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An object grammar for the class of L-convex polyominoes

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Abstract. We consider the class of L-convex polyominoes, i.e. convex polyominoes where any two chosen cells can be connected simply using an internal path having at most one change of direction. The main result of the paper is the definition of an object grammar decomposition of this class, which gives combinatorial evidence of the rationality of the generating function, and suggests some new combinatorial properties.

Mathematics Subject Classifications (2000). Primary 05A15; Secondary 05A17

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1 Basics on L-convex polyominoes

In the plane $\mathbb{Z}\times\mathbb{Z}$ a *cell* is a unit square, and a *polyomino* is a finite connected union of cells having no cut point (see Figure 1 (a)). Polyominoes are defined up to translations. A *column* (*row*) of a polyomino is the intersection between the polyomino and an infinite strip of cells lying on a vertical (horizontal) line. For the main definitions and results concerning polyominoes we refer to [1, 7].

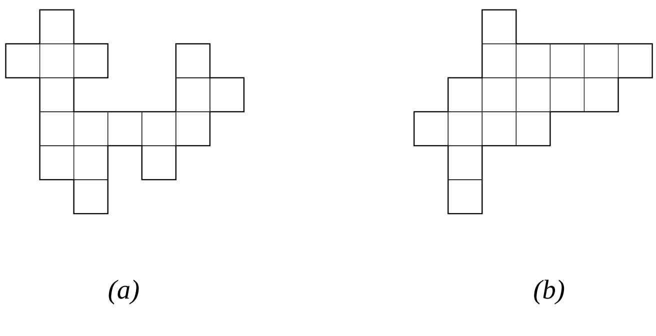
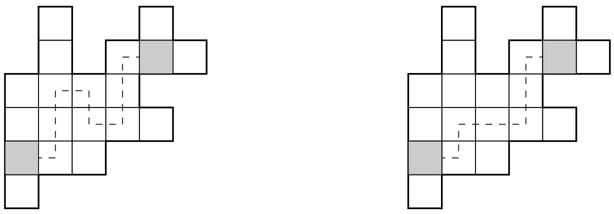


Figure 1: (a) a polyomino; (b) a convex polyomino.

A polyomino is said to be *column convex* (*row convex*) when its intersection with any vertical (horizontal) line of cells in the square lattice is connected, and *convex* when it is both column and row convex (see Figure 1 (b)). The *area* of a polyomino is just the number of cells it contains, while its *semi-perimeter* is half the length of the boundary. Thus, in a convex polyomino the semi-perimeter is the sum of the numbers of its rows and columns.

In [5] the authors observed that convex polyominoes have the property that every pair of cells is connected by a *monotone path*. More precisely, an *internal path* in a polyomino is a self-avoiding sequence of unitary steps of four types: north $N = (0, 1)$, south $S = (0, -1)$, east $E = (1, 0)$, and west $W = (-1, 0)$. A path is *monotone* if it is made with steps of only two types. Given a path $w = u_1 \dots u_k$, with $u_i \in \{N, S, E, W\}$, each pair of steps $u_i u_{i+1}$ such that $u_i \neq u_{i+1}$, $0 < i < k$, is called a *change of direction*. These definitions are illustrated by Fig. 2, in which the non monotone path (a) has 6 changes of direction and the monotone path (b) has 4 changes of direction.



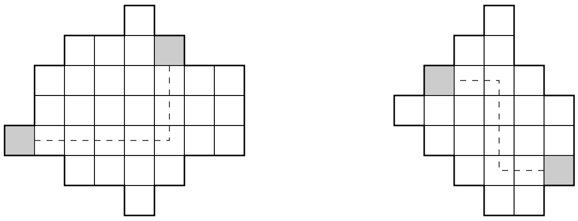
(a) *(b)*

Figure 2: (a) a path between two highlighted cells in a polyomino; (b) a monotone path between the same cells, made only of north and east steps.

