in $\mathbb{Z}$, and let us consider a negative coefficient $M(x,y) = -a$, $a > 0$, if any.

Take any $(0,1)$-matrix M' of the same dimension of M, which is homogeneous of degree 1 and such that $M'(x,y) = 1$. The sum $M + aM'$ is a matrix with coefficients in $\mathbb{Z}$ which is homogeneous of degree $k + a$ and which satisfies

$$(M + aM')(i,j) \geq M(i,j) \text{ for all } (i,j) \in [p] \times [q].$$

By construction, $M + aM'$ has at least one negative coefficient less than M. Summing to M opportune matrices until all its negative coefficients disappear, we succeed in writing M as the difference $M_1 - M_2$ where M_1 and M_2 are two matrices having non negative coefficients, and such that M_1 is homogeneous of degree $k + \sum\{-M(x,y) : M(x,y) < 0, (x,y) \in [p] \times [q]\}$.

It may be more economical to have all the positive coefficient disappear if the sum of the positive coefficients is less than the absolute value of the sum of the negative coefficients.

When M has been written as the difference $M_1 - M_2$, we obtain the thesis by splitting M_1 and M_2 into sums of homogeneous $(0,1)$-matrices of degree 1 by means of Theorem 3. □

3 A deeper analysis of R-null matrices

Clearly, if the two matrices M_1 and M_2 in the proof of Theorem 4 have the same R-projections, then $M = M_1 - M_2$ is R-null. Studying R-null matrices is a natural way to study the equivalence between matrices defined by the equality of their R-projections. In fact we can interpret the problem of constructing a $(0,1)$-matrix with a given R-projection as a problem of discrete tomography: we are given a family of local pieces of information on a set of pixels distributed in a rectangle and the problem is to retrieve such information (see [2] for a complete introduction to discrete tomography).

The first problem of discrete tomography appearing in the literature is the problem of constructing a $(0,1)$-matrix with given row sums and column sums, which is formulated as follows: let $r_0, \ldots, r_{q-1}$ and $c_0, \ldots, c_{p-1}$ be two sequences of non negative integers such that

$$\sum\{r_i : i \in [q]\} = \sum\{c_j : j \in [p]\}.$$

Can one find a $(0,1)$-matrix of size $p \times q$ such that $\forall j \in [q]$ we have $\sum\{M(i,j) : i \in [p]\} = r_j$ and $\forall i \in [p]$ we have $\sum\{M(i,j) : j \in [q]\} = c_i$?

The $r_j's$ are called row sums, and the $c_i's$ are called columns sums (in more recent literature the vectors $\langle r_0, \ldots, r_{p-1}\rangle$ and $\langle c_0, \ldots, c_{q-1}\rangle$ are called the horizontal and the vertical projections, respectively). Solutions to this problem were given by Ryser [4], and independently by Gale [1] in the late fifties.

Our aim here is the study of a more general version of the following equivalence defined by Ryser: *M is equivalent to M' if and only if M and M' have the same horizontal and vertical projections.*

Ryser defined an elementary transformation which acts on the elements of a $(0,1)$-matrix by performing a mutual exchange of two 1's in position (x, y), $(x+h, y+l)$ with two 0's in position $(x+h, y)$ and $(x, y+l)$, if such a configuration exists (see Fig. 4, (a)).

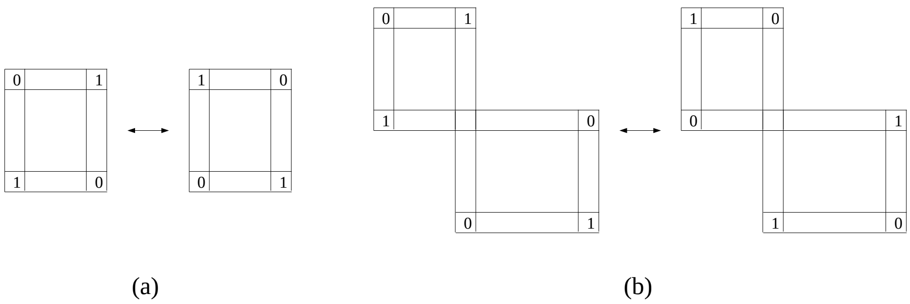


Figure 4: Two examples of Ryser transformations.

