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Regular elements of some transformation semigroups

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Abstract. The full transformation semigroup on a nonempty set X is denoted by $T(X)$. It is well-known that $T(X)$ is a regular semigroup, that is, for every $\alpha \in T(X), \alpha = \alpha\beta\alpha$ for some $\beta \in T(X)$. The subsemigroups of $T(X)$ we consider are $T(X,Y)$ and $\overline{T}(X,Y)$ with $\emptyset \neq Y \subseteq X$ where $T(X,Y) = \{\alpha \in T(X) \mid \operatorname{ran} \alpha \subseteq Y\}$ and $\overline{T}(X,Y) = \{\alpha \in T(X) \mid Y\alpha \subseteq Y\}$. Then $T(X,Y) \subseteq \overline{T}(X,Y)$. In 1966, K.D. Magill has studied the semigroup $\overline{T}(X,Y)$ while J.S.V. Symons has studied the semigroup $T(X,Y)$ in 1975. In this paper, we characterize regular elements of both $T(X,Y)$ and $\overline{T}(X,Y)$. These results are then applied to determine the numbers of regular elements in $T(X,Y)$ and $\overline{T}(X,Y)$ for a finite set X. The numbers are given in terms of $|X|, |Y|$ and their related Stirling numbers.

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1 Introduction

For positive numbers n and r with $n \geq r$, the number of partitions of $\{1, 2, \ldots, n\}$ into r blocks is denoted by $S(n,r)$ and is called a *Stirling number* of the second kind. It is known that

$$S(n,r) = \frac{1}{r!} \sum_{i=0}^{r} (-1)^i \binom{r}{i} (r-i)^n$$

[1, page 12]. Hence the number of maps from $\{1, 2, \ldots, n\}$ onto $\{1, 2, \ldots, r\}$ is $r!S(n,r)$.

The cardinality of a set X is denoted by $|X|$.

An element x of a semigroup S is called a *regular element* of S if there is an element y of S such that $x = xyx$, and S is said to be a *regular semigroup* if every element of S is regular. The set of all regular elements of a semigroup S is denoted by Reg S.

For a nonempty set X, let $T(X)$ be the full transformation semigroup on X, that is, the semigroup under composition of all maps from X into itself. It is known that $T(X)$ is a regular semigroup [2, page 4]. The range of $\alpha \in T(X)$ will be denoted by ran α. The *kernel* of $\alpha \in T(X)$, ker α, is the equivalence relation $\alpha \circ \alpha^{-1}$ on X, that is, ker $\alpha = \{(x, y) \in X \times X \mid x\alpha = y\alpha\}$. Then $x \ker \alpha = (x\alpha)\alpha^{-1}$ for all $x \in X$, in particular, if $x \in$ ran α, $x\alpha^{-1}$ is a ker α-class. Also, the map $x \ker \alpha \mapsto x\alpha$ is a bijection of $X/\ker \alpha$ onto ran α. Hence for any $\alpha \in T(X)$, the set of equivalence classes of ker α and ran α have the same cardinality.

In 1952, A.I. Malcev [4] showed that every automorphism of $T(X)$ is inner. J.S.V. Symons [5] generalized this result in 1975 by considering the semigroup $T(X,Y)$, $\emptyset \neq Y \subseteq X$, including all $\alpha \in T(X)$ whose ranges are in Y, that is,

$$T(X,Y) = \{\alpha \in T(X) \mid \text{ran } \alpha \subseteq Y\}.$$

An interesting subsemigroup of $T(X)$ containing $T(X,Y)$ is

$$\overline{T}(X,Y) = \{\alpha \in T(X) \mid Y\alpha \subseteq Y\}.$$

In fact, K.D. Magill [3] studied the semigroup $\overline{T}(X,Y)$ in 1966.

Elements of the semigroup $T(X,Y)$ and the semigroup $\overline{T}(X,Y)$ need not be regular. Regular elements of these semigroups are characterized in this paper. These characterizations are used to count the numbers of regular elements of $T(X,Y)$ and $\overline{T}(X,Y)$ when X is finite. The numbers are given in terms of $|X|, |Y|$ and their related Stirling numbers.

