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# Bidimensional sand pile and ice pile models

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**Abstract.** In this paper we define an extension of the Sand Pile Model *SPM* and more generally of the Ice Pile Model *IPM* by adding a further dimension to the system. By drawing a parallel between these unidimensional and bidimensional models we will find some common features and some differences. We will show that, like for $SPM(n)$, not all plane partitions are accessible in $BSPM(n)$ starting from the initial state. However, it appears to be much more difficult to characterize the partitions that are accessible in $BSPM(n)$: we will be able to give some necessary but not sufficient conditions for a partition to be accessible. On the other hand, we will show how several properties of the Ice Pile Model in one dimension can be generalized when one adds a second dimension.

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## 1 Introduction

In this paper we introduce the Bidimensional Sand Pile Model *BSPM*, that is, a generalization of the Sand Pile Model *SPM* with the addition of a further dimension. The SPM and some related models have been studied in many different domains. They were considered in the context of integer lattices by Brylawski [3]. From the point of view of physics, Bak, Tang, and Wiesenfeld used them in order to illustrate the important notion of *self organisation criticality*

[2]. Moreover, Anderson et al. [1], Spencer [9], and Goles and Kiwi [5] studied them from a combinatorial point of view.

$SPM(n)$ is a discrete dynamical system describing pilings of $n$ granular objects distributed on an array of columns. More precisely, each state of the system can be described by using a $\ell$-tuple $s = (s_1, s_2, \dots, s_\ell)$, where $s_i \neq 0$ is the number of grains in the column $i$. The system is initially in the state $N = (n)$, that is, all the grains are in the first column. At each step, the system evolves according to the following rule:

$$(s_1, s_2, \dots, s_i, s_{i+1}, \dots, s_\ell) \to (s_1, s_2, \dots, s_i - 1, s_{i+1} + 1, \dots, s_\ell) \quad \text{if } s_i - s_{i+1} >= 2. \tag{1}$$

Because of the evolution rule, we deduce that each state $s = (s_1, s_2, \dots, s_\ell)$ of the system satisfies $s_i \geq s_{i+1}$, for all $i$ and $\sum_{i=1}^{\ell} s_i = n$, where $n$ is the total number of grains, therefore each state can be coded by a partition of the integer $n$. A partition is said to be *accessible* in SPM if it can be obtained by a sequence of applications of rule (1) (also called *transitions*) starting from the state $N$. We denote by $SPM(n)$ the system whose set of configurations is the set of all accessible partitions equiped with the rule of evolution (1). We will omit the integer $n$ and simply write $SPM$ when such notation can be used without ambiguity.

Goles and Kiwi introduced this model in [5] and proved that $SPM(n)$ has a unique *fixed point*, i.e. a configuration in which no grain can fall under the rule (1). Moreover, they showed that the order induced by $SPM(n)$ on accessible partitions is a suborder of the $L_B(n)$ order, introduced by Brylawski in [3]. This is the dominance order on all partitions of $n$, defined as follows: let $a = (a_1, a_2, \dots, a_k)$ and $b = (b_1, b_2, \dots, b_t)$ be two partitions, then

$$a \geq b \Longleftrightarrow \sum_{i=1}^{j} a_i \geq \sum_{i=1}^{j} b_j, \quad \forall j = 1, \dots, \max(k, t).$$

Let us consider the rule:

$$(s_1, \dots, p+1, \underbrace{p, p, \dots, p}_{k}, p-1, \dots, s_\ell) \to (s_1, \dots, p, \underbrace{p, p, \dots, p}_{k}, p, \dots, s_\ell) \quad \text{for any } k. \tag{2}$$

Goles and Kiwi showed that the order $L_B(n)$ coincides with the order defined as follows on the set of all partitions: $a \geq b \Longleftrightarrow b$ can be obtained from $a$ by a sequence of applications of rule (1) or (2) starting from the state $N$. In particular, rule (1) and rule (2) allow to reach all partitions of $n$ from $N$ and the $L_B(n)$ order is a lattice.

Later, Goles, Morvan, and Phan [6] considered a generalisation of $SPM(n)$: the Ice Pile Model (*IPM*). More precisely, for any positive integer $k$, they defined

$IPM_k(n)$, obtained by conserving rule (1) and modifying rule (2) as follows:

$$(s_1, \ldots, p+1, \underbrace{p, p, \ldots p}_{k'}, p-1, \ldots, s_\ell) \to (s_1, \ldots, \underbrace{p, p, \ldots, p}_{k'}, \ldots, s_\ell)$$
$$\text{for all } k' < k. \tag{3}$$

These authors proved that the orders induced by $IPM_k(n)$ on the set of all partitions of $n$ that accessible from the initial configuration $N$ are suborders of the lattice $L_B(n)$, and that they form an increasing sequence of lattices whose min is $SPM(n)$ and whose max is $L_B(n)$. Indeed, $IPM_1(n) \subseteq IPM_2(n) \subseteq \ldots \subseteq IPM_{n-1}(n)$, where $IPM_1(n)$ and $IPM_{n-1}(n)$ correspond to $SPM(n)$ and $L_B(n)$ respectively.

Goles, Morvan, and Phan also gave necessary and sufficient conditions for a partition to be accessible by $IPM_k(n)$ as well as an explicit formula for the unique fixed point of $IPM_k(n)$. Corteel and Gouyou-Beauchamps [4] also studied $IPM_k$ and computed asymptotic bounds for the number of accessible configurations in $IPM_k(n)$ by using the theory of partitions and of $q$-equations. Latapy, Mantaci, Morvan, and Phan [8] extended $SPM(n)$ to SPM($\infty$), a natural extension of $SPM(n)$ when one starts with an infinite number of grains. By using two different approaches they gave recursive formulae for $|SPM(n)|$.

In this paper we define an extension of $SPM(n)$ and more generally of $IPM_k(n)$ by adding a further dimension: the Bidimensional Ice Pile Model (*BIPM*). In order to do it, we place the grains on the vertices of a bidimensional cartesian integer grid and we extend the previous rules so that grains can fall or slide to the east and to the south, and in such a way that the configurations obtained are coded by plane partitions.

DEFINITION 1 *A* plane partition *of $n$ is a matrix $a$ of integers $a_{i,j}$ that are nonincreasing from left to right and from top to bottom, and such that their summation is equal to $n$:*

$$a_{i,j} \geq a_{i+1,j}, \quad a_{i,j} \geq a_{i,j+1}, \text{ for all } i,j \quad \text{and} \quad \sum_{i,j} a_{i,j} = n.$$

DEFINITION 2 *$BIPM_k(n)$ is the dynamical system such that:*

(a) *The system is initially in the configuration:*

$$N = \begin{pmatrix} n & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix}.$$

(b) *At each step, one grain can fall or slide to the east or to the south of its cell by applying locally one of the following rules to a submatrix of the matrix representing the configuration:*

– East transition:

$$\begin{array}{|c|c|}
 & a_{i-1,j+1} \\ \hline
a_{i,j} & a_{i,j+1} \\ \hline
a_{i+1,j} &
\end{array}
\longrightarrow
\begin{array}{|c|c|}
 & a_{i-1,j+1} \\ \hline
a_{i,j}-1 & a_{i,j+1}+1 \\ \hline
a_{i+1,j} &
\end{array}$$

$$\textit{if} \begin{cases} a_{i,j} - a_{i,j+1} \geq 2 \\ a_{i-1,j+1} - a_{i,j+1} \geq 1 \\ a_{i,j} - a_{i+1,j} \geq 1 \end{cases} \tag{4}$$

– South transition:

$$\begin{array}{|c|c|c|}
 & & \\ \hline
 & a_{i,j} & a_{i,j+1} \\ \hline
a_{i+1,j-1} & a_{i+1,j} & \\ \hline
 & &
\end{array}
\longrightarrow
\begin{array}{|c|c|c|}
 & & \\ \hline
 & a_{i,j}-1 & a_{i,j+1} \\ \hline
a_{i+1,j-1} & a_{i+1,j}+1 & \\ \hline
 & &
\end{array}$$

$$\textit{if} \begin{cases} a_{i,j} - a_{i+1,j} \geq 2 \\ a_{i,j} - a_{i,j+1} \geq 1 \\ a_{i+1,j-1} - a_{i+1,j} \geq 1 \end{cases} \tag{5}$$