Clearly such a transformation keeps the two projections invariant. The nice result of Ryser is that if M and M' are equivalent, then one can transform M into M' by a sequence of elementary transformations. As an example, the transformation in Fig. 4, (b), is obtained by performing a sequence of two Ryser elementary transformations.

Introducing matrices with coefficients in $\{-1, 0, 1\}$ we can restate Ryser's result as follows:

THEOREM OF RYSER: *Every matrix M with coefficients in $\{-1, 0, 1\}$ whose horizontal and vertical projections are the constant vector equal to 0 is a sum of $M_1, \ldots, M_k$ matrices such that, for each $1 \leq i \leq k$, the matrix M_i is null but for the four different elements*

$$M_i(x_0, y_0) = M_i(x_1, y_1) = 1 \text{ and } M_i(x_0, y_1) = M_i(x_1, y_0) = -1.$$

We can prove a very similar theorem: let us say that a row of a matrix M is m-null if and only if the sum of any m-consecutive coefficients of that row is 0 (we note that a m-null row is invariant by the translation $(m, 0)$). We define n-null columns in the same way.

Obviously, adding a m-null row or a n-null column to a given matrix M does not change its R-projection. This fact leads to the nice result that whenever two matrices M and M' have the same R-projections then we can pass from one to the other by adding a certain number of m-null rows and n-null columns. In a more formal way, we can state:

THEOREM 5 *The set of all matrices which are full of* 0*'s but for a m-null row or a n-null column is a generating subset of the vector space of R-null matrices.*

Proof. In the decomposition of a R-null matrix as a sum of homogeneous $(0, 1)$ or $(0, -1)$-matrices of degree 1 or -1 it is clear that the number of $(0, 1)$ matrices will be the same as the number of $(-1, 0)$ matrices. Thus we can write $M = H_1 - H'_1 + H_2 - H'_2 + \cdots + H_k - H'_k$, where both H_i and H'_i are $(0, 1)$-matrices homogeneous of degree 1, with $1 \leq i \leq k$. We prove that each difference $M_i = H_i - H'_i$ is a sum of m-null rows and n-null columns.