2 Main results

In what follows, let X be a nonempty set and Y a nonempty subset of X. First, we note that Reg $T(X,Y) \subseteq$ Reg $\overline{T}(X,Y)$ since $T(X,Y)$ is a subsemigroup of $\overline{T}(X,Y)$.

THEOREM 2.1 *For $\alpha \in T(X,Y)$, the following statements are equivalent.*

(i) $\alpha \in$ *Reg* $T(X,Y)$.

(ii) *ran* $\alpha = Y\alpha$.

(iii) x *ker* $\alpha \cap Y \neq \emptyset$ *for every* $x \in X$.

(iv) $x\alpha^{-1} \cap Y \neq \emptyset$ *for every* $x \in$ *ran* α.

Proof. (i) $\Rightarrow$ (ii). Let $\beta \in T(X,Y)$ be such that $\alpha = \alpha\beta\alpha$. Then $X\alpha\beta \subseteq Y$, and so ran $\alpha = X\alpha = X\alpha\beta\alpha = (X\alpha\beta)\alpha \subseteq Y\alpha \subseteq X\alpha =$ ran α. Hence (ii) holds.

(ii) $\Rightarrow$ (iii). For any $x \in X$, $x\alpha \in$ ran $\alpha = Y\alpha$, so $x\alpha = y\alpha$ for some $y \in Y$ which implies that $y \in (x\alpha)\alpha^{-1} = x \ker \alpha$.

(iii) $\Rightarrow$ (iv). This is trivial, since for every $x \in \text{ran } \alpha, x\alpha^{-1}$ is a ker α-class.

(iv) $\Rightarrow$ (i). For each $x \in \text{ran } \alpha$, choose an element $x' \in x\alpha^{-1} \cap Y$. Then $x'\alpha = x$ for every $x \in \text{ran } \alpha$. Let a be a fixed element of Y. Define $\beta : X \to X$ by a bracket notation as follows:

$$\beta = \begin{bmatrix} x & X \setminus \text{ran } \alpha \\ x' & a \end{bmatrix}_{x \,\in\, \text{ran } \alpha},$$

that is, $x\beta = x'$ for all $x \in \text{ran } \alpha$ and $x\beta = a$ for all $x \in X \setminus \text{ran } \alpha$. Then ran $\beta \subseteq Y$ and for every $x \in X$, $x\alpha\beta\alpha = (x\alpha)\beta\alpha = (x\alpha)'\alpha = x\alpha$. Hence $\beta \in T(X, Y)$ and $\alpha = \alpha\beta\alpha$. □

As a consequence of Theorem 2.1, a necessary and sufficient condition for $T(X, Y)$ to be a regular semigroup can be given as follows:

Corollary 2.2 *The semigroup $T(X, Y)$ is regular if and only if either $X = Y$ or $|Y| = 1$.*

Proof. Suppose that $Y \subsetneq X$ and $|Y| > 1$. Let a and b be two distinct elements of Y. Define $\alpha : X \to X$ by

$$\alpha = \begin{bmatrix} Y & X \setminus Y \\ a & b \end{bmatrix}.$$

Then ran $\alpha = \{a, b\} \subseteq Y$ and $b\alpha^{-1} \cap Y = (X \setminus Y) \cap Y = \emptyset$. Hence $\alpha \in T(X, Y)$ but by Theorem 2.1, α is not a regular element of $T(X, Y)$. This proves that if $T(X, Y)$ is a regular semigroup, then $Y = X$ or $|Y| = 1$.

Since $T(X, Y) = T(X)$ if $Y = X$ and $|T(X, Y)| = 1$ if $|Y| = 1$, the converse holds. □

Theorem 2.3 *For $\alpha \in \overline{T}(X, Y)$, the following statements are equivalent.*

(i) $\alpha \in Reg\, \overline{T}(X, Y)$.

(ii) *ran* $\alpha \cap Y = Y\alpha$.

(iii) x *ker* $\alpha \cap Y \neq \emptyset$ *for every* $x \in X$ *with* $x\alpha \in Y$, *that is,* $x \in Y\alpha^{-1}$.