– Slide$_k$

$$\left.\overbrace{\begin{array}{cccccc}
p+1 & p & p & \dots & p & p \\
p & p & p & \dots & p & p \\
\vdots & & & & & \vdots \\
p & p & p & \dots & p & p \\
p & p & p & \dots & p & p-1
\end{array}}^{k'}\right\} k''
\longrightarrow
\left.\overbrace{\begin{array}{cccccc}
p & p & p & \dots & p & p \\
p & p & p & \dots & p & p \\
\vdots & & & & & \vdots \\
p & p & p & \dots & p & p \\
p & p & p & \dots & p & p
\end{array}}^{k'}\right\} k''$$

$$\textit{with } k' + k'' < k, \tag{6}$$

*where $k'$ and $k''$ is respectively the number of columns and of rows of the submatrix.*

Observe that, because the rules preserve the property that the rows and columns in the matrix are non increasing, a newly obtained configuration is still a plane partition. Observe also that a given configuration may be obtained applying different sequences of rules to the initial configuration $N$.

Notice that for $k = 1$ the applicable rules of the corresponding model $BIPM_1(n)$ consist of East transitions and of South transitions only. We also observe that $BIPM_1(n)$ and $BIPM_{n-1}(n)$ represent a natural extension for $SPM(n)$ and for $L_B(n)$ respectively, when one adds a further dimension. For this reason, we decide to rename $BIPM_1(n)$ by $BSPM(n)$ and $BIPM_{n-1}(n)$ by $BL_B(n)$.

By drawing a parallel between these unidimensional and bidimensional models we will find some common features and some differences. We will show that, like for $SPM(n)$, not all plane partitions are accessible in $BSPM(n)$ starting from $N$ (see the example in Fig. 1). However, it appears to be much more difficult to characterize the partitions that are accessible in $BSPM$ than it is in $SPM$: we will give some necessary but not sufficient conditions for a partition to be accessible.

$$\begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Figure 1: A non accessible configuration of $BSPM$ with $n = 9$.

We will also show that, like for $SPM(n)$, the order induced by the rules of $BSPM(n)$ on the set of all accessible configurations is *graded*, that is, if $a$ and $b$ are two elements of $BSPM(n)$, and $\Gamma$ and $\Gamma'$ are two maximal chains having $a$ as maximum and $b$ as minimum, then $|\Gamma| = |\Gamma'|$. In other words, if $b$ is accessible from $a$ then all sequences of transitions that allow to obtain $b$ from $a$ all have the same length. The number of steps needed in order to reach a given configuration from $N$ is equal to a quantity that we call the *energy* of the configuration and for which we give an explicit formula in Section 2. Moreover, we will see that all plane partitions are accessible in $BL_B(n)$, as it is the case in the corresponding unidimensional model $L_B(n)$. We will also find that, unlike $IPM_k(n)$, the order induced by $BIPM_k(n)$ is not a lattice. In particular, this is the case for $k = 1$, as we will see by showing that the system $BSPM(n)$ may have more than one fixed point.

## 2 Definitions and general results on the ordered structure of $BIPM_k(n)$

Definition 3 *A partial order $P$ is a pair $(S, \leq_P)$, where $S$ is a set and $\leq_P$ is a binary relation on $S$ such that $\leq_P$ is reflexive, antisymmetric, and transitive.*

Definition 4 *Let $P = (S, \leq_P)$ and $P' = (S', \leq_{P'})$ be two partial orders. Then $P'$ is a suborder of $P$ if $S' \subseteq S$ and $\forall x, y \in S'$ $x \leq_{P'} y$ if and only if $x \leq_P y$.*

Definition 5 *The relation $\leq_{BIPM_k(n)}$ is the relation defined as follows: for any pair $(a, b)$ of accessible configurations of $BIPM_k(n)$, $a \leq_{BIPM_k(n)} b \iff b$ is obtained from $a$ by a sequence of applications of the rules of $BIPM_k(n)$.*

Proposition 1 *The relation $\leq_{BIPM_k(n)}$ defines an order on the set of all accessible configurations in $BIPM_k(n)$.*

Proof. It is immediate from the previous definition that $\leq_{BIPM_k(n)}$ is reflexive and transitive. Furthermore, since $BIPM_k(n)$ transitions are oriented (they move grains towards east and south but not towards the west and north), then $\leq_{BIPM_k(n)}$ is also antisymmetric. □

While $SPM(n)$ is a suborder of $L_B(n)$ we have that $BSPM(n)$ is not a suborder of $BL_B(n)$. Here is a counterexample: let

$$a = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 1 & 0 \end{pmatrix}$$

and

$$b = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 1 & 0 \end{pmatrix}$$

we have that $a$ and $b$ are accessible in $BL_B(n)$ and in $BSPM(n)$. Moreover we have that $b$ is obtained from $a$ by applying the slide rule of $BL_B(n)$. But $b$ can not obtained from $a$ by $BSPM$ rules. Then $BSPM$ is not a suborder of $BL_B$. Figure 2 shows the structure of the order obtained by applying the rules of $BSPM$ and starting with $n = 5$.

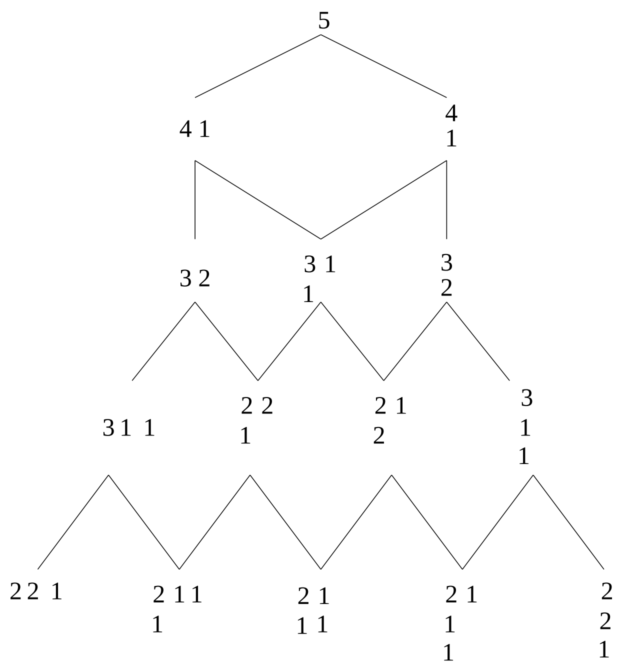


Figure 2: An example of evolution in $BSPM$ with $n = 5$.

Let $P = (S, \leq_P)$ be a finite partial order. For any $x, y \in S$, an element $z \in S$ is said to be an *upper bound* or a *lower bound* of $x$, $y$, respectively, when

$x, y \leq_P z$ or $x, y \geq_P z$. Let us denote by $\sup(x, y)$ and $\inf(x, y)$, respectively, the smallest upper bound and the greatest lower bound of $x$ and $y$ if they exist.

DEFINITION 6 *Let $P = (S, \leq_P)$ be a partial order. Then $P$ is a lattice if for any $x, y \in P$, $\sup(x, y)$ and $\inf(x, y)$ exist.*

In particular this implies that a lattice has a unique absolute minimum and hence, if the order associated with the configuration space of the system is a lattice, then the system has a unique fixed point (i.e. is converging).

*Remark.* $BSPM(n)$ is not a lattice, the example of Figure 2 shows it.

Moreover, we can also show that $BSPM(n)$ does not have a local lattice structure, in the sense that not any interval is a lattice. Here there is a counterexample: let

$$a = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}; \quad b = \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix}; \quad c = \begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix}; \quad d = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}.$$

We have that $c$ and $d$ are both obtained from both $a$ and $b$ using $BSPM$ rules. Moreover the partition $e = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$ is obtained from $c$ and from $d$. Then $a$ and $b$ do not have a infimum in the interval $[(6), e]$.