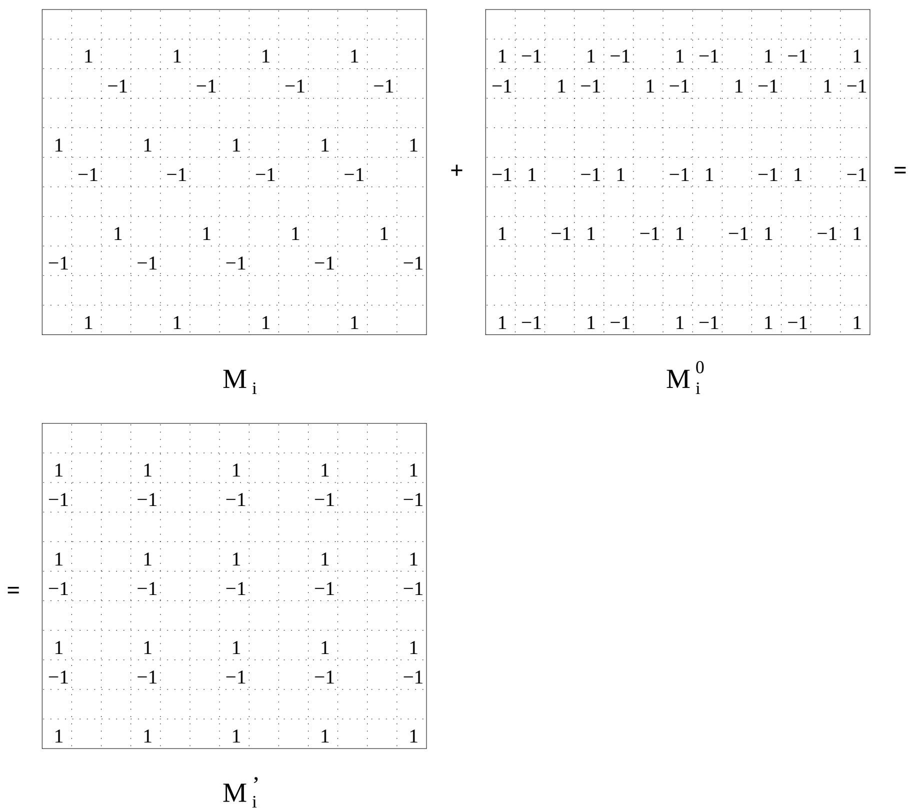


Figure 5: An example of the set of m-null rows M_i^0 which have to be added to M_i, in order to obtain the matrix M'_i having in the first column both elements 1 and -1.

First, we assume that both H_i and H'_i are invariant by the horizontal translation $(m, 0)$ (if they both are invariant by the vertical translation $(0, n)$, a similar argument holds), and also that each row of M_i does not contain both elements 1 and -1 (if so, all the rows of M_i are either full of 0's or contain both 1 and -1, and consequently they are m-null). We add to M_i a set of m-null rows M_i^0 in order to obtain a new R-null matrix M'_i, where each row is either full of 0's or starts with an element 1 or -1 (see Fig. 5 for an example).

Since each column of M'_i is either full of 0's or it contains both 1 and -1, so it is n-null, we obtain the thesis.

Now consider $M_i = H_i - H'_i$ where H_i is invariant by $(m, 0)$ and H'_i is invariant by $(0, n)$ (the choice of H_i invariant by $(0, n)$ and H'_i invariant by $(m, 0)$ is completely analogous). We rely on Fig. 6 for an example of this second case; we point out that the two 0's which appear in M_i, come from two elements 1 and -1 occurring in the same position.

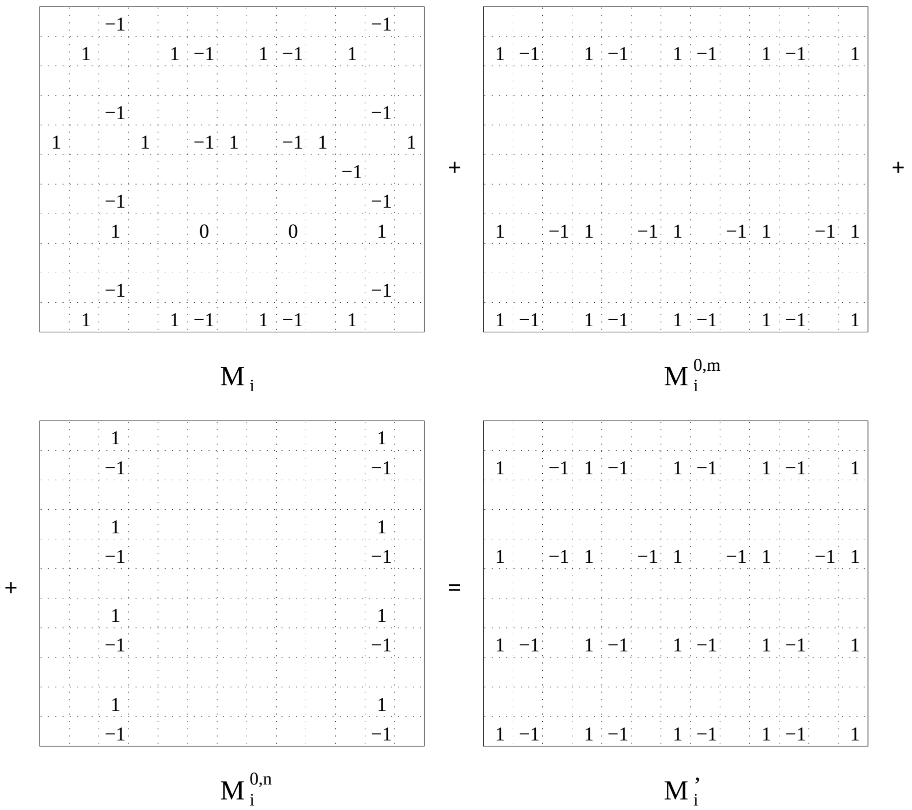


Figure 6: An example of two sets $M_i^{0,m}$ and $M_i^{0,n}$ of m-null rows and n-null columns, respectively, which have to be added to M_i to obtain a m-null matrix.