(iv) $x\alpha^{-1} \cap Y \neq \emptyset$ *for every* $x \in$ *ran* $\alpha \cap Y$.

Proof. (i) $\Rightarrow$ (ii). Let $\beta \in \overline{T}(X, Y)$ be such that $\alpha = \alpha\beta\alpha$. Then $Y\alpha \subseteq X\alpha \cap Y = \text{ran } \alpha \cap Y$. If $x \in \text{ran } \alpha \cap Y$, then $x \in Y$ and $x = a\alpha$ for some $a \in X$. Consequently, $x = a\alpha = a\alpha\beta\alpha = x\beta\alpha \in Y\beta\alpha \subseteq Y\alpha$. Hence (ii) holds.

(ii) $\Rightarrow$ (iii). Let $x \in X$ be such that $x\alpha \in Y$. Then $x\alpha \in \text{ran } \alpha \cap Y = Y\alpha$, so $x\alpha = y\alpha$ for some $y \in Y$. This implies that $y \in (x\alpha)\alpha^{-1} = x \ker \alpha$. Hence $y \in x \ker \alpha \cap Y$.

(iii) $\Rightarrow$ (iv). If $x \in \text{ran } \alpha \cap Y$, then $x = a\alpha$ for some $a \in X$, so by (iii), $a \ker \alpha \cap Y \neq \emptyset$. But $a \ker \alpha = (a\alpha)\alpha^{-1} = x\alpha^{-1}$, so $x\alpha^{-1} \cap Y \neq \emptyset$.

(iv) $\Rightarrow$ (i). For each $x \in \text{ran } \alpha \cap Y$, choose an element $x' \in x\alpha^{-1} \cap Y$. Also, for $x \in \text{ran } \alpha \setminus Y$, choose an element $\overline{x} \in x\alpha^{-1}$. Then $x'\alpha = x$ for every

$x \in \text{ran } \alpha \cap Y$ and $\overline{x}\alpha = x$ for all $x \in \text{ran } \alpha \setminus Y$. Let a be a fixed element of Y. Define $\beta : X \to X$ by

$$\beta = \begin{bmatrix} x & t & X \setminus \text{ran } \alpha \\ x' & \overline{t} & a \end{bmatrix}_{\substack{x \in \text{ran } \alpha \cap Y \\ t \in \text{ran } \alpha \setminus Y}}.$$

Then $Y\beta \subseteq \{x' \mid x \in \text{ran } \alpha \cap Y\} \cup \{a\} \subseteq Y$ and for $x \in X$,

$$x\alpha\beta\alpha = (x\alpha)\beta\alpha = \begin{cases} (x\alpha)'\alpha = x\alpha \text{ if } x\alpha \in \text{ran } \alpha \cap Y \\ \overline{(x\alpha)}\alpha = x\alpha \text{ if } x\alpha \in \text{ran } \alpha \setminus Y. \end{cases}$$

Hence $\beta \in \overline{T}(X, Y)$ and $\alpha = \alpha\beta\alpha$. □

We also have the following corollary which characterizes when $\overline{T}(X, Y)$ is a regular semigroup.

Corollary 2.4 *The semigroup $\overline{T}(X, Y)$ is regular if and only if either $X = Y$ or $|Y| = 1$.*

Proof. Suppose that $Y \subsetneq X$ and $|Y| > 1$. Let $a, b \in Y$ and α be as in the proof of Corollary 2.2. Then $Y\alpha = \{a\} \subseteq Y$, so $\alpha \in \overline{T}(X, Y)$. Since $b \in \text{ran } \alpha \cap Y$ and $b\alpha^{-1} \cap Y = (X \setminus Y) \cap Y = \emptyset$, by Theorem 2.3, α is not a regular element of $\overline{T}(X, Y)$.