# 3 Energy and accessibility of configurations

DEFINITION 7 *Let $a$ be a plane partition, then its energy $E(a)$ is defined as follows:*

$$E(a) = \sum_{i,j} a_{i,j}(i + j - 1).$$

Notice that each time we apply an east or a south transition to a partition $a$, the energy $E(a)$ increases by one. Let us verify this statement in the case of an east transition: suppose we apply rule (4) to a cell $(i, j)$ of $a$, then in the summation expressing the energy, the terms $a_{i,j}(i + j - 1) + a_{i,j+1}(i + j)$ are replaced by $(a_{i,j} - 1)(i + j - 1) + (a_{i,j+1} + 1)(i + j)$, which shows that the energy increases by one. This implies that the order induced by rules (4) and (5) of the $BSPM(n)$ model is graded, that is, a configuration $a$ is always reached from the configuration $N$ by applying the same number of these rules. Such number is the difference $E(a) - E(N) = E(a) - n$. On the other hand, the energy may increase by more than one unit when one applies the slide rule.

DEFINITION 8 *A partition is said to be* accessible *with respect to a set of rules $\Sigma$ (or $\Sigma$-*accessible*) if it can be obtained from $N = (n)$ by applying a sequence of rules of $\Sigma$.*

PROPOSITION 2 *All plane partitions of $n$ are accessible in $BL_B(n)$.*

Proof. Let $a$ be a plane partition such that $a \neq N = (n)$. We want to show that there exists a partition $a'$ whose energy is strictly smaller than that of $a$ and such that $a' \leq_{BL_B(n)} a$. By iterating this process, we will eventually obtain the unique partition having minimal energy, that is, $N$. Let us say the value contained in the cell $(1,1)$ of $a$ is $p$. Then we have the following cases:

*i.* The cell $(1,2)$ contains the value $p$. Then consider the largest rectangle whose top left corner is $(1,1)$ and whose cells all contain the value $p$. If this rectangle only contains the cells $(1,1)$ and $(1,2)$ then $a'$ is obtained by a reverse application of the East transition moving one grain from $(1,2)$ to $(1,1)$. If this rectangle contains more than two cells then $a'$ is obtained by a reverse application of the slide rule.

*ii.* The cell $(1,2)$ contains the value $q \neq p$ and $q \neq 0$. Let $r$ be the value contained in the cell $(1,3)$. Here we distinguish two cases: $r \neq q$, then $a'$ is obtained by a reverse application of the east transition moving one grain from $(1,2)$ to $(1,1)$; $r = q$, then apply the same argument as in case *i.* to the rectangle having the top left corner in $(1,2)$.

*iii.* The cell $(1,2)$ contains the value $q = 0$. In this case we can apply symmetric arguments than those in case *i.* and case *ii.* by focusing on the cell $(2,1)$ instead of the cell $(1,2)$.

In each of these cases we move grains toward west or north, then the energy decreases. □

The same result is not true for $BSPM(n)$. We give now a necessary condition for a plane partition to be accessible in $BSPM$.

PROPOSITION 3 *Each partition containing the square matrix* $\begin{matrix} q & q \\ q & q \end{matrix}$ *(with* $q > 0$*) as submatrix is non accessible in* $BSPM(n)$.

Proof. Let us take a plane partition $a$ of $n$ containing at least one square $\begin{matrix} q & q \\ q & q \end{matrix}$. We want to show that, by applying backwards the rules of $BSPM(n)$, we can not revert to the starting configuration having all $n$ grains in the cell $(1,1)$. We consider the first time that one of such rules modifies the value of one of the cells of the square $\begin{matrix} q & q \\ q & q \end{matrix}$. There are two possibilities:

- a reverse East or South transition moves a grain from a cell adjacent to the square to a cell within the square.

- a reverse East or South transition moves a grain from a cell within the square.

In either cases, it is easy to check that at least two cells within the square would not respect the decreasing condition over the rows and columns. □

Therefore, a partition needs to "avoid the pattern" $\begin{matrix} q & q \\ q & q \end{matrix}$ in order to be accessible in $BSPM(n)$. We want to show that this is the unique forbidden pattern.

DEFINITION 9 *Let $M$ be a rectangular matrix whose entries depend on a set of parameters $q_1, \ldots, q_k$. We say that $M$ is a* forbidden pattern with respect to a set of rules $\Sigma$ *if, regardless of the choice of the values for the parameters $q_1, \ldots, q_k$, no $\Sigma$-accessible configuration contains $M$ as submatrix.*

THEOREM 1 *The pattern $\begin{matrix} q & q \\ q & q \end{matrix}$ is the unique forbidden pattern for BSPM-accessible configurations.*

Proof. First, it is straightforward to see that any pattern with only one row (respectively, one column) is not forbidden. Indeed, it is possible to create such a pattern on the second row (respectively, column) of a plane partition having a sufficiently large number of grains distributed on the first row (respectively, column).

Let us show now that no other pattern of size 2 by 2 is forbidden. If $\begin{matrix} q & r \\ s & t \end{matrix}$ is any pattern different from $\begin{matrix} q & q \\ q & q \end{matrix}$, then $q > t$. We show that the plane partition $\begin{matrix} q & r \\ s & t \end{matrix}$ can be obtained from another plane partition having smaller energy. There are two cases: either $q > r$ or $r > t$. In the first case, $\begin{matrix} q & r \\ s & t \end{matrix}$ can be obtained from $\begin{matrix} q & r+1 \\ s & t-1 \end{matrix}$, in the second case, it can be obtained from $\begin{matrix} q+1 & r-1 \\ s & t \end{matrix}$. By iterating this process we can construct a sequence of plane partitions whose energy strictly decreases, this implies that this process eventually allows to obtain the initial configuration $N = (n)$ and therefore $\begin{matrix} q & r \\ s & t \end{matrix}$ cannot be forbidden.

Suppose now that there exist other forbidden patterns having more than two rows or more than two columns and that do not contain the pattern $\begin{matrix} q & q \\ q & q \end{matrix}$ and let $S$ be the set of such forbidden patterns having minimal area.

If $S$ is not empty, then there must exist a pattern $M \in S$ such that no grain in $M$ can be moved by a reverse $BSPM$-transition. Otherwise, the same energy argument used in the case of patterns of size 2 by 2 would show that it would be possible to obtain the initial configuration by applying a sequence of reverse $BSPM$-transitions to any element of $S$.

Let $M$ be such a pattern and let $a$ be a plane partition containing $M$ as a submatrix:

$$M = (a_{i,j})_{r \le i \le s,\, u \le j \le v}.$$

As first step, we prove that all integers in the first row of $M$ are distinct.

Let us consider the two columns $v$ and $v-1$ of $a$ and let $a_{r,v} = q_1$.

If $a_{r+1,v}$ were smaller than $q_1$, then one could move a grain (by using an inverse transition) from $a_{r,v}$ to $a_{r-1,v}$. Hence $a_{r+1,v}$ must be equal to $q_1$.

Because $M$ can not contain $\begin{matrix} q_1 & q_1 \\ q_1 & q_1 \end{matrix}$, then $a_{r,v-1}$ must be greater than $q_1$, let us denote it by $q_2$.

Now if $a_{r+2,v} < q_1$, one could move a grain from $a_{r+1,v}$ to $a_{r,v}$. Hence $a_{r+2,v}$ must be equal to $q_1$.

Because $M$ does not contain $\begin{matrix} q_1 & q_1 \\ q_1 & q_1 \end{matrix}$, then $a_{r+1,v-1}$ must be greater than $q_1$. Furthermore, if it were smaller than $q_2$, then one could move a grain from $a_{r,v-1}$ to $a_{r-1,v-1}$. Hence, $a_{r+1,v-1}$ must be equal to $q_2$ and therefore $a_{r,v-2}$ must be greater than $q_2$, or $M$ would contain the square $\begin{matrix} q_2 & q_2 \\ q_2 & q_2 \end{matrix}$.