The authors of [5] further proposed a classification of convex polyominoes based on the number of changes of direction in the paths connecting any two cells of a polyomino. More precisely, a convex polyomino is k-convex if every pair of its cells can be connected by a monotone path with at most k changes of direction. In a convex polyomino of the first level of this classification, any two cells can be connected by a path with at most one change of direction: in view of the L-shape of these paths, 1-convex polyominoes are also called L*-convex*. The reader can easily check that in Fig. 3, the polyomino (a) is L-convex, while the polyomino (b) is not, but it is 2-convex.

A useful characterization of L-convex polyominoes in terms of maximal rectangles was also provided in [5]. By abuse of notation, for any two polyominoes P and P' we will write $P \subseteq P'$ to mean that P is geometrically included in P'. A *rectangle*, that we denote by $[x, y]$, with $x, y \in \mathbb{N} \setminus \{0\}$, is a rectangular polyomino with x columns and y rows. We say $[x, y]$ to be *maximal* in P if

$$\forall [x', y'], \qquad [x, y] \subseteq [x', y'] \subseteq P \;\Rightarrow\; [x, y] = [x', y'].$$

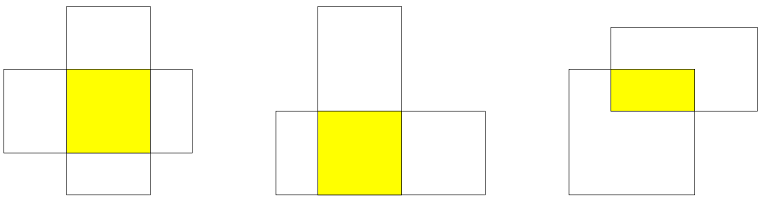


(a) (b)

Figure 3: (a) an L-convex polyomino, and a monotone path with a single change of direction joining two of its cells; (b) a 2-convex but not L-convex polyomino: the two highlighted cells cannot be connected by a path with only one change of direction.

Two rectangles $[x, y]$ and $[x'y']$ have *crossing intersection* if their intersection is a rectangle whose basis is the smallest of the two bases and whose height is the smallest of the two heights (see Fig. 4), i.e.

$$[x, y] \cap [x', y'] = [\min(x, x'), \min(y, y')] \,.$$



(a) (b) (c)

Figure 4: The two rectangles in (a) and in (b) have crossing intersection, while the two rectangles in (c) do not.

The authors of [5] also prove the following remarkable result.

Theorem 1 *A convex polyomino is L-convex if and only if any two of its maximal rectangles have crossing intersection.*

Thus an L-convex polyomino can be seen as the overlapping of maximal rectangles. On the other hand, each finite overlapping of non-comparable rectangles such that any two of them have crossing intersection determines an L-convex polyomino (see Fig. 5 for an example).

This class of polyominoes has been considered from several points of view: in [6] it is shown that the set of L-convex polyominoes is well-ordered with

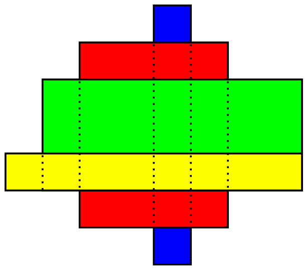


Figure 5: The L-convex polyomino in Fig. 3 (a) defined by the overlapping of four non comparable rectangles having crossing intersection.

respect to the sub-picture order, in [2] the authors have investigated some tomographical aspects of this family. Finally, in [3] it is proved that the number f_n of L-convex polyominoes with semi-perimeter $n+2$ satisfies the recurrence relation:

$$f_n = 4f_{n-1} - 2f_{n-2}, \qquad n \geq 3,$$

with $f_0 = 1$, $f_1 = 2$, $f_2 = 7$. In other terms the generating function of L-convex polyominoes is rational:

$$F(x) = \sum_{n\geq 0} f_n x^{n+2} = x^2 + 2x^3 + 7x^4 + 24x^5 + 82x^6 + 280x^7 + O(x^8)$$
$$= \frac{x^2(1-2x+x^2)}{1-4x+2x^2}.$$

Successively, in [4], the authors have studied the problem of enumerating L-convex polyominoes with respect to the area.

