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From Fibonacci to Catalan permutations

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Abstract. It is well known that permutations avoiding any 3-length pattern are enumerated by the Catalan numbers. If the three patterns 123, 132 and 213 are avoided at the same time we obtain a class of permutations enumerated by the Fibonacci numbers. We start from these permutations and make one or two forbidden patterns disappear by suitably "generalizing" them. In such a way we find several classes of permutations enumerated by integer sequences which lay between the Fibonacci and Catalan numbers. For each class, we provide the generating function according to the length of the permutations. Moreover, as a result, we introduce a sort of "continuity" among the number sequences enumerating these classes of permutations.

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1 Introduction

Fibonacci and Catalan numbers are very well known sequences. They appear in many combinatorial problems as they enumerate a great quantity of combinatorial objects. For instance, Fibonacci numbers are involved in the tiling of a strip, in rabbits' population growth, in bees' ancestors, ..., while Catalan numbers occur in the enumeration of several kinds of paths, trees, permutations, polyominoes and other combinatorial structures. Fibonacci numbers are described by the famous recurrence:

$$\begin{cases} F_0 = 1 \\ F_1 = 1 \\ F_n = F_{n-1} + F_{n-2} \end{cases}$$

from which the generating function:

$$F(x) = \frac{1}{1 - x - x^2}$$

arises, and the sequence begins with $1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$. Catalan numbers have been deeply studied, too: they appear in many relations, also connected to other sequences or by themselves. They are defined by:

$$\begin{cases} C_0 = 1 \\ C_1 = 1 \\ C_n = \sum_{i=0}^{n-1} C_{n-1-i} C_i. \end{cases}$$

The expression

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \quad \text{with } n \geq 0,$$

derived from the generating function

$$C(x) = \frac{1-\sqrt{1-4x}}{2x},$$

is a closed formula for them and the sequence begins with the numbers $1, 1, 2, 5, 14, 42, 132, \ldots$.

Our question is: "What is there between Fibonacci and Catalan numbers?" For instance the following sequences:

- $\{c_n\}_{n\geq 0} = \{1, 1, 2, 4, 7, 13, 24, \ldots\}$, $(c_0 = 1, c_1 = 1, c_2 = 2, c_n = c_{n-1} + c_{n-2} + c_{n-3})$ Tribonacci numbers;
- $\{t_n\}_{n\geq 0} = \{1, 1, 2, 4, 8, 16, 32, \ldots, 2^{n-1}\}$, $(t_0 = 1, t_n = 2^{n-1})$;
- $\{p_n\}_{n\geq 0} = \{1, 1, 2, 5, 12, 29, 70, \ldots\}$, $(p_0 = 1, p_1 = 1, p_2 = 2, p_n = 2p_{n-1} + p_{n-2})$ Pell numbers;
- $\{\bar{F}_n\}_{n\geq 0} = \{1, 1, 2, 5, 13, 34, 89, \ldots\}$, $(\bar{F}_0 = 1, \bar{F}_1 = 1, \bar{F}_n = 3\bar{F}_{n-1} - \bar{F}_{n-2})$ even index Fibonacci numbers,

(for more details see the sequences M1074, M1129, M1413, M1439 in [19], respectively, where they are defined with different initial conditions)

lay between Fibonacci and Catalan numbers (we call the last sequence *even* index Fibonacci numbers while other authors call them *odd* index Fibonacci numbers, but this depends on the initials conditions assumed for the Fibonacci sequence). We are looking for a unifying combinatorial interpretation for all these sequences, and others too. To this aim we will use permutations avoiding forbidden subsequences. Our results can be extended to paths and trees.

The main idea we are going to base on, has already been used in [3]. Here, we briefly recall that. It is well known that $|S_n(123, 213, 312)| = F_n$ and $|S_n(123)| = C_n$ [18], as mentioned in the abstract. The patterns 213 and 312, which are not present in the second equality, can be seen as particular cases of more general patterns. More precisely, 213 can be obtained from the pattern $r_k = k(k-1)(k-2)\ldots 21(k+1)$ with $k = 2$, while 312 is the pattern $q_k = 1(k+1)k(k-1)\ldots 2$ with

$k = 2$, again. When k grows, the patterns r_k and q_k increase their length, then in the limit (k grows to ∞) they can be not considered in the enumeration of the permutations π of $S_n(123, r_k, q_k)$ since, for each $n \geq 0$, any π does not surely contain a pattern of infinite length. In other words, starting from the case $k = 2$ (involving Fibonacci numbers), for each $k > 2$ we provide a class of pattern avoiding permutations where the pattern are suitably generalized in order to make them "disappear" when k grows, leading to the class $S(123)$ enumerated by the Catalan numbers. We say that there is a sort of "continuity" between Fibonacci and Catalan numbers since we provide a succession of generating functions $\{g_k(x)\}_{k\geq 2}$ with $g_2(x) = F(x)$ and whose limit is $C(x)$.

As a matter of fact, in the paper this aim is reached in two steps: first only the pattern 312 is generalized so that we arrive to the class $S(123, 213)$ enumerated by $\{2^{n-1}\}_{n\geq 1}$, then the pattern 213 is increased in order to obtain the class $S(123)$. Nevertheless it is possible to make "disappear" both the patterns at the same time obtaining similar results.

2 Notations and definitions

We denote by S_n the set of permutations on $[n] = \{1, 2, \ldots, n\}$. Let $\pi = \pi_1\pi_2\cdots\pi_n \in S_n$ and $\Gamma = \gamma_1\gamma_2\cdots\gamma_k \in S_k$. We say that π does not contain a subsequence of kind Γ (or the pattern Γ) if no sequence $j_1 < j_2 < \cdots < j_k$ exists such that $\pi_{j_i} < \pi_{j_h}$ if and only if $\gamma_i < \gamma_h$.