If $Y = X$, then $\overline{T}(X, Y) = T(X)$ which is regular. Next, assume that $Y = \{c\}$. Then $c\alpha = c$ for all $\alpha \in \overline{T}(X, Y)$. To show that $\overline{T}(X, Y)$ is regular, let $\alpha \in \overline{T}(X, Y)$. For each $x \in \text{ran } \alpha \setminus \{c\}$, choose an element $x' \in x\alpha^{-1}$. Then $x'\alpha = x$ for all $x \in \text{ran } \alpha \setminus \{c\}$. Let $c' = c$ and define $\beta \in T(X)$ by

$$\beta = \begin{bmatrix} x & X \setminus \text{ran } \alpha \\ x' & c \end{bmatrix}_{x \in \text{ran } \alpha}.$$

Then $Y\beta = \{c\}\beta = \{c'\} = \{c\} = Y$ and for $x \in X, x\alpha\beta\alpha = (x\alpha)'\alpha = x\alpha$. This proves that if $|Y| = 1$, then $\overline{T}(X, Y)$ is a regular semigroup, as required. □

The following result which is obtained from Theorem 2.1 and Theorem 2.3 shows that any nonregular element of $T(X, Y)$ cannot be regular in $\overline{T}(X, Y)$.

Corollary 2.5 *Reg $\overline{T}(X, Y) \subseteq$ Reg $T(X, Y) \cup (\overline{T}(X, Y) \setminus T(X, Y))$, or equivalently,*

$$T(X, Y) \setminus \textit{Reg}\, T(X, Y) \subseteq \overline{T}(X, Y) \setminus \textit{Reg}\, \overline{T}(X, Y).$$

Proof. Let $\alpha \in \text{Reg } \overline{T}(X, Y)$ and assume that $\alpha \in T(X, Y)$. Then $\text{ran } \alpha \cap Y = Y\alpha$ by Theorem 2.3 and $\text{ran } \alpha \subseteq Y$. These imply that $\text{ran } \alpha = Y\alpha$, so $\alpha \in \text{Reg } T(X, Y)$ by Theorem 2.1. □

Next, the cardinalities of regular elements in the semigroups $T(X, Y)$ and $\overline{T}(X, Y)$ are investigated when X is finite. First, we note that if $|X| = n$ and

$|Y| = m$, then

$$\begin{aligned} |T(X)| &= n^n, \\ |T(X,Y)| &= m^n, \\ |\overline{T}(X,Y)| &= m^m \times n^{n-m}. \end{aligned}$$

THEOREM 2.6 *If $|X| = n$ and $|Y| = m$, then*

$$|Reg\, T(X,Y)| = \sum_{r=1}^{m} \binom{m}{r} r!S(m,r)r^{n-m}.$$

Proof. Let $\emptyset \neq Y' \subseteq Y$ and $|Y'| = r$. Then the number of maps from Y onto Y' is $r!S(m,r)$. Consequently, the number of maps α from X onto Y' such that $Y\alpha = Y'$ is $r!S(m,r)r^{n-m}$. Hence

$$|\{\alpha \in T(X,Y) \mid \text{ran } \alpha = Y' = Y\alpha\}| = r!S(m,r)r^{n-m}.$$

But we have from Theorem 2.1 ((i) $\Leftrightarrow$ (ii)) that

$$\{\alpha \in T(X,Y) \mid \text{ran } \alpha = Y' = Y\alpha\} = \{\alpha \in \text{Reg } T(X,Y) \mid \text{ran } \alpha = Y'\},$$

so

$$|\{\alpha \in \text{Reg } T(X,Y) \mid \text{ran } \alpha = Y'\}| = r!S(m,r)r^{n-m}.$$

This implies that for $1 \leq r \leq m$,

$$|\{\alpha \in \text{Reg } T(X,Y) \mid |\text{ran } \alpha| = r\}| = \binom{m}{r} r!S(m,r)r^{n-m}.$$

Therefore it follows that

$$|\text{Reg } T(X,Y)| = \sum_{r=1}^{m} \binom{m}{r} r!S(m,r)r^{n-m},$$

as desired. □

THEOREM 2.7 *If $|X| = n$ and $|Y| = m$, then*

$$|Reg\,\overline{T}(X,Y)| = \sum_{r=1}^{m} \binom{m}{r} r!S(m,r)(n-m+r)^{n-m}.$$