In this paper we give a direct proof of the rationality of the generating function $F(x)$, by defining an object grammar for the class of L-convex polyominoes. The basic idea of an *object grammar* is to define a class of objects by means of terminal objects and some types of operations applied to the objects. Object grammars were first introduced in [9], and have many immediate applications: they let us obtain generating functions for the generated class [8], according to several linear or q-linear parameters; they allow us to determine bijections between classes of different combinatorial objects, [10]; they lead to a simple algorithm for the uniform random generation of the examined class [11].

Here the object grammar approach gives us many helpful properties of the class of L-convex polyominoes, some of which have been proved by means of bijective techniques.

2 An object grammar for L-convex polyominoes

Let us first recall some basic definitions concerning object grammars [9]. An *object grammar* G is a quadruple $\langle \mathbb{O}, \mathbb{E}, \Phi, S \rangle$ where :

- $\mathbb{O} = \{O_i\}_{i \in I}$ is a finite family of sets of objects (where I is a finite subset of $\mathbb{N}$).
- $\mathbb{E} = \{E_{O_i}\}_{i \in I}$ is a finite family of finite subsets of classes belonging to $\mathbb{O}$, called *terminal objects*.
- Φ is a set of *object operations* in $\mathbb{O}$, where an object operation is a mapping $\phi : O_1 \times \ldots \times O_k \to O$, with $O, O_1, \ldots, O_k \in \mathbb{O}$.
- S is a fixed element of $\mathbb{O}$, called the *axiom* of the grammar.

An object o is said to be generated in G, if $o \in S$ and o can be obtained through the application of some object operations starting from the terminal objects of O. The set of objects generated in G is denoted by $O_G(S)$. Object grammars are often graphically described, and in the rest of the paper we adopt this kind of representation. For further definitions and examples on object grammars, we suggest the reader to see [9].

We now introduce the main result of our work, i.e. the definition of an object grammar for L-convex polyominoes. The basic idea is to give a unique decomposition of L-convex polyominoes in four disjoint classes A, B, C, D, according to the position and the dimensions of the maximal rectangle having maximal height (see Fig. 6). The object operations applied on a polyomino of a given class will produce L-convex polyominoes of greater size, by maintaining the fundamental property that each two non comparable maximal rectangles have crossing intersection. So let us consider the object grammar $G_L = \langle \mathbb{O}, \mathbb{E}, \Phi, \mathcal{L} \rangle$,

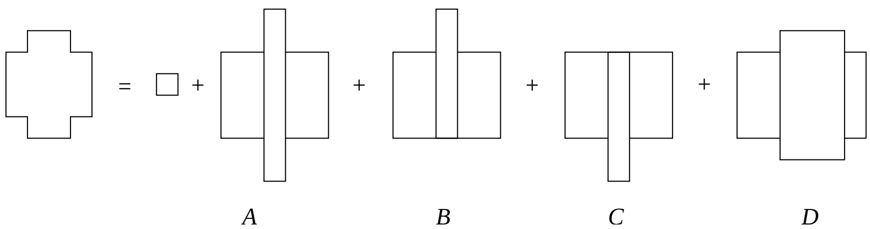


Figure 6: The object grammar for L-convex polyominoes. The four classes of the decomposition are defined by the position and the dimensions of the maximal rectangle having maximal height with respect to the other maximal rectangles.

where $\mathbb{O} = \{\mathcal{L}, A, B, C, D\}$. The axiom $\mathcal{L}$ is the class of L-convex polyominoes, and (see Fig. 6):

1. A is the class of those having one cell in the first row and one cell in the last row (excluding the single cell polyomino).
2. B is the class of those having one cell in the first row but not belonging to A.
3. C is the class of those having one cell in the last row but not belonging to A.

4. D is the class of L-convex polyominoes that are not in A, B, or C.

Moreover, the elements of the set $\mathbb{E}$ are:

- E_L, constituted by the one-cell polyomino,

- E_A, constituted by the vertical domino,

- E_D, constituted by the horizontal domino,

- E_B, and E_C, which are empty sets.