Let $S_n(\Gamma)$ be the set of permutations not containing a subsequence of kind γ. For instance $7465312 \in S_7(123)$ while $7154326 \notin S_7(123)$ since the subsequence 146 is of kind 123. If $\Gamma_1, \Gamma_2, \ldots, \Gamma_j$ are permutations we denote by $S(\Gamma_1, \Gamma_2, \ldots, \Gamma_j) = S(\Gamma_1) \bigcap S(\Gamma_2) \bigcap \cdots \bigcap S(\Gamma_j)$ the set of permutations on $[n]$ that do not contain anyone of the sequences $\Gamma_1, \Gamma_2, \ldots, \Gamma_j$. For instance $6745231 \in S_7(123, 132, 213)$ while $6475231 \notin S_7(123, 132, 213)$, being the sequence 475 of kind 132.

Permutations avoiding forbidden subsequences have been widely studied by many authors [2, 3, 4, 7, 8, 11, 12, 13, 14, 15, 18, 20, 21, 22, 23]. A very efficient and natural method to enumerate classes of permutations was proposed by Chung et al. [8] and Rogers [17], and, later, by West [21]. It consists in generating permutations in S_n from permutations in S_{n-1} by inserting n in all the positions such that a forbidden subsequence does not arise (we denote these positions by a '$\diamond$'). These positions are known as *active sites*, while a *site* is any position between two consecutive elements in a permutation or before the first element or after the last one. If a permutation in $S_{n-1}(\Gamma_1, \ldots, \Gamma_j)$ contains k active sites, it generates k permutations in $S_n(\Gamma_1, \ldots, \Gamma_j)$. In the sequel, we denote the i-th active site as the site located before π_i.

In order to show how we can enumerate classes of permutations by this method, we consider the class $S_n(123)$. Let $\pi = \pi_1\pi_2\cdots\pi_n$ be a permutation in $S_n(123)$ such that $\pi_1 > \pi_2 > \cdots > \pi_{k-1} < \pi_k$. Then the first k sites are active, since the insertion of $n + 1$ in one of these positions does not create a subsequence of kind 123. On the contrary, the sites on the right of π_k are not active because the insertion of $n+1$ produces the subsequence $\pi_{k-1}\pi_k(n+1)$ which is of kind 123. Therefore, from the permutation

$\diamond\pi_1 \diamond \pi_2 \diamond \cdots \diamond \pi_{k-1} \diamond \pi_k\pi_{k+1} \cdots \pi_n$

we obtain the following ones:

$\diamond(n+1)\pi_1 \diamond \pi_2 \diamond \cdots \diamond \pi_{k-1} \diamond \pi_k\pi_{k+1} \cdots \pi_n$
$\diamond\pi_1 \diamond (n+1)\pi_2 \cdot \cdots \cdot \pi_n$
$\diamond\pi_1 \diamond \pi_2 \diamond (n+1)\pi_3 \cdots \pi_n$
$\vdots$
$\diamond\pi_1 \diamond \pi_2 \diamond \cdots \diamond \pi_{k-1} \diamond (n+1)\pi_k \cdots \pi_n$

which have respectively $(k+1), 2, 3, \ldots, k$ active sites. We remark that from a permutation π having k active sites we obtain k permutations having $(k+1), 2, 3, \ldots, k$ active sites, independently from the length of the permutation. Such a permutation is labelled with (k). We can "condense" this property into a *succession rule* (for more details see [22, 23]):

$$\begin{cases} (1) \\ (1) \rightsquigarrow (2) \\ (k) \rightsquigarrow (2)(3)\cdots(k)(k+1), \end{cases} \tag{1}$$

where $(k) \rightsquigarrow (2)(3)\cdots(k)(k+1)$ is the *production* of a permutation π with label (k). The label (1), said the *axiom* of the succession rule, is the number of active sites of the empty permutation ε which is the only permutation with length $n = 0$, meaning that ε generates the minimal permutation $\pi = 1$ with length $n = 1$. In turn, $\pi = \diamond 1 \diamond$ has two active sites, then it produces two permutations: this fact is described by the second line of the rule $(1) \rightsquigarrow (2)$ (the production of the axiom).

The recursive construction of permutation in $S_n(123)$ can also be represented by a *generating tree*, where each node is a permutation, the permutations obtained from π appear as sons of π and the root is the empty permutation ε with length $n = 0$. Therefore, on the n-th level we have all the permutations of length n (if we assume the root level is 0). The succession rule 1 relates the outdegree of each node in the tree to the outdegree of its sons. Usually, from a succession rule we can obtain a functional equation or a system of equations from which one can obtain the generating function $f(x) = \sum_{n\geq 0} a_n x^n$ where a_n is the number of objects on level n. From the above example for $S(123)$, it is possible to obtain (we omit the calculus) the generating function $C(x)$ for Catalan numbers. Moreover, $|S_n(123)| = C_n,$ for $n \geq 0$.

The enumeration of the permutations of $S_n(123, 132, 213)$ is also briefly illustrated, which is the starting point of our argument, as recalled in the Introduction. In the permutations of this class only the first two sites can be active: the insertion of $n+1$ in another site would produce the subsequence $\pi_1\pi_2(n+1)$ which is of kind 123 or 213. If $\pi_1 < \pi_2$ then only the first site is active because the insertion of $n+1$ in the second site would produce the subsequence $\pi_1\pi_2(n+1)$ which is of kind 132. Let $\pi = \pi_1\pi_2\cdots\pi_n$ be a permutation in $S_n(123,132,213)$; if $\pi_1 < \pi_2$, from $\diamond\pi_1\pi_2\cdots\pi_n$ we obtain $\diamond(n+1)\diamond\pi_1\pi_2\cdots\pi_n$ which has two active sites; if $\pi_1 > \pi_2$, from $\diamond\pi_1\diamond\pi_2\cdots\pi_n$ we obtain $\diamond(n+1)\diamond\pi_1\pi_2\cdots\pi_n$ and $\diamond\pi_1(n+1)\pi_2\cdots\pi_n$ having two and one active sites, respectively. This construction can be encoded by the succession rule:

$$\begin{cases} (1) \\ (1) \rightsquigarrow (2) \\ (2) \rightsquigarrow (1)(2). \end{cases} \tag{2}$$

The above succession rule is an example of *finite* succession rule since only a limited number of different labels appear in it. It is easily seen that it leads to Fibonacci numbers and $|S_n(123,213,312)| = F_n$, for $n \geq 0$.