By iterating this process, we obtain the desired step, that is, that all the entries of the first row of $M$ are distinct. Therefore, if the pattern $M$ is included in a partition of a sufficiently large integer $n$ it is possible to bring into the first row of the submatrix $M$ an arbitrary number of grains coming from cells of the partition located east of $M$, using reverse $BSPM$-transitions.

All the patterns obtained this way must be forbidden. Otherwise, if it existed an accessible plane partition containing one of them, then it would be possible to obtain an accessible partition containing $M$. This shows that all patterns having the same shape as $M$ and satisfying:

- the first row contains totally arbitrary values,
- all other rows are equal to those of $M$

are forbidden. This implies that the pattern $M'$ obtained from $M$ by removing the first row would be forbidden as well, which contradicts the minimality of the area of $M$. □

The avoidance of the pattern $\begin{matrix} q & q \\ q & q \end{matrix}$, however, does not completely characterize accessible partitions in $BSPM(n)$. For instance, a partition such as the one showed in Figure 3 is non accessible, even if it does not contain the square $\begin{matrix} q & q \\ q & q \end{matrix}$.

In other terms, an accessible configuration cannot contain three adjacent cells on the first row having the same value and having three empty cells just to the south of them (the same remark can be transposed to the columns, of course). We would like to note that $\begin{matrix} q & q & q \\ 0 & 0 & 0 \end{matrix}$ is not a forbidden pattern,

$$\begin{pmatrix} a_{1,1} & \dots & q & q & q & \dots \\ a_{2,1} & \dots & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \text{with } q > 0.$$

Figure 3: A non accessible configuration in $BSPM$.

in the sense that accessible configurations do not need to avoid it when it is placed at a different location. For example the following partition is accessible in $BSPM(18)$:

$$\begin{pmatrix} 5 & 4 & 3 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

We have determined several other of these submatrices that cannot be found at specific locations of an accessible configuration, e.g.:

$$\begin{pmatrix} a_{1,1} & \dots & q & q & q & \dots \\ a_{2,1} & \dots & q-1 & q-1 & q-1 & \dots \\ a_{3,1} & \dots & 0 & 0 & 0 & \dots \\ \vdots & \dots & \vdots & \vdots & \vdots & \dots \end{pmatrix} \text{with } q > 1.$$

However, the avoidance of these submatrices at their respective forbidden locations does not completely characterize the accessible partition of $BSPM(n)$ either. For instance, the following plane partition:

$$\begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

is not accessible in $BSPM(9)$ even if it does not contain any of the mentioned submatrices.

# 4 Fixed points in $BSPM$

In this section we study the configurations of $BSPM(n)$ to which no transition rule can be applied.

Definition 10 *A partition $a$ is said to be* stable *with respect to a set of rules $\Sigma$ if none of the rules of $\Sigma$ can be applied to $a$.*

Definition 11 *A partition is said to be a* fixed point *with respect to a set of rules $\Sigma$ if it is accessible and stable with respect to the rules of $\Sigma$.*

Definition 12 *A stable partition $a$ is said to be* smooth *if $a_{i,j} - a_{i,j+1} \leq 1$ and $a_{i,j} - a_{i+1,j} \leq 1$ for all $i, j$.*

A smooth partition is obviously a fixed point but there are also non-smooth fixed points in $BSPM(n)$. In Figure 4 there is an example of non smooth fixed point in $BSPM(n)$.

$$\begin{pmatrix} 7 & 6 & 1 & 1 \\ 6 & 6 & 1 & 0 \\ 3 & 3 & 1 & 0 \end{pmatrix}$$

Figure 4: A non smooth fixed point of $BSPM$ with $n = 35$.

Definition 13 *For a positive integer $\ell$, we will call $\ell$-th diagonal of a matrix (or of the corresponding partition) the set of all cells having indices $(i, j)$ with $i + j - 1 = \ell$.*

*Remark.* The energy of a plane partition $a$ is defined in such a way that the integers on the $\ell$-th diagonal contribute to the sum $E(a)$ with a weight $\ell$.

*Notation.* For a positive integer $\ell$, let $S(\ell) = \sum_{i=1}^{\ell} i = \frac{\ell(\ell+1)}{2}$ and $T(\ell) = \sum_{i=1}^{\ell} i(\ell - i + 1) = \sum_{j=1}^{\ell} \sum_{i=1}^{j} i = \frac{\ell(\ell+1)(\ell+2)}{6}$. Then for each $n \in \mathbb{N}$ there exists a unique triple $(k, m, q)$ of non negative integers such that:

$$n = T(k) + S(m) + q \text{ with } 0 \leq S(m) < \sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2} \text{ and } 0 \leq q < m+1.$$

Example 1 *If $n = 103$, the unique decomposition is $103 = T(7) + S(5) + 4$ where $T(7) = 84$ and $S(5) = 15$.*

Definition 14 *Let $n$ be an integer and let $n = T(k) + S(m) + q$. A* pyramidal *partition of $n$ is a partition obtained by taking the following steps (see Figure 5) in this order:*

- ***Main staircase.*** *For $\ell = 1, 2, \ldots, k$, fill all the cells of the $\ell$-th diagonal of the partition with value $k - \ell + 1$. That is, the cell $(i, j)$ contains $k - i - j + 2$ grains for $i + j \leq k + 1$.*
- ***Additional triangle.*** *If $m \neq 0$, then choose any of the cells of the $(k - m + 2)$-th diagonal (these cells contain the value $m - 1$). Let us denote $(i_0, j_0)$ the coordinates of this cell. Take the sub-partition having this cell as top-left angle. Then add 1 to each cell of the first $m$ diagonals of this sub-partition, that is, to all cells $(i, j)$ with $i_0 \leq i \leq i_0 + m - 1$ and $j_0 \leq j \leq j_0 + m - 1$.*

- ***Additional row or column.*** *If $q \neq 0$, then take one of the two cells $(i_1, j_1)$ or $(i_2, j_2)$ on the $(k-q+2)$-th diagonal and having either $j_1 = j_0 - 1$ or $i_2 = i_0 - 1$ (one of these cells may not exist if $i_0 = 1$ or $j_0 = 1$). Then add 1 to the values in the first $q$ cells that are either south of $(i_1, j_1)$ or east of $(i_2, j_2)$, respectively, depending on whether the cell you choose is $(i_1, j_1)$ or $(i_2, j_2)$. (The explicit values of the indices of this cell can be computed as follows: since $i_1 + j_1 - 1 = k - q + 2$, with $j_1 = j_0 - 1$, and $i_2 + j_2 - 1 = k - q + 2$, with $i_2 = i_0 - 1$, then $(i_1, j_1) = (k - j_0 - q + 4, j_0 - 1)$ or $(i_2, j_2) = (i_0 - 1, k - i_0 - q + 4)$).*

$$\begin{array}{ccccccc} 7&6&5&4&3&2&1\\ 6&5&4&3&2&1&\\ 5&4&3&2&1&&\\ 4&3&2&1&&&\\ 3&2&1&&&&\\ 2&1&&&&&\\ 1&&&&&& \end{array} \quad \rightarrow \quad \begin{array}{ccccccc} 7&6&5&4&3&2&1\\ 6&5&\boxed{5}&4&3&2&1\\ 5&4&4&3&2&1&\\ 4&3&3&2&1&&\\ 3&2&2&1&&&\\ 2&1&1&&&&\\ 1&&&&&& \end{array} \quad \rightarrow$$

main staircase ... ... with the additional triangle ...

$$\rightarrow \quad \begin{array}{ccccccc} 7&6&5&4&3&2&1\\ 6&5&\boxed{5}&4&3&2&1\\ 5&4&4&3&2&1&\\ 4&\boxed{4}&3&2&1&&\\ 3&3&2&1&&&\\ 2&2&1&&&&\\ 1&1&&&&& \end{array}$$

with the additional column

Figure 5: A pyramidal partition of $n = 103$, where $n = T(k) + S(m) + q$ with $k = 7$, $m = 5$, and $q = 4$.