For simplicity's sake, we prefer to omit a formal definition of the object operations on the classes $\mathcal{L}$, A, B, C, and D, and present simply a pictorial description. Figures 6, 7, 8 show object-grammar constructions for the classes $\mathcal{L}$, A, B, C, or D, respectively. The one depicted in Fig. 6 reads as follows: an L-convex polyomino is the single cell polyomino, or it is a polyomino in A, B, C, or D.

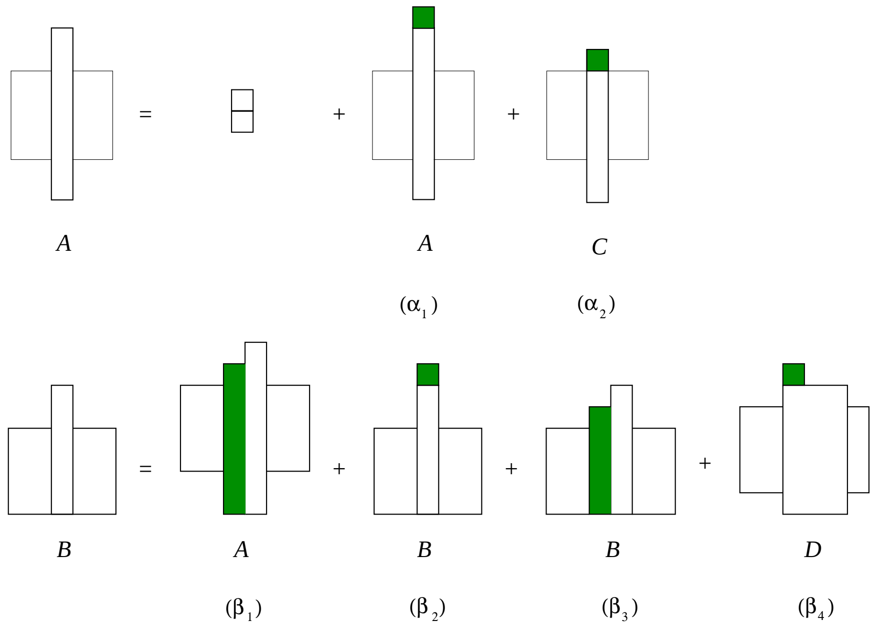


Figure 7: The construction for the classes A and B.

Just to give an example, let us explain in detail the operations performed to obtain a polyomino of the class B, represented in Fig. 7. It is simple to observe that a polyomino of the class B can be obtained uniquely in only one possible way from a polyomino having immediately lower semi-perimeter, by performing one of the four operations depicted in Fig. 7:

1. from a polyomino P in A adding a column of length $h_P - 1$ on the left of c_P, with c_P being the highest column of P, and h_P be the length of c_P (operation β_1);

2. from a polyomino P in B adding a cell on the top of c_P (operation β_2);

3. from a polyomino P in B adding a column of length $h_P - 1$ on the left of c_P (operation β_3);

4. from a polyomino P in D adding a cell onto the leftmost cell of the first row (operation β_4).