In the last part of this section, we only note that the permutations of the class $S(123,213)$, which is the intermediate step between the above considered classes (see the Introduction), have exactly two active sites (the first two sites), so that the corresponding succession rule is

$$\begin{cases} (1) \\ (1) \rightsquigarrow (2) \\ (2) \rightsquigarrow (2)(2). \end{cases} \tag{3}$$

It is easy to prove that the related enumerating sequence $\{t_n\}_{n\geq 0}$ is defined by

$$\begin{cases} t_0 = 1 \\ t_n = 2^{n-1}, \quad n \geq 1 \end{cases}$$

and $|S_n(123,213)| = t_n$. The corresponding generating function is $t(x) = \frac{1-x}{1-2x}$. In the sequel, we refer to this sequence simply with $\{2^{n-1}\}_{n\geq 0}$.

We conclude by observing that all the considered sequences take into account the empty permutation which is enumerated by C_0, F_0 and t_0. Moreover, in each presented succession rule the axiom refers to it and the production $(1) \rightsquigarrow (2)$ describes its behavior.

3 From Fibonacci to 2^{n-1}

Consider a permutation $\pi \in S_n(123,213,1(k+1)k\ldots 2)$. His structure is essentially known thanks to [10], where the author analyzes the permutations of $S_n(123,132,k(k-1)\ldots 21(k+1))$ which is equivalent to the class we are considering (the permutations of the former are the reverse complement of the latter). In the same paper the author shows that those permutations are enumerated by the sequence of k-generalized Fibonacci numbers, providing also the related generating function. Here, we give an alternative proof of the same facts by using the ECO method [6]. To this aim, we recall the structure of the permutations referring directly to the class $S_n(123,213,1(k+1)k\ldots 2)$, nevertheless we omit the easy proofs that one can recover from [10].

If $\pi \in S_n(123, 213, 1(k+1)k\dots2)$, then either $\pi_1 = n$ or $\pi_2 = n$;

- if $\pi_1 = n$, then $\pi = n\tau$, with $\tau \in S_{n-1}(123, 213, 1(k+1)k\dots2)$;
- if $\pi_2 = n$, then $\pi_1 = n-j$, with $j \in \{1, 2, \dots, k-1\}$, and $\pi = (n-j)n(n-1)\dots(n-j+1)\sigma$, with $\sigma \in S_{n-j-1}(123, 213, 1(k+1)k\dots2)$.

If $\pi \in S_n(123, 213, 1(k+1)k\dots2)$, denote $\pi^{(i)}$ the permutations such that $\pi_1 = n-i$. The active sites of π are the first two sites: the insertion of $n+1$ in any other site would create the forbidden pattern 123 or 213. More precisely, the permutations $\pi^{(j)}$ with $j \in \{0, 1, 2, \dots, k-2\}$ have label (2) (the first two sites are active), while $\pi^{(k-1)}$ has label (1) (the first site is active). The son of the permutation $\pi^{(k-1)}$ is the permutation of $S_{n+1}(123, 213, 1(k+1)k\dots2)$ obtained from π by inserting $n+1$ in its first active site, which we denote $\bar{\pi}^{(0)}$. It is easily seen that $\bar{\pi}^{(0)}$ has, in turn, label (2). The two sons of the permutations with label (2) are $\bar{\pi}^{(0)}$ and $\bar{\pi}^{(j+1)}$ ($\bar{\pi}^{(j+1)}$ is obtained from π by inserting $n+1$ in the second active site). Therefore, all these permutations have, in turn, label (2) but $\bar{\pi}^{(k-1)}$ which has label (1). Since all the labels (2) have not the same production, it is suitable to label each permutation $\pi^{(j)}$ ($j \in \{0, 1, 2, \dots, k-2\}$) with (2_j) in order to recognize the permutation $\pi^{(k-2)}$ whose sons have labels (1) and (2). Then, the above description can be encoded by:

$$\begin{cases} (1) & & \\ (1) & \rightsquigarrow & (2_0) \\ (2_j) & \rightsquigarrow & (2_0)(2_{j+1}), \qquad \text{for } \; j = 0, 1, 2, \dots, k-3 \\ (2_{k-2}) & \rightsquigarrow & (2_0)(1). \end{cases}$$

We now deduce the generating function $T^k(x, y)$ of the permutations of $S(123, 213, 1(k+1)k\dots2)$, according to their length and number of active sites. To this aim we consider the subsets T_1 of the permutations with label (1) and T_{2_j}, with $j = 0, 1, 2, \dots, k-2$, of the permutations with label (2_j). It is obvious that these subsets form a partition of $S(123, 213, 1(k+1)k\dots2)$. Denote with $T_1(x, y) = \sum_{\pi \in T_1} x^{n(\pi)} y^{f(\pi)}$ the generating function of T_1 and $T_{2_j}(x, y) = \sum_{\pi \in T_{2_j}} x^{n(\pi)} y^{f(\pi)}$ the generating function of T_{2_j} ($j = 0, 1, \dots, k-2$), where $n(\pi)$ and $f(\pi)$ are the length and the number of active sites of a permutation π, respectively. From the above succession rule the following system is derived:

$$\begin{cases} T_1(x, y) &= y + xy \sum_{\pi \in T_{2_{k-2}}} x^{n(\pi)} \\ T_{2_0}(x, y) &= xy^2 (T_1(x, 1) + \sum_{i=0}^{k-2} T_{2_i}(x, 1)) \\ T_{2_j}(x, y) &= xy^2 T_{2_{j-1}}(x, 1), \quad j = 1, 2, \dots, k-2\,. \end{cases}$$