*Remark.* By construction, pyramidal partitions are smooth fixed points.

PROPOSITION 4 *The number of pyramidal partitions of $n = T(k) + S(m) + q$ is given by:*

$$\begin{cases} 2(k-m)+2 & \text{if } m, q > 0 \\ k-m+2 & \text{if } m > 0 \text{ and } q = 0 \\ 1 & \text{if } m = 0. \end{cases}$$

Proof. If $m > 0$ and $q > 0$, there are $(k - m + 2)$ choices for the position of the top-left angle of the additional triangle on the $(k - m + 2)$-th diagonal. For $(k - m)$ of such choices, it is possible to obtain two pyramidal partitions (one having an additional row, and one having an additional column), but for the two extremal cells of the diagonal, only one pyramidal partition can be obtained (in one case, one can only add the additional column and in the other case, one can only add the additional row).

If $m > 0$ and $q = 0$, then a pyramidal partition is determined only by the choice of the position of the additional triangle.

If $m = 0$ then the only pyramidal partition is the staircase. □

PROPOSITION 5 *Every pyramidal partition of $n$ is accessible in $BIPM_k(n)$ for all $k$.*

Proof. Observe that we just need to prove it for $BIPM_1(n)$, i.e. for $BSPM(n)$. Let $p$ be a pyramidal partition of $n = T(k) + S(m) + q$. Then we have the following cases:

- **The partition $p$ consists of the main staircase only.** That is, $m = 0$. We want to show that $p$ is accessible:

  *i.* Start with placing the $T(k)$ grains in the cell $(1, 1)$.

  *ii.* By applying $BSPM$'s East transitions, construct the partition:

  $$\begin{pmatrix} T(k) - \frac{k(k-1)}{2} & k-1 & k-2 & \dots & 2 & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}$$

  This is possible because

  $$\begin{pmatrix} T(k) - \frac{k(k-1)}{2} & k-1 & k-2 & \dots & 2 & 1 & 0 \end{pmatrix}$$

  is an accessible configuration of $SPM(n)$.

  *iii.* For $j = 2, \dots, k$, construct the $j$-th row as follows: move one grain from the cell $(1, 1)$ to the south until it reaches the cell $(j, 1)$. Then move it to the east as far as possible. Repeat the previous operation until the cells $(j, 1), (j, 2), \dots, (j, k-j+1)$ contain the values $k-j+1, k-j, \dots, 1$.

- **The partition $p$ consists of the main staircase and of the additional triangle.** That is, $m \neq 0$ and $q = 0$. Let $i$ be the row index of the cell of the $(k-m+2)$-th diagonal where the vertex of the additional triangle lays. In other terms, the cell containing the vertex of the additional triangle is $(i, k-m+3-i)$. Start by constructing the main staircase as described in the previous case. At the end of this step, the cell $(1, 1)$ will contain the value $k + S(m)$ and all the others will contain the values of the main staircase. To construct the $\ell$-th row of the additional triangle, for $\ell = 1, 2, \dots, m-1$, do as follows:

  - Let one grain fall from the cell $(1, 1)$ to the east until it reaches the cell $(1, k-m+3-i-1)$ (note that this cell is on the column immediately preceding the one containing the vertex of the additional triangle), then move it to the south until it reaches the cell $(i + \ell - 1, k - m + 3 - i - 1)$. Then move it to the east as far as possible. Repeat the previous operation until the cells $(i + \ell - 1, k - m + 3 - i), (i + \ell - 1, k - m + 4 - i), \ldots, (i + \ell - 1, k + 2 - i - \ell + 1)$ contain the values $m - \ell + 1, m - \ell, \ldots, 1$.

- **The partition $p$ consists of the main staircase, of the additional triangle, and of the additional row or column.** That is, $m \neq 0$ and $q \neq 0$. Start by constructing the main staircase and the additional triangle as in the previous cases. Let us suppose the additional row starts at the cell $(i - 1, m - q + i + 1)$. Let one grain fall from the cell $(1, 1)$ to the east until it reaches the cell $(1, m - q + i + 1)$, then move it to the south until it reaches the cell $(i - 1, m - q + i + 1)$. Then move it to the east as far as it can. Repeat the previous operation until the cells $(i - 1, m - q + i + 1), (i - 1, m - q + i + 2), \ldots, (i - 1, m + i)$ contain the values $q, q - 1, \ldots, 1$. The construction is analoguous if an additional column needs to be added instead of an additional row.

It is easy to verify that all the described moves are permitted under the set of rules of $BSPM$. □

## 5 Pyramidal partitions and energy

We want to show that pyramidal partitions of $n$ correspond to the smooth fixed points of $BSPM(n)$ having minimal energy.

Proposition 6 *The energy of pyramidal partitions of $n = T(k) + S(m) + q$ is given by:*

$$\frac{k(k+1)^2(k+2)}{12} + \frac{(3k - m + 4)(m + 1)m}{6} + \frac{(2k + 3 - q)q}{2}.$$

Proof. The contribution of the main staircase is:

$$\sum_{i=1}^{k} i(k - i + 1)^2 = \frac{k(k+1)^2(k+2)}{12}.$$

If $p > 0$, the contribution of the additional triangle is:

$$\sum_{i=1}^{m} i(k - m + i + 1) = \frac{(3k - m + 4)(m + 1)m}{6}.$$

Finally if $q > 0$, the contribution of the additional column or row is:

$$\sum_{i=1}^{q} (k - q + i + 1) = \frac{(2k + 3 - q)q}{2}.$$ □

*Notation.* Let us denote by $d_i(a)$ the sum of the entries in the $i$-th diagonal of a matrix $a$.

LEMMA 1 *Let $a$ be a smooth partition of $n = T(k) + S(m) + q$ and let $p$ be a pyramidal partition of $n$. We have the following for all $s$:*

$$\sum_{j=1}^{s} d_j(p) \geq \sum_{j=1}^{s} d_j(a).$$

Proof. We give a proof by recurrence on the integer $s$.

- For $s = 1$, since the cell $(1, 1)$ of $p$ contains $k$, we must show that $a_{1,1} \leq k$. Let us suppose we have a value at least equal to $k + 1$ in this cell. This means that the two cells on the second diagonal contain a value greater than or equal to $k$, otherwise $a$ would not be smooth. For the same reason the cells of the third diagonal of $a$ contain a value at least equal to $k - 1$ and so on. But then the sum of the values of the partition would be at least $\sum_{i=1}^{k+1} i(k-i+2) = T(k+1)$. But $T(k+1) > n$, since $k$ is the largest number such that $T(k) \leq n$, which is a contradiction.

- For $s \neq 1$, let us suppose that

  $$\sum_{j=1}^{i} d_j(p) \geq \sum_{j=1}^{i} d_j(a)$$

  for $i = 1, \ldots, s - 1$. Then we want to prove that

  $$\sum_{j=1}^{s} d_j(p) \geq \sum_{j=1}^{s} d_j(a). \qquad (7)$$

  Let us suppose (7) is not true, that is, we have:

  $$\sum_{j=1}^{s} d_j(p) < \sum_{j=1}^{s} d_j(a), \qquad (8)$$

  therefore, since by the recurrence hypothesis $\sum_{j=1}^{s-1} d_j(p) \geq \sum_{j=1}^{s-1} d_j(a)$, we have necessarily $d_s(p) < d_s(a)$. Consequently, there exist some entries on the $s$-th diagonal of $a$ that are larger than the corresponding entries of $p$.

  This is the argument we are going to use in the proof in order to obtain a contradiction.

  We distinguish two cases:

  (a) The $s$-th diagonal of the pyramidal partition $p$ is entirely in the main staircase (see Figure 6).