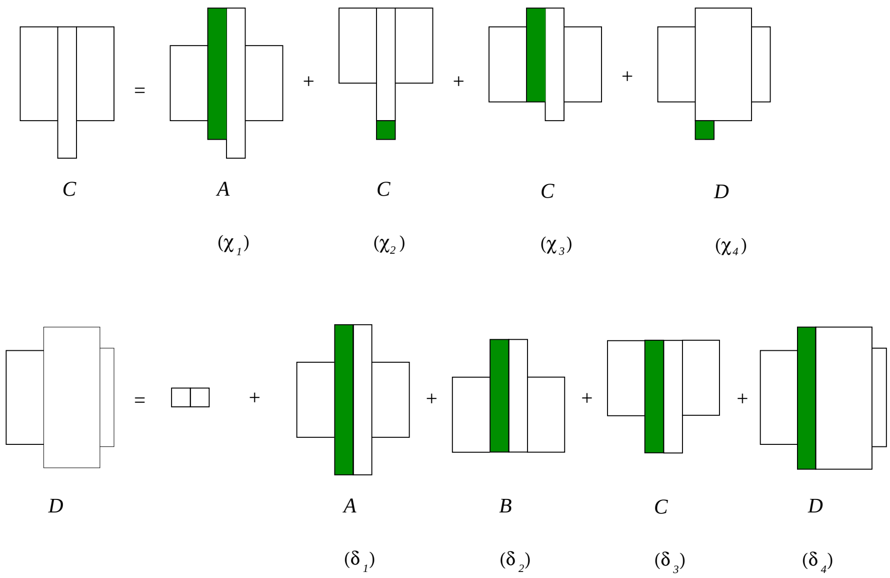


Figure 8: The construction for the classes C and D.

Figure 9 shows, from right to left, the sequence of grammar operations applied to the vertical domino in order to generate a polyomino belonging to class A. The reader can easily check, using the characterization of Theorem 1, that every L-convex polyomino is generated exactly once through the defined object grammar, more precisely one can notice that in order to avoid ambiguities, at each step either a single column of dimension $h_P - 1$ is added on the left of c_P, or the maximum height h_P of the polyomino is uniquely increased by one.

Let $L(x, y, z)$, $A(x, y, z)$, $B(x, y, z)$, $C(x, y, z)$, $D(x, y, z)$ be the generating functions of L-convex polyominoes, and of the classes A, B, C, D, respectively, according to the number of rows, columns, and area. First we easily observe that for symmetry reasons $B(x, y, z) = C(x, y, z)$. Then, translating the grammar

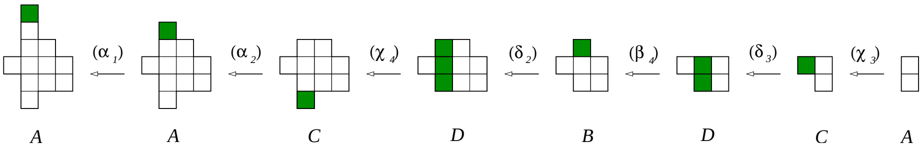


Figure 9: The grammar operations applied for the generation of a polyomino in the class A, and the belonging class of the polyomino obtained at each step.

productions into equations, we obtain the following system:

$$\begin{aligned} L(x,y,z) &= xyz + A(x,y,z) + 2B(x,y,z) + D(x,y,z) \\ A(x,y,z) &= x^2yz^2 + xzA(x,y,z) + xzB(x,y,z) \\ B(x,y,z) &= \frac{yA(xz,y,z)}{z} + xzB(x,y,z) + \frac{yB(xz,y,z)}{z} + xzD(x,y,z) \\ D(x,y,z) &= xy^2z^2 + yA(xz,y,z) + 2yB(xz,y,z) + yD(xz,y,z). \end{aligned}$$

The solution of the system according to the area and semi-perimeter (i.e. setting $x = y$) leads to the same solution derived in [4], while setting $z = 1$ we obtain the generating function for L-convex polyominoes according to rows and columns,

$$L(x,y) = \frac{(1-x)(1-y)\,xy}{1-2\,x-2\,y+x^2+y^2}. \tag{1}$$

Then letting $x = y$ we obtain the generating function for L-convex polyominoes according to the semi-perimeter,

$$L(x) = \frac{x^2(1-x)^2}{1-4x+2x^2},$$

and the other useful generating functions:

$$\begin{aligned} A(x) &= \frac{x^3(1-3x+x^2)}{1-4x+2x^2}, \\ B(x) = C(x) &= \frac{x^4(2-x)}{1-4x+2x^2}, \\ D(x) &= \frac{x^3(1-x)^2}{1-4x+2x^2}. \end{aligned}$$

Let f_n be the number of L-convex polyominoes having semi-perimeter equal to $n+2$. From the generating function obtained above we have that:

$$\begin{aligned} &f_0 = 1,\ f_1 = 2,\ f_2 = 7, \\ &f_n = 4f_{n-1} - 2f_{n-2}, \qquad n \geq 3 \end{aligned} \tag{2}$$

which is equivalent to the renewal recurrence relation:

$$\begin{aligned} &f_0 = 1,\ f_1 = 2, \\ &f_n = 3f_{n-1} + f_{n-2} + f_{n-3} + \ldots + f_0, \qquad n \geq 2. \end{aligned} \tag{3}$$

In the next Section we will prove bijectively (3) using some combinatorial relations between the classes A, B, C and D.