Clearly, it is $T^k(x, y) = T_1(x, y) + \sum_{j=0}^{k-2} T_{2_j}(x, y)$ and, if $y = 1$, $T^k(x, 1)$ is the generating function of the permutations of $S(123, 213, 1(k+1)k\dots2)$ according to their length. From the above system (we omit the calculus), it follows:

$$T^k(x, 1) = \frac{1-x}{1-2x+x^{k+1}}\,.$$

Note that if k grows to ∞, the generating function $t(x)$ related to the sequence $\{2^{n-1}\}_{n\geq 0}$ (enumerating the permutations of $S(123,213)$, see Section 2) is obtained. For each $k\geq 2$, we get an expression which is the generating function of the k-generalized Fibonacci numbers. For $k=1$, the formula leads to $\frac{1}{1-x}$ which is the generating function of the sequence $\{1\}_{n\geq 0}$ enumerating the permutations of $S_n(123,213,12)=S_n(12)=n\ (n-1)\ \dots 2\ 1$. For $k=3$ the succession is

$$\begin{cases} (1) & \\ (1) & \rightsquigarrow \quad (2_0) \\ (2_0) & \rightsquigarrow \quad (2_0)(2_1) \\ (2_1) & \rightsquigarrow \quad (2_0)(1)\,, \end{cases}$$

which defines the Tribonacci numbers, whose generating function is $T^3(x,1)=\frac{1}{1-x-x^2-x^3}$.

Notice that in [1] the authors reached similar results.

4 From 2^{n-1} to Catalan

Let π be a permutation of $S_n(123,k(k-1)\dots 21(k+1))$. Then if $\pi_i=n$ it is $i\in\{1,2,\dots,k\}$, otherwise if $\pi_j=n$ with $j\geq k+1$, it should be $\pi_1>\pi_2>\dots>\pi_k$ in order to avoid the pattern 123. But in this way the entries $\pi_1,\pi_2,\dots,\pi_k,\pi_j$ are a pattern $k(k-1)\dots 21(k+1)$ which is forbidden.

If α_π denotes the minimum index j such that $\pi_{j-1}<\pi_j$, we can describe the active sites of π by using α_π.

1. If $\alpha_\pi=j\leq k$, then the active sites are the first j sites of π. The insertion of $n+1$ in any other site would create the pattern 123. In this case π as label (j).

2. If $\alpha_\pi>k$, then the active sites of π are the first k sites since the insertion of $n+1$ in any other site would lead to the occurrence of the forbidden patterns $k(k-1)\dots 21(k+1)$ or 123. In this case π has label (k).

In order to describe the labels of the sons of π, in the sequel we denote $\bar{\pi}^{(i)}$ the permutation $\bar{\pi}\in S_{n+1}(123,k(k-1)\dots 21(k+1))$ obtained from π by inserting $n+1$ in the i-th active site of π.

1. If π has label (k), it is not difficult to see that $\alpha_{\bar{\pi}^{(1)}}=\alpha_\pi+1>k$, then $\bar{\pi}^{(1)}$ has label (k) again. While, if we consider $\bar{\pi}^{(i)}$, with $i=2,3,\dots,k$, then $\alpha_{\bar{\pi}^{(i)}}=i$ and $\bar{\pi}^{(i)}$ has label (i). Therefore the production of the label (k) is $(k)\rightsquigarrow(2)(3)\dots(k)(k)$.

2. If π has label (j) with $j\in\{2,3,\dots,k-1\}$, then it is easily seen that $\alpha_{\bar{\pi}^{(1)}}=\alpha_\pi+1\leq k$ and $\bar{\pi}^{(1)}$ has label $(j+1)$ (note that in this case $\alpha_\pi=j$). While if we consider $\bar{\pi}^{(i)}$, with $i=2,3,\dots,j$, then $\alpha_{\bar{\pi}^{(i)}}=i$ and $\bar{\pi}^{(i)}$ has label (i). Therefore the production of (j) is $(j)\rightsquigarrow(2)(3)\dots(j)(j+1)$.

The above construction can be encoded by the succession rule:

$$\begin{cases} (1) & & \\ (1) & \rightsquigarrow & (2) \\ (j) & \rightsquigarrow & (2)(3)\dots(j)(j+1), \qquad \text{for } \; j=2,3,\dots,k-1 \\ (k) & \rightsquigarrow & (2)(3)\dots(k)(k)\,, \end{cases}$$

where the axiom and its production refer to the empty permutation generating the permutation $\pi = 1$, which, in turn, produces two sons: $\pi = 12$ and $\pi = 21$. Using the theory developed in [9], the production matrix related to the above succession rule is

$$P_k = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & 1 & 0 & \cdots & \cdots & \cdots \\ 0 & 1 & 1 & 1 & 0 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & & \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \\ 0 & 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & 1 & 1 & \cdots & 2 \end{pmatrix},$$

with k rows and columns. For each $k \geq 2$, it is easy to see that the matrix P_k can be obtained from P_{k-1} as follows:

$$P_k = \begin{pmatrix} 0 & u^T \\ 0 & P_{k-1} + eu^T \end{pmatrix},$$

where u^T is the row vector $(1, 0, \dots, 0)$ and e is the column vector $(1, 1, \dots, 1)^T$ (both $k-1$-dimensional). If $f_{P_k}(x)$ is the generating function according to the length of the permutations associated to P_k, from a result in [9] (more precisely Proposition 3.10), the following functional equation holds:

$$f_{P_k}(x) = \frac{1}{1 - x f_{P_{k-1}}(x)} \; .$$

In the limit, we have $f(x) = \frac{1}{1-xf(x)}$ which is the functional equation verified by the generating function of the Catalan numbers $C(x)$.
As a particular case, it is possible to check that for $k = 3$, the sequence of the even index Fibonacci numbers is involved. The obtained succession rule is

$$\begin{cases} (1) & & \\ (1) & \rightsquigarrow & (2) \\ (2) & \rightsquigarrow & (2)(3) \\ (3) & \rightsquigarrow & (2)(3)(3)\,, \end{cases}$$

leading to the related generating function $\bar{F}(x) = \frac{1-2x}{1-3x+x^2}$.

