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Two new characterizations for sporadic simple groups

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Abstract. In this paper we give two new characterizations for the sporadic simple groups, based on the orders of the normalizers of the Sylow subgroups.

Let S be a sporadic simple group and p be the greatest prime divisor of $|S|$. In this paper, we prove that S is uniquely determined among finite groups by $|S|$ and $|N_S(P)|$, where $P \in \mathrm{Syl}_p(S)$. Also we prove that if G is a finite group, then $G \cong S$ if and only if for every prime q, $|N_S(Q)| = |N_G(Q')|$, where $Q \in \mathrm{Syl}_q(S)$ and $Q' \in \mathrm{Syl}_q(G)$.

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1 Introduction

Let n be an integer. The set of all prime divisors of n is denoted by $\pi(n)$. If G is a finite group, the set $\pi(|G|)$ is denoted by $\pi(G)$. It was proved that if G is an alternating group, a Janko group, a finite projective special linear group or a finite projective symplectic group, then G is characterizable by the orders of normalizers of its Sylow subgroups (see [1, 2, 3, 4, 5, 12]).

Mazurov and Shi in [14, 15, 16, 17] and Deng in [8] proved that some of the almost sporadic simple groups are characterizable by the set of element orders. Chen in [6] and the authors in [11] proved that some of the almost sporadic simple groups are characterizable by the set of order components. Xianhau Li and Dianjun Wang in [13] proved that a finite group G, with the property that the set of indices of the maximal subgroups of G is equal to the corresponding set for a sporadic simple group S, is actually isomorphic to S.

Most of these characterizations are based on the classification of finite simple groups. In this paper, using the classification of finite simple groups, we give two new characterizations for the sporadic simple groups.

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For $q \in \pi(G)$, we define $n_q(G) = |N_G(Q)|$, where $Q \in \mathrm{Syl}_q(G)$. In this paper, we prove that the sporadic simple groups are characterizable by the normalizers' orders of their Sylow subgroups. Also we prove that if S is a sporadic simple group and p is the greatest element in $\pi(S)$, then S is uniquely determined by $|S|$ and $n_p(S)$. Note that $\mathbb{Z}_6$ and S_3 have the same orders and $n_3(S_3) = n_3(\mathbb{Z}_6) = 6$. But $\mathbb{Z}_6$ and S_3 are not characterizable by these conditions.

In this paper, all groups are finite and by simple groups we mean non-abelian simple groups. All further unexplained notations are standard and refer to [7].

We have to thank the authors of the atlas for the rich mine of information they have provided. Also we want to thank professor Thomas Breuer for his helps about computing the orders of normalizers of Sylow subgroups of the sporadic simple groups.

2 Preliminary Results

In this section we give a few lemmas.

Lemma 2.1 ([9, 19]) *Using the atlas of finite simple groups* [7]*, a program in GAP and the information about the structure of $N_S(P)/P'$, where S is a sporadic simple group in* [19] *we can determine the orders of normalizers of Sylow subgroups of the sporadic simple groups. We omit the details of computations and collect these information in Tables 1–2.*

In the proof of the main results, we use the following well known properties of normalizers:

Lemma 2.2 *If G is a finite group and H, K are subgroups of G, then we have:*

(i) *if $H \leq K$, then $N_K(H) = N_G(H) \cap K$,*

(ii) *if $g \in G$, then $N_G(H^g) = N_G(H)^g$,*

(iii) $N_G(H) \cap N_G(K) \leq N_G(H \cap K)$,

(iv) *if $H \lhd G$ and P is a Sylow subgroup of G, then*

$$N_{G/H}(PH/H) = N_G(P)H/H,$$

(v) *(Frattini argument) if Q is a Sylow subgroup of H and $H \lhd K$, then $K = N_K(Q)H$.*

By Theorem 2.1.17 in [18], we deduce the following lemma.

Lemma 2.3 *Let G be a q-group and $|G| = q^k$, for some $k > 0$. If $p \neq q$ divides $|Aut(G)|$, then p is a divisor of $\prod_{i=1}^{k}(q^i - 1)$.*

Notation. Let m be a positive integer and q be a prime number. Then $(m)_q$ denotes the $q-$part of m. In other words, $(m)_q = q^k$ if $q^k \| m$ (i.e. $q^k \mid m$ but $q^{k+1} \nmid m$).