3 A proof of recurrence (3)

Let $\mathcal{L}_n$ be the set of L-convex polyominoes with semi-perimeter $n+2$, and A_n (resp. B_n, C_n, D_n) the class of polyominoes in $\mathcal{L}_n \cap A$ (resp. B, C, D). The table in Fig. 10 gives the first terms of the sequences $|A_n|$, $|B_n|$, $|C_n|$, and $|D_n|$, $n \geq 1$ (in fact the single cell polyomino is not in any of the four classes).

n	$\|A_n\|$	$\|B_n\|$	$\|C_n\|$	$\|D_n\|$	f_n
1	1	0	0	1	2
2	1	2	2	2	7
3	3	7	7	7	24
4	10	24	24	24	82
5	34	82	82	82	280
6	116	280	280	280	956
7	396	956	956	956	3264
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Figure 10: The first terms of $|A_n|$, $|B_n|$, $|C_n|$, and $|D_n|$, $n \geq 1$.

Some facts are trivial:

$$|B_n| = |C_n|, \qquad |D_n| = |\mathcal{L}_{n-1}| = f_{n-1} \qquad n \geq 1.$$

Moreover, we will bijectively prove the following statement, that one can check looking at the Table in Fig. 10.

Proposition 1 *For any $n \geq 2$ we have:*

i) $|B_n| = |C_n| = f_{n-1}$.

ii) $|A_n| = f_{n-2} + f_{n-3} + \ldots + f_0$.

Proof.

i) we prove that each polyomino in $\mathcal{L}_{n-1}$ uniquely produces a polyomino in B_n. So let $P \in \mathcal{L}_{n-1}$, and consider the four cases:

1. $P \in A$: in this case there is unique column in P having maximal length, say $l \geq 2$; we then add a new column of length $l-1$ on the left of such a column, from the bottom to height $l-1$ (see Fig. 11 (i));
2. $P \in B$: in this case we add a cell onto the top of the first row of P (see Fig. 11 (ii));

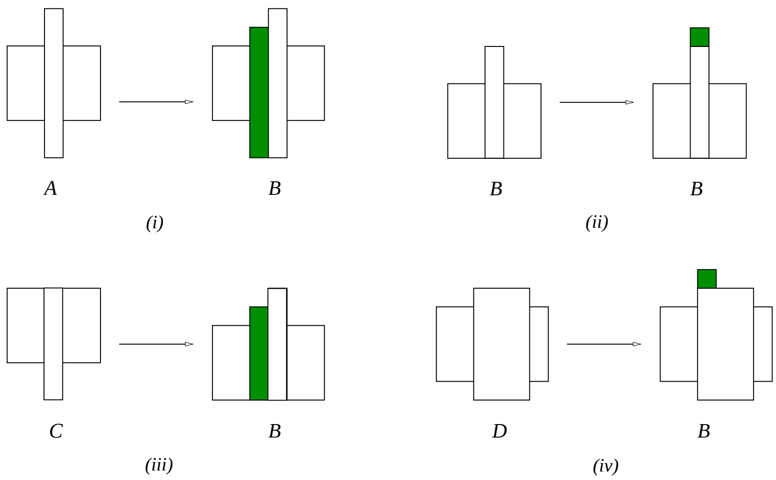


Figure 11: The four operations that produce the polyominoes in B_n from those in $\mathcal{L}_{n-1}$.