5 Another way for the same goal

In Section 3, starting from $S(123,\ 213, 132)$ and using the knowledge that $S(123, 213)$ is enumerated by $\{2^{n-1}\}_{n\geq 0}$, the pattern 132 has been generalized in $1(k+1)k\ldots 2$, in order to make it "disappear". Since the class $S(123, 132)$ is enumerated by $\{2^{n-1}\}_{n\geq 0}$, too, one can choose the pattern 213 instead of 132 (among the forbidden patterns of the permutations of $S(123, 213, 132)$) as the one to be generalized. Indeed, there is no a particular reason why we chose the pattern 132 to make it disappear.

Similarly, starting from $S(123, 132)$ and recalling that $|S_n(p)| = C_n\ \forall\, p \in S_3$, either the pattern 123 or the pattern 132 can be generalized in order to find a class enumerated by the Catalan numbers.

The difference between a choice with respect to another one lies in the fact that different ECO construction for the permutations are expected. Therefore, different succession rules for the same sequence could be found.

5.1 From Fibonacci to 2^{n-1}

Starting from $S(123, 213, 132)$, here we generalize the pattern 213 considering the class $S(123, 132, k(k-1)\ldots 21(k+1))$, for $k \geq 3$. This class has already been described in [10], where the author provides the structure of its permutations. From his results, it is possible to deduce the following succession rule (similarly to Section 3, the details are omitted), encoding the construction of those permutations:

$$\begin{cases} (1) \\ (1) \rightsquigarrow (2) \\ (h) \rightsquigarrow (1)^{h-1}(h+1) \quad \text{for } h < k \\ (k) \rightsquigarrow (1)^{k-1}(k)\,. \end{cases}$$

In [10] the author shows also that the k-generalized Fibonacci numbers are the enumerating sequence of the permutations of $S(123, 132, k(k-1)\ldots 21(k+1))$. This fact can be derived also by solving the system that can be obtained from the above succession rule, with a technique similar to that one used in Section 3 leading to the same generating function $T^k(x, 1) = \frac{1-x}{1-2x+x^{k+1}}$. This agrees with the fact that in the limit for $k \to \infty$, the class to be considered is $S(123, 132)$, enumerated by $\{2^{n-1}\}_{n\geq 0}$ [18]. We note that it is possible to describe the permutations of $S(123, 132)$ with the succession rule

$$\begin{cases} (1) \\ (1) \rightsquigarrow (2) \\ (h) \rightsquigarrow (1)^{h-1}(h+1)\,, \end{cases}$$

from which one can get that the related generating function is, again, $t(x) = \frac{1-x}{1-2x}$.

The particular case $k = 3$ is marked: the obtained succession rule is

$$\begin{cases} (1) & & \\ (1) & \rightsquigarrow & (2) \\ (2) & \rightsquigarrow & (1)(3) \\ (3) & \rightsquigarrow & (1)(1)(3) \end{cases}$$

corresponding to the sequence of Tribonacci numbers, as one can check by deriving the related generating function $T^3(x,1) = \frac{1}{1-x-x^2-x^3}$.

5.2 From 2^{n-1} to Catalan

Starting from $S(123, 132)$, the pattern 132 is generalized in $(k-1)(k-2)\dots 21(k+1)k$, with $k \geq 3$. Moreover, the construction of the permutations of $S(123, (k-1)(k-2)\dots 21(k+1)k)$ is described and the corresponding succession rule is showed. Finally, we prove that the corresponding generating function is, in the limit for $k \to \infty$, the generating function of the Catalan numbers $C(x)$.

Let π be a permutation of $S_n(123, (k-1)(k-2)\dots 21(k+1)k)$. We denote:

- $r = \min\{1, 2, \dots, n\}$ such that $\pi_{r-1} < \pi_r$;
- $s = \min\{1, 2, \dots, n\}$ and $t = \min\{1, 2, \dots, n\}$ such that, fore some indexes $m_1 < m_2 < \dots < m_{k-2} < s < t$, it is $\pi_{m_1}\pi_{m_2}\dots\pi_{m_{k-2}}\pi_s\pi_t \simeq (k-1)(k-2)\dots 21k$ (the two subsequences are order-isomorphic and π_s and π_t correspond to the 1 and to the k of the pattern $(k-1)(k-2)\dots 21k$);
- $\alpha_\pi = \min\{r, s\}$;
- $\bar{\pi}^{(l)}$ the permutation of $S_{n+1}(123, (k-1)(k-2)\dots 21(k+1)k)$ obtained from π by inserting $n+1$ in the l-th site.

We prove that π has α_π active sites which are the first α_π sites of π.

It is easily seen that the insertion of $n+1$ in any site among the first α_π sites of π, does not induce either the pattern 123 or the pattern $(k-1)(k-2)\dots 21(k+1)k$. On the other hand, if $\alpha_\pi = r$, then the insertion of $n+1$ in the l-th site, $l > \alpha_\pi$, would create the pattern 123 in the entries $\bar{\pi}^{(l)}_{r-1}\bar{\pi}^{(l)}_r\bar{\pi}^{(l)}_l$. While, if $\alpha_\pi = s$, then the insertion of $n+1$ in the i-th site, $\alpha_\pi+1 \leq i \leq t$, would create the pattern $(k-1)(k-2)\dots 21(k+1)k$ in the entries $\bar{\pi}^{(i)}_{m_1}\bar{\pi}^{(i)}_{m_2}\dots\bar{\pi}^{(i)}_{m_{k-2}}\bar{\pi}^{(i)}_{\alpha_\pi}\bar{\pi}^{(i)}_i\bar{\pi}^{(i)}_{t+1}$ (recall that $\bar{\pi}^{(i)}_i = n+1$ and $\bar{\pi}^{(i)}_{t+1} = \pi_t$). Finally, if $i \geq t+1$, the pattern 123 would appear in the entries $\bar{\pi}^{(i)}_{\alpha_\pi}\bar{\pi}^{(i)}_t\bar{\pi}^{(i)}_i$.