Table 1: The orders of Sylow normalizers of sporadic simple groups

	n_2	n_3	n_5	n_7	n_{11}	n_{13}	n_{17}	n_{19}	n_{23}	n_{29}	n_{31}
Co_1	2^{21}	157464	10000	3528	660	1872	-	-	253	-	-
Co_2	262144	23328	12000	336	110	-	-	-	253	-	-
Co_3	1024	69984	6000	252	110	-	-	-	253	-	-
Fi'_{24}	2^{21}	8×3^{16}	28800	12348	1320	2808	272	-	253	406	-
Fi_{22}	2^{17}	78732	2400	252	110	78	-	-	-	-	-
Fi_{23}	2^{18}	8×3^{13}	2400	5040	440	468	272	-	253	-	-
HN	3×2^{14}	5832	125000	2520	220	-	-	171	-	-	-
HS	512	288	2000	42	55	-	-	-	-	-	-
He	1024	216	1200	6174	-	-	136	-	-	-	-
J_1	168	60	60	42	110	-	-	114	-	-	-
J_2	384	216	300	42	-	-	-	-	-	-	-
J_3	384	1944	60	-	-	-	136	171	-	-	-
M_{11}	16	144	20	-	55	-	-	-	-	-	-
M_{12}	64	108	40	-	55	-	-	-	-	-	-
M_{22}	128	72	20	21	55	-	-	-	-	-	-
M_{23}	128	144	60	42	55	-	-	-	253	-	-
M_{24}	1024	216	240	126	110	-	-	-	253	-	-
McL	128	5832	3000	42	55	-	-	-	-	-	-
ON	512	25920	720	8232	110	-	-	114	-	-	465
Ru	16384	432	4000	168	-	624	-	-	-	406	-
Suz	24576	34992	600	504	110	78	-	-	-	-	-
Th	32768	236196	12000	7056	-	468	-	342	-	-	465

Table 2: The orders of Sylow normalizers of sporadic simple groups

	n_2	n_3	n_5	n_7	n_{11}	n_{13}	n_{17}	n_{19}	n_{23}
B	2^{41}	8×3^{13}	16×5^6	28224	13200	3744	1088	684	506
J_4	2^{21}	864	26880	2520	319440	-	-	-	506
Ly	256	69984	250000	1008	330	-	-	-	-
M	2^{46}	64×3^{20}	96×5^9	36×7^6	72600	632736	45696	20520	6072

Table 2: (continued). The orders of Sylow normalizers of sporadic simple groups

	n_{29}	n_{31}	n_{37}	n_{41}	n_{43}	n_{47}	n_{59}	n_{67}	n_{71}
B	-	465	-	-	-	1081	-	-	-
J_4	812	310	444	-	602	-	-	-	-
Ly	-	186	666	-	-	-	-	1474	-
M	2436	2790	-	1640	-	2162	1711	-	2485

3 Characterizations of the sporadic simple groups

In this section we introduce two new characterizations for the sporadic simple groups.

Lemma 3.1 *Let G be a finite group and $p \in \pi(G)$. If $p^2 \nmid |G|$, then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that K/H is a simple group and $p \in \pi(K/H)$.*

Proof. Since G is a finite group, G has a chief series. So let $G_0 \leq G_1 \leq G_2 \leq \cdots \leq G_r = G$ be a chief series of G. There exists some t, such that $1 \leq t \leq r$ and $p \in \pi(G_t) \setminus \pi(G_{t-1})$. If $K = G_t$ and $H = G_{t-1}$, then $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ is a normal series of G and K/H is a chief factor of G. Therefore K/H is a minimal normal subgroup of G/H. We know that the chief factors are characteristically simple. Also every characteristically simple group is a simple group or a product of isomorphic simple groups. So K/H is a simple group or a product of isomorphic simple groups. Since $p \in \pi(K/H)$ and $p^2 \nmid |K/H|$, it follows that K/H is a simple group. □

Now as the main result we prove the following theorem:

Theorem 3.2 *Let S be a sporadic simple group and p be the greatest element of $\pi(S)$. Then S is uniquely determined by $|S|$ and $n_p(S)$.*

Proof. Let G be a finite group such that $|G| = |S|$ and $n_p(G) = n_p(S)$. By [7], it follows that $p^2 \nmid |S|$. Now Lemma 3.1, implies that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a simple group and $p \in \pi(K/H)$. By our assumptions, every Sylow $p-$subgroup of G is a Sylow $p-$subgroup of K. So let P be a Sylow $p-$subgroup of G. By Lemma 2.2, we know that $N_K(P) = K \cap N_G(P)$. Hence

$$\frac{N_G(P)}{N_K(P)} = \frac{N_G(P)}{K \cap N_G(P)} \cong \frac{KN_G(P)}{K},$$

and by the Frattini argument it follows that $G = KN_G(P)$. Therefore $N_G(P)/N_K(P) \cong G/K$ and so $|G/K|$ is a divisor of $n_p(G)$. We know that $p \mid |K|$, and hence $|G/K|$ divides $n_p(G)/p$. Also note that $p \nmid |H|$. Now we prove that $|H|$ is a divisor of $n_p(G)/p$.