Proof. Let $\emptyset \neq Y' \subseteq Y$ and $|Y'| = r$. Then the number of maps from Y onto Y' is $r!S(m,r)$. Therefore it follows that the number of maps $\alpha : X \to X$ such that $Y\alpha = Y'$ and ran $\alpha \cap Y = Y'$ is $r!S(m,r)(n-m+r)^{n-m}$ since $|(X \backslash Y) \cup Y'| = |X \backslash Y| + |Y'| = n - m + r$. Hence

$$|\{\alpha \in \overline{T}(X,Y) \mid \text{ran } \alpha \cap Y = Y' = Y\alpha\}| = r!S(m,r)(n-m+r)^{n-m}.$$

We have from Theorem 2.3 ((i) $\Leftrightarrow$ (ii)) that

$$\{\alpha \in \overline{T}(X,Y) \mid \text{ran } \alpha \cap Y = Y' = Y\alpha\} = \{\alpha \in \text{Reg } \overline{T}(X,Y) \mid \text{ran } \alpha \cap Y = Y'\}$$

which implies that

$$|\{\alpha \in \text{Reg } \overline{T}(X,Y) \mid \text{ran } \alpha \cap Y = Y'\}| = r!S(m,r)(n-m+r)^{n-m}.$$

Consequently, for $1 \leq r \leq m$,

$$|\{\alpha \in \text{Reg } \overline{T}(X,Y) \mid |\text{ran } \alpha \cap Y| = r\}| = \binom{m}{r} r!S(m,r)(n-m+r)^{n-m},$$

whence

$$|\text{Reg } \overline{T}(X,Y)| = \sum_{r=1}^{m} \binom{m}{r} r!S(m,r)(n-m+r)^{n-m}. \qquad \square$$

EXAMPLE 2.8 Since $S(n,r) = \frac{1}{r!}\sum_{i=0}^{r}(-1)^i \binom{r}{i}(r-i)^n$, we have $S(3,1) = 1$, $S(3,2) = 3$ and $S(3,3) = 1$.

(1) Let $|X| = 4$ and $|Y| = 3$. By Theorem 2.6 and Theorem 2.7, we have respectively that

$$|\text{Reg } T(X,Y)| = \sum_{r=1}^{3} \binom{3}{r} r!S(3,r)r = 57,$$
$$|\text{Reg } \overline{T}(X,Y)| = \sum_{r=1}^{3} \binom{3}{r} r!S(3,r)(1+r) = 84.$$

Hence

$$|T(X,Y) \setminus \text{Reg } T(X,Y)| = 81 - 57 = 24,$$
$$|\overline{T}(X,Y) \setminus \text{Reg } \overline{T}(X,Y)| = 108 - 84 = 24,$$

and so by Corollary 2.5, $T(X,Y) \setminus \text{Reg } T(X,Y) = \overline{T}(X,Y) \setminus \text{Reg } \overline{T}(X,Y)$. Since $|\overline{T}(X,Y) \setminus T(X,Y)| = 108 - 81 = 27$, we deduce that $|\text{Reg } T(X,Y) \cup (\overline{T}(X,Y) \setminus T(X,Y))| = 57 + 27 = 84 = |\text{Reg } \overline{T}(X,Y)|$, so by Corollary 2.5, we have that $\text{Reg } \overline{T}(X,Y) = \text{Reg } T(X,Y) \cup (\overline{T}(X,Y) \setminus T(X,Y))$. Therefore every element in $\overline{T}(X,Y) \setminus T(X,Y)$ is regular in $\overline{T}(X,Y)$.

(2) Assume that $|X| = 5$ and $|Y| = 3$. Then

$$|\text{Reg } T(X,Y)| = \sum_{r=1}^{3} \binom{3}{r} r!S(3,r)r^2 = 129,$$
$$|\text{Reg } \overline{T}(X,Y)| = \sum_{r=1}^{3} \binom{3}{r} r!S(3,r)(2+r)^2 = 465,$$

$$|T(X,Y) \setminus \text{Reg } T(X,Y)| = 243 - 129 = 114,$$
$$|\overline{T}(X,Y) \setminus \text{Reg } \overline{T}(X,Y)| = 675 - 465 = 210,$$
$$|\overline{T}(X,Y) \setminus T(X,Y)| = 675 - 243 = 432,$$
$$|\text{Reg } T(X,Y) \cup (\overline{T}(X,Y) \setminus T(X,Y))| = 129 + 432 = 561.$$