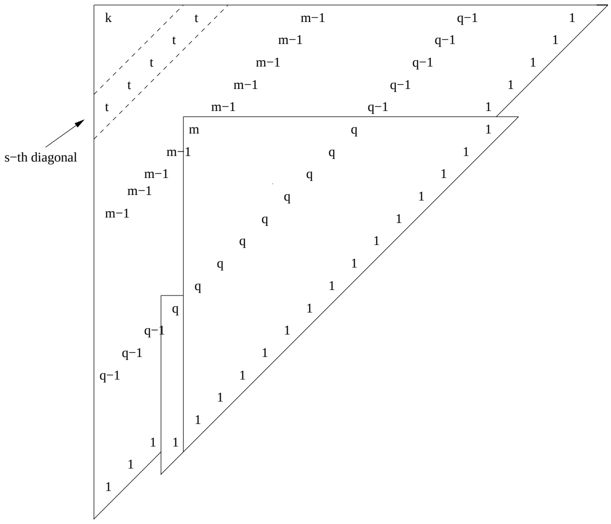


Figure 6: Case in which the $s$-th diagonal is entirely in the main staircase of $p$.

Recall that the entries of the $s$-th diagonal of $p$ are all equal to $k-s+1$. Let us call $t$ the integer $k-s+1$, in order to simplify notations in the remainder of the proof. The $s$-th diagonal of $a$ contains then values that are larger than $t$.

The proof of this lemma is particularly technical and computational. Therefore, for sake of clarity, we have decided to provide first a complete proof in the simple case where only one entry of the $s$-th diagonal of $a$ is larger than the corresponding entry of $p$ and it contains the value $t+1$. We will then provide the proof in the general case.

Let us then suppose we are in the simplest case: equation (8) holds with only one term $a_{i_0,j_0}$ of $d_s(a)$ such that $a_{i_0,j_0} > t$ and in particular $a_{i_0,j_0} = t+1$. Let us denote by $a'$ the smooth partition having the entries in the first $s$ diagonals equal to those of the partition $a$ and whose rows and columns "decrease as fast as possible" (that is, by one unit) in the remaining diagonals. More precisely, the values of the cells of $a'$ that are south of the $s$-th diagonal are defined as follows (see Figure 7,(b)):

* The cells $(i,j)$ with $i \geq i_0$ and $j \geq j_0$ form a perfect staircase of height $t+1$ whose highest point is placed in $(i_0, j_0)$;
* If $i < i_0$ and $j > s+1-i$, then $a'_{i,j} = a'_{i,j-1} - 1$;
* if $j < j_0$ and $i > s+1-j$, then $a'_{i,j} = a'_{i-1,j} - 1$;

Let us denote by $|a|$ and $|a'|$ respectively the sum of all the elements of $a$ and $a'$. Then we have $|a'| \leq |a| = n$, because $a$ is smooth and therefore the rows and the columns of $a'$ decrease at least as fast as

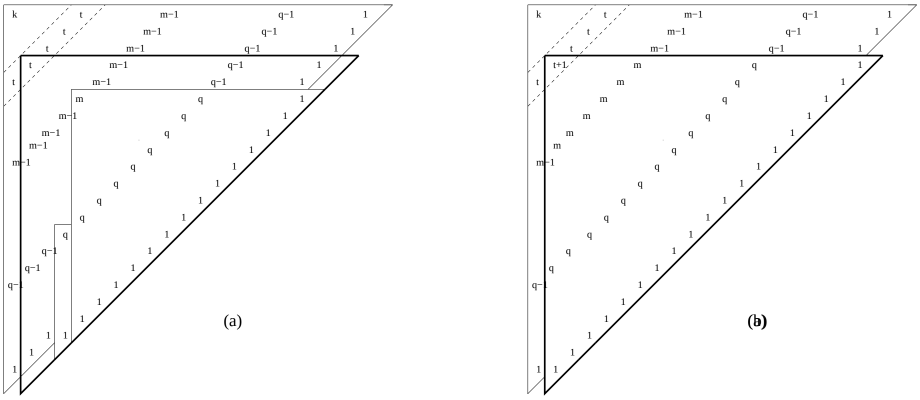

Figure 7: The pyramidal partition $p$ on the left and the partition $a'$ on the right.

those of $a$.

Figure 7,(b) illustrates that we can write $|a'|$ as:

$$\begin{aligned}|a'| &= \sum_{j=1}^{s} d_j(a') + T(t+1) - (t+1) + (s-1)S(t-1)\\ &= \sum_{j=1}^{s} d_j(a) + T(t) + S(t) + (s-1)S(t-1),\end{aligned}$$

where the term $T(t+1)$ comes from the triangular region with a thicker border in the figure and the term $(s-1)S(t)$ comes from the remaining rows and columns. The term $-(t+1)$ is a correcting term that is necessary because the contribution of the cell $(i_0, j_0)$ is included both in $\sum_{j=1}^{s} d_j(a')$ and in $T(t+1)$. Similarly, we can compute $|p|$ as (see Figure 7, (a)):

$$|p| = \sum_{j=1}^{s} d_j(p) + T(t) - t + S(m) + q + (s-1)S(t-1),$$

where $t > m$, because we are supposing that the $s$-th diagonal is entirely in the main staircase. Since $S(t) > S(m)+q$ and $\sum_{j=1}^{s} d_j(a) > \sum_{j=1}^{s} d_j(p)$ then $|a| \geq |a'| > |p|$ and we get a contradiction, since $|a| = |p| = n$.

Let us now give the proof in the case where more than one cell of the $s$-th diagonal contains a value greater than $t$.

Suppose there are $u$ cells containing the values $t - j_i$, with $j_i \geq 0$ for $i = 1, \ldots, u$ and $v$ cells containing the values $t + l_i$, with $l_i \geq 1$ for $i = 1, \ldots, v$. Since we are supposing that $d_s(a) > d_s(p)$, at least one cell contains a value larger than $t$ and, for the same reason, we have $\sum_{i=1}^{v} l_i - \sum_{i=1}^{u} j_i > 0$.

We construct again a new smooth partition $a'$, having the entries in the first $s$ diagonals equal to those of the partition $a$ and whose cells placed south of the $s$-th diagonal contain values that we are going to define next.

Let $(i_0, j_0)$ be the southmost cell on the $s$-th diagonal such that $a_{i_0,j_0} > p_{i_0,j_0} = t$, and suppose this cell contains the value $t + l_1$, then the values of the cells of $a'$ that are south of the $s$-th diagonal are defined as follows:

* The cells $(i, j)$ with $i \geq i_0$ and $j \geq j_0$ form a perfect staircase of height $t + l_1$ whose highest point is placed in $(i_0, j_0)$;
* If $i < i_0$ and $j > s + 1 - i$, then $a'_{i,j} = a'_{i,j-1} - 1$;
* if $j < j_0$ and $i > s + 1 - j$, then $a'_{i,j} = a'_{i-1,j} - 1$.

Like in the simple case, we have $|a'| \leq |a| = n$, because $a$ is smooth and therefore the rows and the columns of $a'$ decrease at least as fast as those of $a$.

Recall that

$$\begin{aligned}
|p| &= \sum_{j=1}^{s} d_j(p) + T(t) - t + S(m) + q + (s-1)S(t-1) \\
&= \sum_{j=1}^{s} d_j(p) + T(t-1) + S(t-1) + S(m) + q + sS(t-1) \\
&= \sum_{j=1}^{s} d_j(p) + T(t-1) + S(m) + q + sS(t-1).
\end{aligned}$$

Note that $S(t+l_i) > S(t) + l_i t$ and $S(t-j_i) \geq S(t) - j_i t$ for all $t > 0$, for all $l_i > 0$ and for all $j_i \geq 0$. We will use this fact when we now compute $|a'|$ and we compare it to $|p|$.

$$\begin{aligned}
|a'| &= \sum_{j=1}^{s} d_j(a) + T(t+l_1) - (t+l_1) + \sum_{i=2}^{v} S(t+l_i-1) \\
&\quad + \sum_{i=1}^{u} S(t-j_i-1) \\
&= \sum_{j=1}^{s} d_j(a) + T(t+l_1-1) + S(t+l_1-1) + \sum_{i=2}^{v} S(t+l_i-1) \\
&\quad + \sum_{i=1}^{u} S(t-j_i-1)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{s} d_j(a) + T(t+l_1-1) + \sum_{i=1}^{v} S(t+l_i-1) + \sum_{i=1}^{u} S(t-j_i-1) \\
&\geq \sum_{j=1}^{s} d_j(a) + T(t+l_1-1) + \sum_{i=1}^{v} [S(t-1) + l_i(t-1)] \\
&\quad + \sum_{i=1}^{u} [S(t-1) - j_i(t-1)] \\
&= \sum_{j=1}^{s} d_j(a) + T(t+l_1-1) + sS(t-1) + (t-1)\left(\sum_{i=1}^{v} l_i - \sum_{i=1}^{u} j_i\right).
\end{aligned}
$$