Let $q \in \pi(H)$ and Q be a Sylow $q-$subgroup of H. Similarly to the above discussion it follows that $K = HN_K(Q)$ and $N_K(Q)/N_H(Q) \cong K/H$. Therefore $N_K(Q)/N_H(Q)$ is a simple group and $N_H(Q) \trianglelefteq C_K(Q)N_H(Q) \trianglelefteq N_K(Q)$. Hence $C_K(Q)N_H(Q) = N_H(Q)$ or $C_K(Q)N_H(Q) = N_K(Q)$. Now we show that the former case leads to a contradiction.

Case 1. If $C_K(Q)N_H(Q) = N_H(Q)$, then $C_K(Q) \leq N_H(Q)$. It is a well known result that $N_K(Q)/C_K(Q)$ is isomorphic to a subgroup of Aut (Q). Also $|N_K(Q)/N_H(Q)|$ is a divisor of $|N_K(Q)/C_K(Q)|$ and so $|K/H|$ is a divisor of $|\mathrm{Aut}\ (Q)|$. Now by Lemma 2.3, if $|Q| = q^k$, then every prime divisor of $|K/H|$ divides $t(q,k) := \prod_{i=1}^{k}(q^i - 1)$. We know that $p \in \pi(K/H)$ and so $p \mid t(q,k)$. Also as we mentioned above $\pi(G/K) \subseteq \pi(n_p(G)/p)$.

For every sporadic simple group S, using [7] first we determine all possibilities for (q,k) such that $q^k \mid |G|$ and $p \mid t(q,k)$. So $|H|$ and specially $\pi(H)$ is determined. Now note that

$$\pi(S) = \pi(G) = \pi(G/K) \cup \pi(K/H) \cup \pi(H),$$

and $\pi(G/K) \subseteq \pi(n_p(G)/p)$. Again using [7], for every sporadic simple group S, we can find a prime number

$$p' \in \pi(K/H) = \pi(S) - (\pi(H) \cup \pi(n_p(G)/p)),$$

such that $p' \nmid t(q,k)$, where $q^k \| \, |H|$, which is a contradiction.

As an example, we illustrate this method for the Monster group (M). By assumptions $71 \mid |K/H|$ and so $71 \mid t(q,k)$. We note that

$$|M| = 2^{46} \times 3^{20} \times 5^9 \times 7^6 \times 11^2 \times 13^3 \times 17 \times 19 \times 23 \times 29 \times 31 \times 41 \times 47 \times 59 \times 71.$$

If (q,k) belongs to

$$\{(3,20);(7,6);(11,2);(13,3);(17,1);(19,1);(23,1);(29,1);(31,1);(41,1);\\ (47,1);(59,1)\},$$

then $71 \nmid t(q,k)$. Therefore $(q,k) = (2,46)$ or $(5,9)$, which implies that $\pi(H) \subseteq \{2,5\}$. Also we proved that $|G/K| \mid 35$. Hence $59 \mid |K/H|$. But $59 \nmid t(2,46)$ and $59 \nmid t(5,9)$, which is a contradiction.

The conclusion so far is that $C_K(Q)N_H(Q)$ must be equal to $N_K(Q)$.

Case 2. If $C_K(Q)N_H(Q) = N_K(Q)$, then

$$\frac{N_K(Q)}{N_H(Q)} = \frac{C_K(Q)N_H(Q)}{N_H(Q)} \cong \frac{C_K(Q)}{C_K(Q) \bigcap N_H(Q)},$$

and since $K/H \cong N_K(Q)/N_H(Q)$, we conclude that $K/H \cong C_K(Q)/(C_K(Q) \bigcap N_H(Q))$. Also $p \mid |K/H|$, and hence $p \mid |C_K(Q)|$. Let P be a Sylow p-subgroup of $C_K(Q)$. Then $Q \leq C_K(P)$, and hence $Q \leq N_G(P)$. Therefore $|Q|$ is a divisor of $n_p(G)$. We know that $p \nmid |H|$, which implies that $|Q|$ be a divisor of $n_p(G)/p$. Hence $|H|$ divides $n_p(G)/p$. As we proved, $|G/K|$ is a divisor of $n_p(G)/p$, too. Hence $|G/K| \cdot |H|$ is a divisor of $(n_p(G)/p)^2$. On the other hand, $|G| = |G/K| \cdot |K/H| \cdot |H|$ and so $|K/H| = |S|/r$, where $r \mid (n_p(G)/p)^2$. Now K/H is a simple group of order $|S|/r$. But by using the classification of finite simple groups, we can see that there exists no simple group of order $|S|/r$, where $r \mid (n_p(G)/p)^2$ and $r > 1$ (see [7, pp. 239-241]). Therefore $r = 1$ and hence $|G/K|$=1 and $|H| = 1$. Hence G is a simple group of order S. But there exist only one simple group of order $|S|$. Therefore $G \cong S$. □

Now we can give a new characterization for the sporadic simple groups based on the orders of the normalizers of Sylow subgroups. First we prove the following lemma.