It follows from Corollary 2.5 that $T(X,Y)\setminus\text{Reg } T(X,Y) \subsetneq \overline{T}(X,Y)\setminus\text{Reg } \overline{T}(X,Y)$ and Reg $\overline{T}(X,Y) \subsetneq$ Reg $T(X,Y) \cup (\overline{T}(X,Y) \setminus T(X,Y))$. Since Reg $T(X,Y) \subseteq$ Reg $\overline{T}(X,Y)$, we deduce that there is an element of $\overline{T}(X,Y) \setminus T(X,Y)$ which is not regular in $\overline{T}(X,Y)$.

From Example 2.8 (1), it is natural to ask whether it is true that for a set X and $\emptyset \neq Y \subseteq X$, if $|X \setminus Y| = 1$, then Reg $\overline{T}(X,Y) =$ Reg $T(X,Y) \cup (\overline{T}(X,Y) \setminus T(X,Y))$. Also, does the converse hold if $Y \neq X$ and $|Y| > 1$? The later question is motivated by Example 2.8 (2). The following theorem shows that these are true in general. Note that by Corollary 2.2 and Corollary 2.4, if $X = Y$ or $|Y| = 1$, then both $T(X,Y)$ and $\overline{T}(X,Y)$ are regular which implies that Reg $\overline{T}(X,Y) =$ Reg $T(X,Y) \cup (\overline{T}(X,Y) \setminus T(X,Y))$.

Theorem 2.9 *If $|X \setminus Y| = 1$, then Reg $\overline{T}(X,Y) =$ Reg $T(X,Y) \cup (\overline{T}(X,Y) \setminus T(X,Y))$, and the converse holds if $Y \subsetneq X$ and $|Y| > 1$.*

Proof. Assume that $X \setminus Y = \{c\}$ and let $\alpha \in \overline{T}(X,Y) \setminus T(X,Y)$ be given. Then $Y\alpha \subseteq Y$ and $X\alpha \nsubseteq Y$. But $X = Y \cup \{c\}$, so $c\alpha = c$. Hence ran $\alpha \cap Y = (Y \cup \{c\})\alpha \cap Y = (Y\alpha \cup \{c\}) \cap Y = Y\alpha \cap Y = Y\alpha$. By Theorem 2.3, $\alpha \in$ Reg $\overline{T}(X,Y)$. Hence Reg $T(X,Y) \cup (\overline{T}(X,Y) \setminus T(X,Y)) \subseteq$ Reg $\overline{T}(X,Y)$. The reverse inclusion is obtained from Corollary 2.5.

Conversely, let $Y \subsetneq X$ and $|Y| > 1$ and assume that $|X \setminus Y| > 1$. Let $a, b \in X \setminus Y$ be distinct and c and d be distinct elements of Y. Define $\alpha : X \to X$ by

$$\alpha = \begin{bmatrix} a & b & X \setminus \{a,b\} \\ c & b & d \end{bmatrix}.$$

Since $Y \subseteq X \setminus \{a,b\}$, $Y\alpha = \{d\} \subseteq Y$ and ran $\alpha = \{c,b,d\} \nsubseteq Y$, we have that $\alpha \in \overline{T}(X,Y) \setminus T(X,Y)$. Also, ran $\alpha \cap Y = \{c,d\} \neq \{d\} = Y\alpha$. By Theorem 2.3, $\alpha \notin$ Reg $\overline{T}(X,Y)$.

Hence the proof is complete. □

Remark 2.10 Let X be infinite. We shall give some remarks relating to the cardinalities of Reg $T(X,Y)$ and Reg $\overline{T}(X,Y)$. First, we note that if $|Y| = 1$, then $|\text{Reg } T(X,Y)| = |T(X,Y)| = 1$. The following three facts are provided.