Now we use the fact that $\sum_{i=1}^{v} l_i - \sum_{i=1}^{u} j_i > 0$, as well as the fact that when $l_1 > 0$, one has $T(t+l_1-1) \geq T(t-1) + S(t-1)$ and the fact that, by recurrence hypothesis, $\sum_{j=1}^{s} d_j(a) > \sum_{j=1}^{s} d_j(p)$.

$$
\begin{aligned}
|a'| &> \sum_{j=1}^{s} d_j(p) + T(t-1) + sS(t-1) + S(t-1) + (t-1) \\
&\geq \sum_{j=1}^{s} d_j(p) + T(t-1) + sS(t-1) + S(m) + q = |p|,
\end{aligned}
$$

the last inequality being justified by the fact that $S(t-1) \geq S(m)$ and $t-1 \geq q$.

We have then that $|a| \geq |a'| > |p|$, which is a contradiction.

(b) The $s$-th diagonal of the pyramidal partition $p$ is not entirely in the main staircase (see Figure 8). In this case we have to distinguish between cells of the main staircase that are equal to $t$ and those that are equal to $t+1$ but otherwise we can apply the same arguments as before. □

PROPOSITION 7 *Let $a$ be a plane partition and let us denote by $\ell$ its last non zero diagonal. Then its energy $E(a) = \sum_{k,t} a_{k,t}(k+t-1)$ can be rewritten as*

$$
\ell \sum_{j=1}^{\ell} d_j(a) - \sum_{i=1}^{\ell-1} \sum_{j=1}^{i} d_j(a) = \ell n - \sum_{i=1}^{\ell-1} \sum_{j=1}^{i} d_j(a).
$$

Proof. We have that

$$
\begin{aligned}
\sum_{k,t} a_{k,t}(k+t-1) &= \sum_{j=1}^{\ell} j d_j(a) \\
&= \sum_{i=1}^{\ell} \sum_{j=i}^{\ell} d_j(a)
\end{aligned}
$$

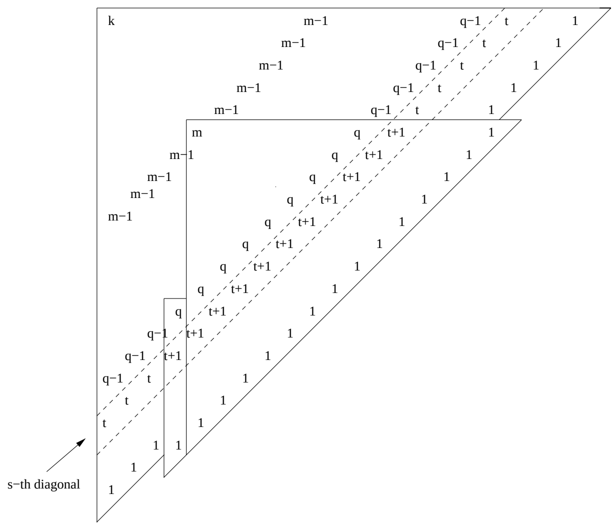


Figure 8: Case in which the $s$-th diagonal is not entirely in the main staircase of $p$.

$$= \sum_{i=1}^{\ell} \left( \sum_{j=1}^{\ell} d_j(a) - \sum_{j=1}^{i-1} d_j(a) \right)$$

$$= \ell \left( \sum_{j=1}^{\ell} d_j(a) \right) - \sum_{i=1}^{\ell} \sum_{j=1}^{i-1} d_j(a)$$

$$= \ell n - \sum_{i=1}^{\ell} \sum_{j=1}^{i-1} d_j(a).$$

It is straightforward to verify that the ranges of the indices $i$ and $j$ in these sums can be modified to make them equal to those of the claim of the proposition without modifying the values of the sums themselves. □

PROPOSITION 8 *Let $a$ be a smooth partition of $n$ and let $p$ be a pyramidal partition of $n$. If $\sum_{j=1}^{i} d_j(p) = \sum_{j=1}^{i} d_j(a)$ for each $i$, then $a$ is a pyramidal partition.*

Proof. For simplicity we prove the result for the simplest case where $p$ consists of the main staircase only. The other cases can be proved by using the same arguments.

Since $d_1(a) = d_1(p)$ then $a_{1,1} = p_{1,1}$. Moreover $d_2(a) = d_2(p)$, that is

$$a_{1,2} + a_{2,1} = p_{1,2} + p_{2,1} = 2(p_{1,1} - 1).$$

Let us suppose now that $a_{1,2} > p_{1,2}$, then from the previous equation we must have that $a_{2,1} < p_{2,1} = p_{1,1} - 1$, but this is impossible since $a_{1,1} = p_{1,1}$ and $a$ is smooth. Therefore

$$a_{1,2} = a_{2,1} = p_{1,1} - 1.$$

By iterating the same argument on the remaining diagonals we obtain that $a = p$. □

PROPOSITION 9 *In the set of all accessible smooth partitions, pyramidal partitions have minimal energy.*

Proof. Let $p$ be a pyramidal partition of $n$, by construction $p$ is smooth. We want to show that it has minimal energy among all smooth partitions, that is, $E(p) \leq E(a)$ for all smooth partition $a$. From Proposition 7, we have that $E(p) = \ell n - \sum_{i=1}^{\ell-1} \sum_{j=1}^{i} d_j(p)$. Then from Lemma 1 we have the result. □

PROPOSITION 10 *A smooth partition having minimal energy among all smooth partitions is pyramidal.*

Proof. Let $a$ be a smooth partition of $n$ with minimal energy among all smooth partitions. In Proposition 9 we proved that pyramidal partitions all have minimal energy. Therefore for any pyramidal partition $p$ one has $E(p) = E(a)$. Using Proposition 7 this means that:

$$n\ell - \sum_{i=1}^{\ell-1} \sum_{j=1}^{i} d_j(p) = n\ell' - \sum_{i=1}^{\ell'-1} \sum_{j=1}^{i} d_j(a), \tag{9}$$

where $\ell$ and $\ell'$ denote the last non zero diagonal of $p$ and $a$ respectively. Note that $\ell' \geq \ell$, otherwise, if $\ell' < \ell$ then $\sum_{j=1}^{\ell'} d_j(a) = n$, while $\sum_{j=1}^{\ell'} d_j(p) < n$ which contradicts Lemma 1. Identity (9) can be rewritten as

$$n(\ell' - \ell) + \sum_{i=1}^{\ell-1} \left( \sum_{j=1}^{i} d_j(p) - \sum_{j=1}^{i} d_j(a) \right) - \sum_{i=\ell}^{\ell'-1} \sum_{j=1}^{i} d_j(a) = 0. \tag{10}$$

Observe that $n(\ell' - \ell) - \sum_{i=\ell}^{\ell'-1} \sum_{j=1}^{i} d_j(a) >= 0$ because for all $i$, the term $\sum_{j=1}^{i} d_j(a)$ is smaller than or equal to $n$ and the sum $\sum_{i=\ell}^{\ell'-1}(\sum_{j=1}^{i} d_j(a))$ has $(\ell'-\ell)$ terms. From Lemma 1 we have that the remaining term $\sum_{i=1}^{\ell-1}(\sum_{j=1}^{i} d_j(p)$ $- \sum_{j=1}^{i} d_j(a))$ of the left-hand side of equation (10) is also non negative. Therefore both terms $n(\ell'-\ell) - \sum_{i=\ell}^{\ell'-1} \sum_{j=1}^{i} d_j(p)$ and $\sum_{i=1}^{\ell-1}(\sum_{j=1}^{i} d_j(p) - \sum_{j=1}^{i} d_j(a))$ must be equal to 0.