3. $P \in C$: in this case we first reflect P with respect to the x-axis; now there is unique column in P having maximal length, say $l \geq 2$; we then add a new column of length $l-1$ on the left of such a column, from the bottom to height $l-1$ (see Fig. 11 (iii));
4. $P \in D$: in this case we add a cell onto the leftmost cell of the first row of P (see Fig. 11 (iv)).

It should be clear that each polyomino in B_n can be obtained in a unique way from a polyomino in $\mathcal{L}_{n-1}$.

ii) It is equivalent to prove that $|A_n| = f_{n-2} + |A_{n-1}|$, $n \geq 2$. We prove this by showing that each polyomino of A_n can be obtained either from a polyomino P in A_{n-1}, by adding a cell on its first row, or from one in $\mathcal{L}_{n-2}$ by performing one of the following operations:

1. $P \in A$: there is unique column in P having maximal length, say $l \geq 2$; we then add a new column of length $l-1$ on the left of such a column, and one cell on its bottom (see Fig. 12 (i));
2. $P \in B$: there is unique column in P having maximal length, say $l \geq 2$; we then add a new column of length $l-1$ on the left of such a column, and one cell on its bottom (see Fig. 12 (ii));
3. $P \in C$: we add cell on the bottom and one on the top of the column having maximal length (see Fig. 12 (iii));
4. $P \in D$: in this case we add a cell onto the leftmost cell of the first row, and one cell below the leftmost cell of the last row (see Fig. 12 (iv)). ■

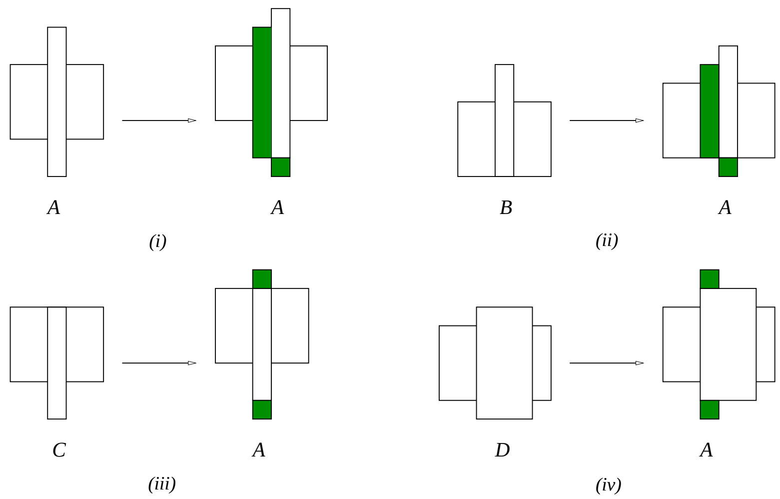


Figure 12: The four operations that produce polyominoes in A_n from those in $\mathcal{L}_{n-2}$.

Figure 13 shows how each polyomino in A_4 can be obtained in a unique way either from a polyomino in A_3 or one in $\mathcal{L}_2$.

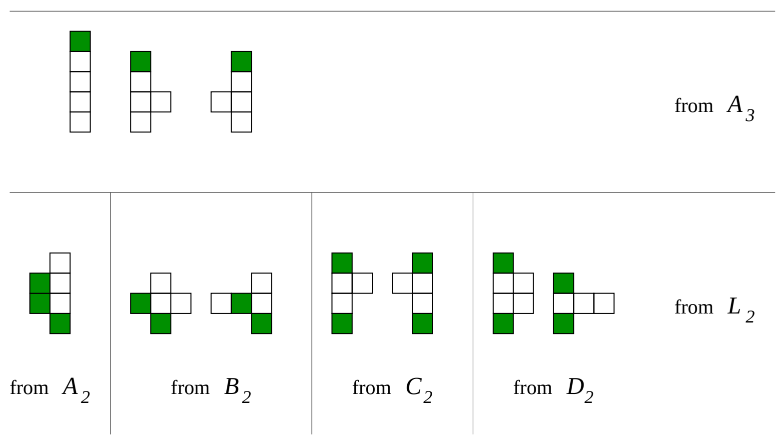


Figure 13: The ten polyominoes in A_4 obtained from those in A_3 and those in $\mathcal{L}_2$.