(1) If $|Y| > 1$, then $|\text{Reg } T(X,Y)| \geq 2^{|X|}$. To see this, let a and b be distinct elements of Y. For any $A \in P(X \setminus \{a,b\})$ (the power set of $X \setminus \{a,b\}$), define $\alpha_A : X \to X$ by

$$\alpha_A = \begin{bmatrix} A \cup \{a\} & X \setminus (A \cup \{a\}) \\ a & b \end{bmatrix}.$$

Then ran $\alpha_A = \{a,b\} = (\{a,b\})\alpha_A = Y\alpha_A$ for every $A \in P(X \setminus \{a,b\})$, so $\{\alpha_A \mid A \in P(X \setminus \{a,b\})\} \subseteq \text{Reg } T(X,Y)$ by Theorem 2.1. Since for distinct $A, B \in P(X \setminus \{a,b\}), \alpha_A \neq \alpha_B$, it follows that $|P(X \setminus \{a,b\})| \leq |\text{Reg } T(X,Y)|$. However, $|X| = |X \setminus \{a,b\}|$, so $|P(X)| = |P(X \setminus \{a,b\})|$. Therefore it follows that

$$|\text{Reg } T(X,Y)| \geq |P(X)| = 2^{|X|}.$$

(2) If $|Y| = |X|$, then $|\text{Reg } T(X,Y)| = |T(X)|$. To prove this, assume that $|Y| = |X|$. Then $|T(Y)| = |T(X)|$ through a map $\alpha \mapsto \varphi^{-1}\alpha\varphi$ where $\varphi : X \to Y$ is a bijection. For $\alpha \in T(Y)$, define a map $\alpha' : X \to X$ by $\alpha'|_Y = \alpha$ and $(X \setminus Y)\alpha' \subseteq \text{ran } \alpha$. Hence for every $\alpha \in T(Y), \alpha' \in T(X,Y)$ and ran $\alpha' =$ ran $\alpha = Y\alpha'$, so $\alpha' \in \text{Reg } T(X,Y)$ for all $\alpha \in T(Y)$ by Theorem 2.1. Moreover, $\alpha \mapsto \alpha'$ is an injective map from $T(Y)$ into Reg $T(X,Y)$, so

$$|T(X)| \geq |\text{Reg } T(X,Y)| \geq |\{\alpha' \mid \alpha \in T(Y)\}| = |T(Y)| = |T(X)|,$$

and the required result is obtained.

(3) $|\text{Reg } \overline{T}(X,Y)| = |T(X)|$. If $|Y| = |X|$, then by (2), $|\text{Reg } T(X,Y)| = |T(X)|$. Since Reg $T(X,Y) \subseteq$ Reg $\overline{T}(X,Y) \subseteq \overline{T}(X,Y) \subseteq T(X)$, we have that $|\text{Reg } \overline{T}(X,Y)| = |T(X)|$ when $|Y| = |X|$. Next, assume that $|Y| < |X|$. Then $|X| = |X \setminus Y| + |Y| = |X \setminus Y|$ since X is infinite, and hence $|T(X \setminus Y)| = |T(X)|$. For $\alpha \in T(X \setminus Y)$, define a map $\overline{\alpha} : X \to X$ by $\overline{\alpha}|_{X\setminus Y} = \alpha$ and $Y\overline{\alpha} \subseteq Y$. Thus for every $\alpha \in T(X \setminus Y), \overline{\alpha} \in \overline{T}(X,Y)$ and ran $\overline{\alpha} \cap Y = (\text{ran } \alpha \cup Y\overline{\alpha}) \cap Y = Y\overline{\alpha}$. It follows from Theorem 2.3 that $\{\overline{\alpha} \mid \alpha \in T(X \setminus Y)\} \subseteq \text{Reg } \overline{T}(X,Y)$. Since $\alpha \mapsto \overline{\alpha}$ is an injective map from $T(X \setminus Y)$ into Reg $\overline{T}(X,Y)$, we have

$$|T(X)| \geq |\text{Reg } \overline{T}(X,Y)| \geq |\{\overline{\alpha} \mid \alpha \in T(X \setminus Y)\}| = |T(X \setminus Y)| = |T(X)|,$$

and thus $|\text{Reg } \overline{T}(X,Y)| = |T(X)|$.

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