Denote (h) the label of π, whit $h = \alpha_\pi$. In order to describe the labels of the sons $\bar{\pi}^{(l)}$, $l = 1, 2, \dots, h$, of π, we have:

1. If $h < k$ (note that on this case $\alpha_\pi = r$ or, if $\alpha_\pi = s$, then $s = k-1$), then the permutation $\bar{\pi}^{(1)} = (n+1)\pi_1\pi_2\dots\pi_{\alpha_\pi}\dots\pi_k\dots\pi_n$, so that $\alpha_{\bar{\pi}^{(1)}} = \alpha_\pi + 1$. Therefore $\bar{\pi}^{(1)}$ has label $(h+1)$. While if we consider the permutations $\bar{\pi}^{(j)}$, $j = 2, 3, \dots, h$, it is $\alpha_{\bar{\pi}^{(j)}} = j$ since $\bar{\pi}^{(j)}_{j-1} < \bar{\pi}^{(j)}_j (=$

$n+1$). So $\bar{\pi}^{(j)}$ has label (j) and we conclude that the production of (h) is $(h) \rightsquigarrow (2)(3)\dots(h)(h+1)$.

2. If $h \geq k$, then $\bar{\pi}^{(1)} = (n+1)\pi_1\pi_2\dots\pi_k\dots\pi_{\alpha_\pi}\dots\pi_n$, so that $\alpha_{\bar{\pi}^{(1)}} = \alpha_\pi+1$. Therefore $\bar{\pi}^{(1)}$ has label $(h+1)$. Note that in both cases $\alpha_\pi = r$ or $\alpha_\pi = s$ it is $\pi_1 > \pi_2 > \dots > \pi_{\alpha_\pi - 1}$. Then, if we consider the permutations $\bar{\pi}^{(j)}$, $j = k, k+1, \dots, \alpha_\pi$, we obtain $\alpha_{\bar{\pi}^{(j)}} = k-1$, regardless of j, since $\bar{\pi}^{(j)}_1\bar{\pi}^{(j)}_2\dots\bar{\pi}^{(j)}_{k-1}\bar{\pi}^{(j)}_j \simeq (k-1)(k-2)\dots1k$. Then $\bar{\pi}^{(j)}$ has label $(k-1)$, for $j = k, k+1, \dots, \alpha_\pi$. For the remaining sons $\bar{\pi}^{(j)}$, $j = 2, 3, \dots, k-1$, it is easily seen that $\bar{\pi}^{(j)}_{j-1} < \bar{\pi}^{(j)}_j (= n+1)$. So, $\bar{\pi}^{(j)}$ has label (j). We conclude that, in this second case, the production of (h) is $(h) \rightsquigarrow (2)(3)\dots(k-2)(k-1)^{h-k+2}(h+1)$.

The above description of the generation of the permutations of $S(123, (k-1)(k-2)\dots21(k+1)k)$ can be then encoded in the following succession rule Ω_k:

$$\Omega_k = \begin{cases} (1) \\ (1) \rightsquigarrow (2) \\ (h) \rightsquigarrow (2)\cdots(h)(h+1) \quad \text{for } h < k \\ (h) \rightsquigarrow (2)\cdots(k-2)(k-1)^{h-k+2}(h+1) \quad \text{for } \ h \geq k\,. \end{cases}$$

For $k = 2$, the class $S(123, 132)$ is obtained, whose corresponding succession rule has been considered in Section 5.1. Note that it does not correspond with the one obtained from the above one poising $k = 2$.

For $k = 3$ (the class is $S(123, 2143)$) we get the succession rule:

$$\begin{cases} (1) \\ (1) \rightsquigarrow (2) \\ (h) \rightsquigarrow (2)^{h-1}(h+1)\,, \end{cases}$$

leading to the even index Fibonacci numbers. Note that it is different from the succession rule corresponding to the same numbers of Section 4. Its associated production matrix [9] is:

$$M_3 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & \cdots \\ 0 & 2 & 0 & 1 & 0 & \cdots \\ 0 & 3 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

For each $k \geq 4$, it is easy to check that the production matrix related to Ω_k satisfies

$$M_k = \begin{pmatrix} 0 & u^T \\ 0 & M_{k-1} + eu^T \end{pmatrix},$$

where $u^T = (1, 0, 0, \dots)$ and $e = (1, 1, 1, \dots)^T$. Then, if $g_{M_k}(x)$ is the corresponding generating function, we deduce [9]:

$$g_{M_k}(x) = \frac{1}{1 - xg_{M_{k-1}(x)}}\ .$$

If $g(x)$ denotes the limit of $g_{M_k}(x)$, the functional equation $g(x) = \frac{1}{1-xg(x)}$ is obtained, which is verified by the generating function $C(x)$ of the Catalan numbers.

6 From Fibonacci to Catalan directly

This section summarizes the results found when the two patterns 132 and 213 are generalized at the same time, considering the class $S(123, (k-1)(k-2)\dots 21(k+1)k), k(k-1)\dots 21(k+1))$ in order to obtain the class $S(123)$, when k grows to ∞. Most of the proofs are omitted but they can easily recovered by the reader. At the first step, for $k = 3$, we find the succession rule:

$$\begin{cases} (1) \\ (1) \rightsquigarrow (2) \\ (2) \rightsquigarrow (2)(3) \\ (3) \rightsquigarrow (2)(2)(3) \end{cases}$$

corresponding to $S_n(123, 2143, 3214)$. This class is enumerated by Pell numbers which we define with the recurrence:

$$\begin{cases} p_0 = 1 \\ p_1 = 1 \\ p_2 = 2 \\ p_n = 2p_{n-1} + p_{n-2}, \qquad \text{for } n \geq 3. \end{cases}$$

Note that the initial conditions are different from the usual ones (which are $p_0 = 0$ and $p_1 = 1$) in order to consider the empty permutation ε, for $n = 0$.