Now, since it is clear that

$$f_n = |B_n| + |C_n| + |D_n| + |A_n|,$$

from **i)** and **ii)** we derive (3).

4 Further work

In this final section we would like to present a problem which relates the class of L-convex polyominoes to a recently studied class.

In [12] the authors studied some combinatorial and enumerative properties of the class of permutations in S_n which are smaller than or equal to the involution $(n, 2, 3, \ldots, n-1, 1)$ (briefly, $(1, n)$) according to the *Bruhat order* [14]. In practice, by recalling the definition of the Bruhat order $<_B$ on S_n, we have that all the permutations of S_n are comparable with $(1, n)$, and that

$$\pi = (\pi_1, \ldots, \pi_2) <_B (n, 2, 3, \ldots, n-1, 1)$$

if and only if

1. $\pi_1 \leq 2$ or $\pi_2 \leq 2$, and
2. the permutation π' of $n-1$, obtained by removing from π the minimum among π_1 and π_2, is under $(1, n-1)$.

Thus, starting from a permutation π of n, condition 1. must be tested on n permutations, of lengths $n, n-1, \ldots, 1$ respectively. So, for instance, using the scheme below we can easily test that $\pi = (4, 2, 3, 5, 1, 6) <_B (6, 2, 3, 4, 5, 1)$ (observe that the minimal element satisfying condition 1. is underlined):

$$\begin{array}{l} 4\underline{2}3516 \\ 3\underline{2}415 \\ \underline{2}314 \\ 2\underline{1}3 \\ \underline{1}2 \\ \underline{1} \end{array}$$

All the permutations of 3 are less than or equal to $(1, 3)$, while, among the 24 permutations of 4 the following four are not under $(1, 4)$:

$$(3, 4, 1, 2), \quad (3, 4, 2, 1), \quad (4, 3, 1, 2), \quad (4, 3, 2, 1).$$

In [12] the authors proved that the number p_n of the permutations in S_n which are smaller than or equal to $(1, n)$ is defined by the recurrence relation:

$$\begin{aligned} p_0 &= 1 \\ p_1 &= 2 \\ p_n &= 4p_{n-1} - 2p_{n-2}, \end{aligned}$$

giving the numbers $1, 2, 6, 20, 68, 232, \ldots$ (sequence A006012 in [13]), and with generating function

$$\frac{1 - 2x}{1 - 4x + 2x^2}.$$

We remark that the previous recurrence relation differs from (2) only for the initial conditions. Therefore a remarkable problem is to find a combinatorial reason to the fact that this class of permutations and L-convex polyominoes satisfy the same recurrence relation, and in particular, since $p_n \leq f_n$, this could be done by determining a coding of each permutation in terms of an L-convex polyomino.

Moreover, by simple computations we obtain that the two sequences are related by the following identity:

$$p_n = 3f_{n-1} - 2f_{n-3}, \qquad n \geq 4. \tag{4}$$

Equation (4) requires a combinatorial explanation, which can be suggested by the following property:

$$3f_{n-1} = |B_n| + |C_n| + |D_n|, \qquad 2f_{n-3} = |A_{n-1}| + |A_{n-2}| + \ldots + |A_2|.$$

Another interesting question comes from the study of the sequence $\{f_n - q_n\}_{n\geq 0}$, giving the numbers $1, 4, 14, 48, 184, \ldots$, since it holds that

$$f_n - p_n = |B_{n-1}| + |C_{n-1}|, \qquad n \geq 3.$$

In practice it means that the set of L-convex polyominoes which cannot be encoded in terms of permutations are those of the classes B and C. Finally, combining the last equations we obtain that

$$4f_n = p_2 + p_3 + \ldots + p_{n+1},$$

which also begs for a combinatorial explanation, that can perhaps be given using the classes of the object grammar decomposition we have presented in our paper.

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