We proceed by adding to M_i a set of m-null rows (the matrix $M_i^{0,m}$ in Fig. 6) which move the first element 1, if any, of each row to the first position. The obtained matrix is clearly invariant by $(0, n)$.

Now it suffices to move all -1's of the columns of M'_i in the rows containing 1's by adding n-null columns (the matrix $M_i^{0,n}$ in Fig. 6), and we finally obtain a matrix which has only m-null rows. □

Since the addition operation is commutative, and a matrix whose rows [resp. columns] are all m-null [resp. n-null] is invariant by the translation $(m, 0)$ [resp. $(0, n)$] we can state:

Corollary 1 *Every R-null matrix M is the sum of M_1 and M_2 where M_1 is m-null and M_2 is n-null.*

Remark 1 The set of m-null rows and n-null columns is not a basis of the vector space of R-null matrices, since they are not linearly independent.

As an example, the following matrix can be obtained both as sum of 3-null rows and as a sum of 3-null columns.

	1	−1		1	−1		1
	−1	1		−1	1		−1
	1	−1		1	−1		1
	−1	1		−1	1		−1
	1	−1		1	−1		1

4 An alternate proof of Theorem 5

Let us consider a R-null matrix M of size $p \times q$, and let us compute a matrix M_0 of the same size, with m-null rows, and such that $M + M_0$ has only 0's in its leftmost column, as follows: for each $i \in [q]$,

- if $M(0, i) = a_i$ is different from 0, then the row i of M_0 is

$$-a_i\, 0 \,\ldots\, a_i\, 0 \,\ldots\, -a_i\, 0 \,\ldots\, a_i\, 0 \,\ldots,$$

  where the $-a_i$'s appear in positions $(\beta m, i)$, with $\beta \in \mathbb{N}$, $\beta m < p$, and the a_i's in positions $(\beta m + h, i)$ for some h between 1 and $m - 1$, $\beta m + h < p$. Clearly such a row is m-null;

- if $M(0, i)$ is equal to 0, then the row i of M_0 is full of 0's.

The first column of $M + M_0$ is full of 0's, and this implies that all the columns in position βm of $M + M_0$ are n-null columns. In fact, for all $i \in [q]$, $(M + M_0)(0, i) = 0$ implies

$$\sum\{(M + M_0)(k, i + h) : h \in [n], 1 \leq k \leq m - 1\} = 0$$

and, since $\sum\{(M+M_0)(k+1, i+h) : h \in [n], k \in [m]\} = 0$, we have

$$\sum\{(M+M_0)(i+h, m) : h \in [n]\} = 0.$$

So, the sum of n consecutive coefficients in the column m is equal to 0, and obviously this is also true for all the columns in position βm.

Let $\overline{M}_0$ be the matrix whose columns are full of 0's except for those in position βm, where the elements are defined to be the opposite of the correspondent ones in $M+M_0$. Finally, the matrix $M+M_0+\overline{M}_0$ has all the columns in position βm full of 0's.

The process described up to now can be repeated for each column j, with $0 < j < m$, in order to compute the matrices M_j and $\overline{M}_j$, having m-null rows and n-null columns respectively, and such that $M+M_j+\overline{M}_j$ has all the columns in positions $\beta m + j$ full of 0's, with $\beta m + j < p$.

Eventually we can write M as the sum of a matrix with m-null rows and a matrix with n-null columns. □

EXAMPLE 3 Let us consider the matrix M of Fig. 7 which is 4×3-null.