For a general k we have the class $S_n(123, (k-1)\cdots 1(k+1)k, k(k-1)\cdots 1(k+1))$. We briefly describe the construction of the permutations of the class (the details are omitted). Let π be a permutation of the class. It is easily seen that if $\pi_l = n$, then $l \leq k$. Therefore, if (h) denotes the label of π, it is $h \in \{1, 2, \dots, k\}$. Now, if $h < k$, then $\bar{\pi}^{(1)}$ has label $(h+1)$ and $\bar{\pi}^{(j)}$, $j = 2, 3, \dots, h$, has label (j). While, if $h = k$, then $\bar{\pi}^{(1)}$ has label (k), $\bar{\pi}^{(j)}$, $j = 2, 3, \dots, k-1$, has label (j) and $\bar{\pi}^{(k)}$ has label $(k-1)$, again. The construction can be encoded in the succession rule:

$$\begin{cases} (1) \\ (1) \rightsquigarrow (2) \\ (h) \rightsquigarrow (2)(3)\cdots(h-1)(h)(h+1) \text{ for } h < k \\ (k) \rightsquigarrow (2)(3)\cdots(k-1)(k-1)(k)\,. \end{cases}$$

For each k, considering the associated production matrices [9] and the corresponding generating functions, it possible to prove that, in the limit, the generating function of the Catalan numbers is obtained.

6.1 A continuity between Pell numbers and even index Fibonacci numbers

We conclude by showing that it is possible to find a “continuity” between Pell and even index Fibonacci numbers. We start from the class $S_n(123, 2143, 3214)$ (obtained by posing $k = 3$ in the preceding succession rule) enumerated by Pell numbers, then we generalize the pattern 2143, so obtaining the classes $S(123, 3214, 21(k+1)k(k-1)\dots 43)$.

Let $\pi \in S_n(123, 3214, 21(k+1)k(k-1)\dots 43)$. Then, if $\pi_l = n$, it is $l \leq 3$ in order to avoid the patterns 123 and 3214. Therefore, π has at most 3 active sites (the first three sites of π). We denote r_π the number of entries of π with index $j \geq 3$ such that $\pi_j > \pi_1$ (note that if $\pi_1 > \pi_2$, then $r_\pi = 0$). It is:

- $\pi_{j_1} > \pi_{j_2} > \dots > \pi_{j_{r_\pi}}$ (the pattern 123 is forbidden);
- $r_\pi \leq (k-2)$ (the pattern $21(k+1)k\dots 43$ is forbidden);
- the elements π_{j_i} are adjacent in π in order to avoid 123 or $21(k+1)k\dots 43$.

If π starts with an ascent (i.e. $\pi_1 < \pi_2$), then only the first two sites are active, since the insertion of $n+1$ in any other site would create the pattern 123: the permutation π has label (2).

If π starts with a descent (i. e. $\pi_1 > \pi_2$), then the number of its active sites depends on r_π:

1. If $r_\pi = h < k-2$, then π has three active sites. Let (3_h) be its label. The permutation $\bar{\pi}^{(1)}$ (obtained by π by inserting $n+1$ in the first site) starts with a descent and $r_{\bar{\pi}^{(1)}} = 0$ (since $\bar{\pi}^(1)_1 = n+1$); therefore, $\bar{\pi}^{(1)}$ has label (3_0). The son $\bar{\pi}^{(2)}$ starts with an ascent and its label is (2). The last son $\bar{\pi}^{(3)}$ starts with a descent and $r_{\bar{\pi}^{(3)}} = h+1$, so its label is (3_{h+1}). The production of (3_h) is $(3_h) \rightsquigarrow (2)(3_0)(3_{h+1})$.
2. If $r_\pi = k-2$, then π has two active sites, since the insertion in the third site would create the pattern $21(k+1)k\dots 43$, while the insertion in any other site surely creates the pattern 123. Its son $\bar{\pi}^{(1)}$ has label (3_0) since it starts with a descent and $r_{\pi^{(1)}} = 0$. While the other son $\bar{\pi}^{(2)}$ starts with an ascent and has label (2). Therefore, the production of label (2) is $(2) \rightsquigarrow (2)(3_0)$.

The following succession rule:

$$\begin{cases} (1) \\ (1) \rightsquigarrow (2) \\ (2) \rightsquigarrow (2)(3_0) \\ (3_j) \rightsquigarrow (2)(3_0)(3_{j+1}), \quad \text{for} \quad j = 0, 1, 2, \dots, k-3 \\ (3_{k-3}) \rightsquigarrow (2)(2)(3_0) \end{cases}$$

summarizes the construction of the class $S(123, 3214, 21(k+1)k\dots 43)$. Solving the system one can deduce from the above rule, the generating function $\bar{F}_k(x) =$

$\frac{1-2x+x^k}{1-3x+x^2+x^k}$ is obtained, which in the limit is the generating function of the even index Fibonacci numbers $\bar{F}(x)$.

Starting from the class $S(123, 2143, 3214,)$, one can generalize the pattern 3214 instead of 2143. The class we get is $S(123, 2143, k(k-1)\dots 32(k+1)1)$ and the succession rule describing its construction is (the easy proof is omitted):

$$\begin{cases} (1) \\ (1) \rightsquigarrow (2) \\ (h) \rightsquigarrow (2)^{h-1}(h+1) \quad \text{for} \;\; h < k \\ (k) \rightsquigarrow (2)^{k-1}(k)\,. \end{cases}$$

Once again, one can prove that the corresponding generating function is $\bar{F}_k(x)$, leading, in the limit, to $\bar{F}(x)$.

7 Remarks

In order to summarize the several "continuities" we have proposed in the paper, we condense our results in Figure 1 where a straight line represents a direct step and a dashed line represents a family of permutations obtained by generalizing one or two patterns.

The results we found for permutations can be easily extended to Dyck paths and planar trees by means of ECO method [5, 6]. We can find classes of paths and trees described by the finite succession rules we introduced by imposing some conditions on the height of paths and the level of their valleys and on the outdegree and level of nodes in the trees.