However, $\sum_{i=1}^{\ell-1}(\sum_{j=1}^{i} d_j(p) - \sum_{j=1}^{i} d_j(a)) = 0$ implies

$$\sum_{j=1}^{i} d_j(p) - \sum_{j=1}^{i} d_j(a) = 0 \text{ for all } i$$

because, by Lemma 1, all terms $\sum_{j=1}^{i} d_j(p) - \sum_{j=1}^{i} d_j(a)$ are greater than or equal to 0. We deduce that $\sum_{j=1}^{i} d_j(p) = \sum_{j=1}^{i} d_j(a)$ for all $i$ and consequently that $\ell = \ell'$. By using Proposition 8, this implies that $a$ is pyramidal. □

From these two propositions we can deduce the following theorem.

THEOREM 2 *The set of smooth partitions with minimal energy is the set of pyramidal partitions.*

## 6 Longest and shortest maximal chains in $BL_B(n)$

Our goal in this section is to determine the minimal and the maximal number of transitions of $BL_B(n)$ that allow to reach a fixed point of $BL_B(n)$ starting from the initial configuration.

DEFINITION 15 *Let $P = (S, \leq_P)$ a partially ordered set and let $x_1, x_2, \ldots, x_m$ be elements of $S$. We say that $x_1, x_2, \ldots, x_m$ form a* chain *if $x_1 \leq_P x_2 \leq_P \ldots \leq_P x_m$. The integer $m-1$ is called the* length *of the chain.*

DEFINITION 16 *We say a chain $\Gamma$ is* maximal *if $\Gamma$ cannot be included as sub-chain in any other chain.*

*Remark.* Note that if a chain $\Gamma = x_1, x_2, \ldots, x_m$ is a maximal chain of $BIPM_k(n)$, then $x_1$ is the initial configuration $N = (n)$ and $x_m$ is a fixed point of $BIPM_k(n)$.

The length of the shortest and of the longest maximal chain in $BIPM_{n-1}(n) = BL_B(n)$ are in fact the minimal and the maximal number of applications of $BL_B(n)$ rules that allow to reach a fixed point of $BL_B(n)$ starting from the initial configuration.

*Remark.* We recall that all plane partitions are accessible in $BL_B(n)$. Furthermore, it is easy to see that a plane partition $a = (a_{i,j})$ having $a_{1,1} > 1$ cannot be a fixed point. Consequently, the fixed points of $BL_B(n)$ are precisely the plane partitions whose parts are all equal to 1.

DEFINITION 17 *We define two special fixed points of $BL_B(n)$ as follows. Let $P_R$ be the plane partition defined by $P_R(1, j) = 1$ for $1 \leq j \leq n$ and let $P_C$ be the plane partition defined by $P_C(i, 1) = 1$ for $1 \leq i \leq n$. The fixed point $P_R$ is obviously represented by a one-row matrix, while $P_C$ is represented by a one-column matrix.*

Note that, if $n \leq 3$, the longest and the shortest maximal chains in $BL_B(n)$ have the same length and this length is $n-1$. The next proposition considers the cases where $n \geq 4$.

PROPOSITION 11 *For $n \geq 4$, the shortest maximal chains in $BL_B(n)$ have length $2n-5$ and more precisely:*

- *If $P$ is any fixed point of $BL_B(n)$ with $P \neq P_R$ and $P \neq P_C$, then it is possible to construct a maximal chain of length $2n-5$ ending with $P$.*
- *If $P = P_R$ or $P = P_C$, then the shortest maximal chain ending with $P$ has length $2n-4$.*

Proof. We first prove that at least $2n-5$ transitions are necessary to obtain a fixed point $P$ from the initial configuration. We will count how many transitions are needed to move a grain form the cell $(1,1)$, where it is initially placed, to an empty cell $(i,j) \neq (1,1)$.

Let us observe that one (*East* or *South*) transition is needed to move one grain from the cell $(1,1)$ to the cell $(1,2)$ or to the cell $(2,1)$.

Let $(i,j)$ be a cell not in $\{(1,1),(1,2),(2,1)\}$. While the cell $(1,1)$ contains more than 2 grains, one needs at least two transitions to move one grain from this cell to the cell $(i,j)$. It is clear indeed that one transition would not be sufficient to move a grain from the cell $(1,1)$ to the cell $(i,j)$.

At the end of this process, when all cells except one have been filled and there are only 2 grains left in the cell $(1,1)$, then it is only possible to apply one *Slide* transition, thus obtaining a fixed point.

So, in total, one needs at least one transition to bring one grain to each of the cells $(1,2),(2,1)$ and the last cell to be filled, and two transitions for each of the $n-4$ remaining cells, i.e., $3+2(n-4) = 2n-5$ transitions.

We shall construct now a sequence of $2n-5$ transitions allowing to obtain $P$ from $N = (n)$. Let us consider first the case where $P$ is a fixed point different from $P_R$ and $P_C$. Let $r$ be the number of rows of $P$ and for $1 \leq i \leq r$, denote by $j_i$ the largest integer such that $P(i,j_i) = 1$.

The first transition (an East transition) transfers one grain from the cell $(1,1)$ to the cell $(1,2)$ and the second one (a South transition) transfers one grain from the cell $(1,1)$ to the cell $(2,1)$. The last transition of the sequence is a Slide transition, moving one grain from the cell $(1,1)$ to the cell $(r,j_r)$. The remaining $2n-8$ transitions are described as follows.

For any of the cells $(1,j)$, with $2 < j \leq j_i$, we move a grain from cell $(1,1)$ to cell $(1,j)$ by applying two transitions: an East transition moving the grain from $(1,1)$ to $(1,2)$, then a Slide transition moving the grain from $(1,2)$ to $(1,j)$.

For any other cell $(i,j)$, we first move a grain from cell $(1,1)$ to the cell $(2,1)$ using a South transition, then we move it from $(2,1)$ to $(i,j)$ using a Slide transition.

The total number of transitions in the sequence is clearly $2n-5$.

It is easy to see that a shortest maximal chain ending with $P_R$ or$P_C$ has length $2n-4$. □

The following proposition deals with the length of longest maximal chains in $BL_B(n)$ and proves that their length is the same as the length of longest maximal chains in $L_B(n)$. Longest maximal chains in $L_B(n)$ were studied by

Greene and Kleitman in their paper [7]. We refer the reader to this article for more details on longest maximal chains in $L_B(n)$.

PROPOSITION 12 *Longest maximal chains in $BL_B(n)$ have the same length as longest maximal chains in $L_B(n)$.*

Proof. We will show first that the length of any longest maximal chain in $BL_B(n)$ is smaller than or equal to the length of a longest maximal chain in $L_B(n)$. Then we will show that there exists a fixed point $P$ such that the longest maximal chain ending with $P$ has the same length as a longest maximal chain of $L_B(n)$.

Let us first consider the following map $\varphi$ from $BL_B(n)$ to $L_B(n)$: for each plane partition $a$, define $\varphi(a)$ as the partition obtained from $a$ by sorting the parts of $a$ in decreasing order. The map $\varphi$ is clearly surjective. Now, let $a \to b$ be a transition in $BL_B(n)$. It is clear that $\varphi(b)$ is smaller than $\varphi(a)$ by the dominance order in $L_B(n)$, so $\varphi(b)$ can be obtained from $\varphi(a)$ by a transition in $L_B$ (see [3]). This implies that every chain in $BL_B(n)$ can be mapped onto a chain of $L_B(n)$ having at least the same length. Therefore the length of a longest maximal chain in $BL_B(n)$ is smaller than or equal to the length of a longest maximal chain in $L_B(n)$.

On the other hand, if we consider only plane partitions having one row, the evolution of the system in $BL_B(n)$ is analoguous to the one in $L_B(n)$. So the length of a longest maximal chain ending with $P_L$ is equal to the length of any longest maximal chain in $L_B(n)$. □

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