−2				2	1			−3
1						−1		4
2	2	−3		−1	1	−2		
1	−2		−1	5	−1		−1	
		4	−3	−1		3	−3	3
4	−1	−2		1	−2	−1		2
−2				2	1			−3

M

+

2	−2			2	−2			2
−1		1		−1		1		−1
−2		2		−2		2		−2
−1			1	−1			1	−1
−4		4		−4		4		−4
2		−2		2		−2		2

M_0

+

				−4				1
				1				−3
				3				2
				−4				1
				1				−3
				3				2
				−4				1

$\overline{M}_0$

=

	−2				−1			
		1						
	2	−1			1			
	−2				−1			
		4	−3			3	−3	
	−1	2			−2	3		
		−2			1	−2		

Figure 7: The first step in the decomposition of the matrix M into m-null rows and n-null columns.

We show how to decompose it into matrices having 4-null rows and 3-null columns: first we compute one of the matrices M_0 whose sum with M produces a matrix having all 0's in its leftmost column. Then we compute the matrix $\overline{M}_0$ which is full of 0's but for the elements in columns 4 and 8, which are the opposite of the correspondent ones of $M + M_0$, as depicted in Fig. 7.

We repeat this process for the columns 1 and 2 of M and we obtain four matrices M_1, $\overline{M}_1$, M_2, and $\overline{M}_2$. Eventually we have

$$M = (-M_0 - M_1 - M_2) + (-\overline{M}_0 - \overline{M}_1 - \overline{M}_2),$$

where the first sum has only 4-null rows, while the second has only 3-null columns.

REMARK 2 We can describe a basis of the vector space of $m \times n$-null matrices.

We take all the matrices which have only one row which is not full of 0's and this row is of the form

$$(1\,0\,\ldots\,1\,0\,\ldots)(1\,0\,\ldots\,1\,0\ldots)\ldots$$

with 1's in position βm and $\beta m + h$ for some h between 1 and $m-1$, $\beta \in \mathbb{N}$ and $\beta m < p$. There are $(m-1)q$ such matrices, $m-1$ for each one of the q rows.

We take all the matrices which have only one column which is not full of 0's and this column is of the form

$$(\ldots)^T(0\,1\,\ldots\,0\,1)^T(\ldots\,0\,1\,\ldots\,0\,1)^T$$

with 1's in positions αn and $\alpha n + k$ for some k between 1 and $n-1$, $\alpha \in \mathbb{N}$ and $\alpha n < q$. There are $(n-1)p$ such matrices, $n-1$ for each one of the p columns. The set of these matrices certainly generates the whole vector space.

But we can express the last $m-1$ columns as a linear combination of the other columns and of the rows, as shown in Fig. 8.

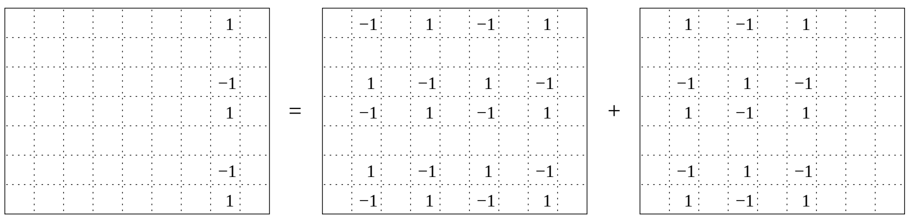


Figure 8: The matrices having one m-null row and one n-null column are not a linearly independent set.

The dimension of the vector space is then

$$(n-1)p + (m-1)q - (m-1)(n-1)$$

and this is compatible with the fact that if we know the elements in the $(m-1)$ first columns and the $(n-1)$ first rows of a $m \times n$-null matrix, then we know the whole matrix.

References

[1] D. Gale, *A theorem on flows in networks*, Pacif. J. Math., **7** (1957), 1073–1082.

[2] G.T. Herman and A. Kuba (eds.), *Discrete Tomography: Foundations, Algorithms and Applications*, Birkhauser Boston, Cambridge, MA (1999).

[3] M. Nivat, *Sous-ensembles homogènes de $\mathbb{Z}^2$ et pavages du plan*, C. R. Acad. Sci. Paris, Ser. I, **335** (2002), 83–86.

[4] H. Ryser, *Combinatorial properties of matrices of zeros and ones*, Canad. J. Math., **9** (1957), 371–377.