Figure 1 allows to see the different three ways we have followed to describe a discrete "continuity" between Fibonacci and Catalan numbers: the generalization of a single pattern (the rightmost and the leftmost path from the top to the bottom in the figure) and the generalization of a pair of patterns (central path in the figure). In particular, following the rightmost and the leftmost path in the graph, the intermediate level of the permutations enumerated by $\{2^{n-1}\}_{n\geq 0}$ is encountered. For each k, our approach produces two different class of permutations enumerated by the same sequence, indeed the two corresponding generating functions are the same for each k. We note that, in this way, we can provide two different succession rules encoding the same sequence. An instance can be seen by looking at the succession rules the reader can find at the end of the Sections 3 and 5.1.

The same happens with the succession rule at the end of Section 5.2 and the succession rule of the particular case ($k = 3$) of Section 5.2, which encode the sequence of the even index Fibonacci numbers. Really, we did not prove that this is the case for each k related to the classes of permutations used to describe the discrete continuity between $\{2^{n-1}\}_{n\geq 0}$ and Catalan numbers, since we did not get the explicit formulas of the generating functions.

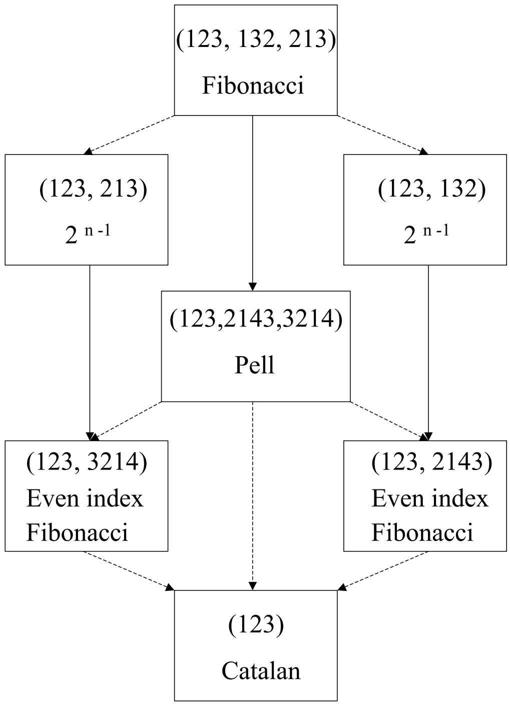

Figure 1: The graph of permutations.

References

[1] J.-L. Baril and P.-T. Do, *Eco-generation of Fibonacci and Lucas permutations*, Proceedings of GASCom and Bijective Combinatorics 2006, 5^{th} Edition.

[2] J. Bandlow and K. Killpatrick, *An area-to-inv bijection Dyck paths and 312-avoiding permutations*, Electron. J. Combin., **8** (2001).

[3] E. Barcucci, A. Del Lungo, E. Pergola and R. Pinzani, *From Motzkin to Catalan permutations*, Discrete Math., **217** (2000), 33–49.

[4] E. Barcucci, A. Del Lungo, E. Pergola and R. Pinzani, *Some permutations with fobidden subsequences and their inversion number*, Discrete Math., **234** (2001), 1–15.

[5] E. Barcucci, A. Del Lungo, E. Pergola and R. Pinzani, *A methodology for plane trees enumeration*, Discrete Math., **180** (1998), 45–64.

[6] E. Barcucci, A. Del Lungo, E. Pergola and R. Pinzani, *ECO: A methodology for enumeration of combinatorial objects*, J. Diff. Equ. Appl., **5** (1999), 435–490.

[7] W.Y.C. Chen, Y.P. Deng and L.L.M. Yang, *Motzkin paths and reduced decompositions for permutations with forbidden patterns*, Elect. J. Combin., **9** (2003).

[8] F.R.K. Chung, R.L. Graham, V.E. Hoggatt Jr. and M. Kleiman, *The number of Baxter permutations*, J. Combin. Theory Ser. A, **24** (1978), 382–394.

[9] E. Deutsch, L. Ferrari and S. Rinaldi, *Production matrices*, Adv. in Appl. Math., **34** (2005), 101–122.

[10] E.S. Egge and T. Mansour, *Restricted permutations, Fibonacci numbers and k-generalized Fibonacci numbers*, Integers, **5** (2005), A1.

[11] E.S. Egge and T. Mansour, *Permutations which avoid 1243 and 2143, continued fractions, and Chebyshev polynomials*, Electron. J. Combin., **9** (2) (2003) #R7.

[12] S. Gire, *Arbres, permutations à motifs exclus et cartes planaires: quelques problèmes algorithmiques et combinatoires*, Thèse de l'Université de Bordeaux I, 1993.

[13] O. Guibert, *Combinatoire des permutations à motifs exclus en liaison avec mots, cartes planaires et tableaux de Young*, PH. D. Thesis, Université de Bordeaux I, 1995.

[14] C. Krattenthaler, *Permutations with restricted patterns and Dyck paths*, Adv. Applied Math., **27** (2001), 510–530.

[15] D. Kremer, *Permutations with forbidden subsequences and a generalized Schröder number*, Discrete Math., **218** (2000), 121–130.

[16] T. Mansour, *Permutations avoiding a pattern from S_k and at least two patterns from S_3*, Ars Combin., **62** (2005).

[17] D.G. Rogers, *Ascending sequences in permutations*, Discrete Math., **22** (1978), 35–40.

[18] R. Simion and W. Schmidt, *Restricted permutations*, Europ. J. Combin., **6** (1985), 383–406.

[19] N.J.A. Sloane and S. Plouffe, *The Encyclopedia of Integer Sequences*, Academic press, 1996.

[20] Z.E. Stankova, *Forbidden subsequences*, Discrete Math., **132** (1994), 291–316.

[21] J. West, *Permutations with forbidden subsequences and stack-sortable permutations*, Ph.D. Thesis, M.I.T. Cambridge, MA, 1990.

[22] J. West, *Generating trees and the Catalan and Schröder numbers*, Discrete Math., **146** (1995), 247–262.

[23] J. West, *Generating trees and forbidden subsequences*, Discrete Math., **157** (1996), 363–374.