LEMMA 3.3 *Let G_1 and G_2 be finite groups. If for every prime number q we have $n_q(G_1) = n_q(G_2)$, then $|G_1| = |G_2|$.*

Proof. If Q is a Sylow q-subgroup of G_1, then $|Q|$ divides $|N_{G_1}(Q)|$. Also $(|G_1|)_q = |Q|$. Since $n_q(G_1) = n_q(G_2)$, we conclude that $(n_q(G_1))_q = (n_q(G_2))_q$. Therefore for every prime q, we have $(|G_1|)_q = (|G_2|)_q$. Hence $|G_1| = |G_2|$. □

THEOREM 3.4 *Let S be a sporadic simple group and G be a finite group. If for every prime number q, $n_q(G) = n_q(S)$, then $G \cong S$.*

Proof. By Lemma 3.3, we conclude that $|G| = |S|$. Now by Theorem 3.2 we get the result. □

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References

[1] J. BI, *On the characteristic property of the alternating groups*, (in Chinese) Chinese Sci. Bull., **34** (1989), 11–17.

[2] J. BI, *A characterization of $L_2(q)$*, (Chinese), J. Liaoning Univ., (Natural Sciences Edition) **19** (2) (1992), 1–4.

[3] J. BI, *A characterization of $L_n(q)$ by the normalizers' orders of their Sylow subgroup*, Acta Math. Sinica (New Ser.), **11 (4)** (1995), 300–306.

[4] J. BI, *On the group with the same orders of Sylow normalizers as the finite simple group $S_4(q)$*, Algebras, Groups and Geom., **18** (3) (2001), 349–355.

[5] J. BI, *Characterization of alternating groups by orders of normalizers of Sylow subgroups*, Algebra Colloq., **8** (3) (2001), 249–256.

[6] G.Y. CHEN, *A new characterization of sporadic simple groups*, Algebra Colloq., **3** (1) (1996), 49–58.

[7] J. CONWAY, R. CURTIS, S. NORTON, R. PARKER and R. WILSON, *Atlas of finite groups*, Clarendon Press, Oxford (1985).

[8] H. DENG, *The number of isomorphism classes of finite groups with the set of element orders of almost sporadic simple groups*, Southeast Asian Bull. Math., **22** (3) (1998), 265–271.

[9] [GAP 00] The GAP Group, GAP — Groups, Algorithms, and Programming, Version 4.2; Aachen, St Andrews, 1999. (http://www-gap.dcs.st-and.ac.uk/ gap)

[10] D. Gorenstein, *Finite groups*, Harper and Row, New York, 1968.

[11] Amir Khosravi and Behrooz Khosravi, *A new characterization of almost sporadic groups*, J. Algebra and Its Appl., **1 (3)** (2002), 267–279.

[12] Behrooz Khosravi and Bahman Khosravi, *Groups With the Same Orders of Sylow Normalizers as the Janko groups*, Journal of Applied Algebra and Discrete Structures, **3 (1)** (2005), 23–31.

[13] Li, Xianhua and Wang, Dianjun, *A quantitative property of sporadic simple groups*, J. Group Theory, **5** (1) (2002), 41–52.

[14] V.D. Mazurov and W. Shi, *A note to the characterization of sporadic simple groups*, Algebra Colloq., **5** (3) (1998), 285–288.

[15] W. Shi, *A new characterization of the sporadic simple groups*, Group Theory, Proceeding of the 1987 Singapore Group Theory Conference, Walter de Gruyter, Berlin, New York, 1989, 531–540.

[16] W. Shi, *Pure quantitative characterization of finite simple groups* (I), Progress in Natural Science, **4** (3) (1994), 316–326.

[17] W. Shi, *The characterization of the sporadic simple groups by their element orders*, Algebra Colloq., **1** (2) (1994), 159–166.

[18] M. Suzuki, *Group Theory* (I), Springer-Verlag, New York, 1982.

[19] R.A. Wilson, *The McKay conjecture is true for sporadic simple groups*, *J. Algebra,* **207** (1) (1998), 294–305.