k-fixed-points-permutations

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(Received: October 31, 2006)

Abstract. In this paper we study the distribution of *k-fixed-points* over symmetric group. We will give some combinatorial interpretations to the relations defining them as well as their generating functions. A combinatorial interpretation directly on derangements of the famous relation on derangement numbers $d_n = nd_{n-1} + (-1)^n$ will be given.

Résumé. Nous étudions dans ce papier la distribution de la statistique *k-points-fixes* sur le groupe symétrique. Nous donnerons des interprétations combinatoires de leurs relations de récurrence. Une interprétation combinatoire directe sur les dérangements de la célèbre relation sur les nombres des dérangements $d_n = nd_{n-1} + (-1)^n$ sera donnée.

Mathematics Subject Classifications (2000). 05A15; 05A19

Keywords: k-fixed points, generating function.

1 Introduction

Euler [1], [4] introduced the difference table $(e_n^k)_{0\le k\le n}$ where the coefficients e_n^k are defined by

$$e_n^n = n! \text{ and } e_n^{k-1} = e_n^k - e_{n-1}^{k-1} \text{ for } 1 \le k \le n$$

without giving their combinatorial interpretation. In our previous paper [11], we have studied these numbers which generalise the derangement theory through the study of k-successions. The first values of the numbers e_n^k are given in the following table

	e_n^k					
	$k=0$	1	2	3	4	5
$n=0$	0!					
1	0	1!				
2	1	1	2!			
3	2	3	4	3!		
4	9	11	14	18	4!	
5	44	53	64	78	96	5!

and their generating functions are defined by

$$\begin{cases} E^{(k)}(u) = \displaystyle\sum_{n\ge 0} e_{n+k}^k \frac{u^n}{n!} = k!\frac{\exp(-u)}{(1-u)^{k+1}} \\ E(x,u) = \displaystyle\sum_{k\ge 0}\sum_{n\ge 0} e_{n+k}^k \frac{x^k}{k!}\frac{u^n}{n!} = \frac{\exp(-u)}{1-x-u}. \end{cases}$$

The motivation of this paper is to study the numbers d_n^k which are obtained from the numbers e_n^k by dividing them by $k!$. It follows straightforwardly that their generating functions are defined by

$$\begin{cases} D^{(k)}(u) = \sum_{n\geq 0} d_{n+k}^k \frac{u^n}{n!} = \frac{\exp(-u)}{(1-u)^{k+1}} \\ D(x,u) = \sum_{k\geq 0}\sum_{n\geq 0} d_{n+k}^k x^k \frac{u^n}{n!} = \frac{\exp(-u)}{1-x-u}. \end{cases}$$

We obtain then the following table for some first values of the numbers d_n^k

d_n^k						
	$k=0$	1	2	3	4	5
$n=0$	1					
1	0	1				
2	1	1	1			
3	2	3	2	1		
4	9	11	7	3	1	
5	44	53	32	13	4	1

By a simple computation, we can find that the numbers d_n^k satisfy the following recurrences

$$\begin{cases} d_k^k = 1 \\ d_n^k = (n-1)d_{n-1}^k + (n-k-1)d_{n-2}^k \text{ for } n > k \geq 0. \end{cases}$$

The aims of this paper are to give combinatorial interpretations of these numbers. We will give a combinatorial bijection to the unexpected relation

$$d_n^k + d_{n-2}^{k-1} = nd_{n-1}^k$$

which is a generalization of the famous recurrence on derangement numbers [2], [5], [14]

$$d_n = nd_{n-1} + (-1)^n.$$

Désarmenien [2] gave a combinatorial proof of this last relation with other objects which are in bijection with derangements, and many authors [3], [6], [7], [8], [9], [10], [12], [15] have studied in depth the numbers d_n. A bijective proof directly over derangements or permutations without fixed points for this last relation of derangement numbers will be given in a separate section. Let us denote by $[n]$ the interval $\{1, 2, \cdots, n\}$, by σ a permutation of the symmetric group $\mathfrak{S}_n$. In this paper, we will use the linear notation $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ and the notation of the decomposition in a product of disjoint cycles to write a permutation σ.

Definition 1.1 We say that an integer i is a fixed point of a permutation σ if $\sigma(i) = i$. We will denote by $Fix(\sigma)$ the set of fixed points of the permutation σ.

Definition 1.2 We say that a permutation σ is a k-fixed-points-permutation if for all integers i in the interval $[k]$, $\sigma^p(i) \notin [k] \setminus \{i\}$ for all integer p and $Fix(\sigma) \subseteq [k]$.

We will denote by D_n^k the set of k-fixed-points-permutations of the symmetric group $\mathfrak{S}_n$.

Example 1.3 We have

$$D_1^0 = \{\},\ D_1^1 = \{1\},$$
$$D_2^0 = D_2^1 = \{21\},\ D_2^2 = \{12\}.$$
$$D_3^0 = \{231, 312\},\ D_3^1 = \{132, 231, 312\},\ D_3^2 = \{132,\ 312\},\ D_3^3 = \{123\}.$$

Remark 1.4 The permutation $12 \cdots k$ is the only k-fixed-points-permutation of the symmetric group $\mathfrak{S}_k$.

2 Numbers d_n^k

2.1 First relation of the numbers d_n^k

Theorem 2.1 *For* $0 \leq k \leq n-1$, *we have*

$$d_n^k = (n-1)d_{n-1}^k + (n-k-1)d_{n-2}^k.$$

Proof. Let us consider the map $\varphi : D_n^k \to [n-1] \times D_{n-1}^k \cup [n-k-1] \times D_{n-2}^k$ which associates to each permutation σ a pair $(m, \sigma') = \varphi(\sigma)$ defined as follows

(1) If the integer n is in a cycle of length greater or equal to 3, or the length of the cycle which contains the integer n is equal to 2 and $\sigma(n) \leq k$, then the integer m is equal to $\sigma^{-1}(n)$ and the permutation σ' is obtained from the permutation σ by removing the integer n from his cycle. The permutation σ' is indeed an element of the set D_{n-1}^k.

(2) If the length of the cycle which contains the integer n is equal to 2 and $\sigma(n) > k$, then the integer m is equal to $\sigma(n)$ and the permutation σ' is obtained from the permutation σ by removing the cycle $(\sigma(n), n)$ and then decreasing by 1 all integers between $\sigma(n)+1$ and $n-1$ in each cycle. The permutation σ' is indeed an element of the set D_{n-2}^k.

The map φ is bijective. Notice that a pair (m, σ') in the image $\varphi(D_n^k)$ is contained either in the set of all pairs of $[n-1] \times D_{n-1}^k$ if the integer n lies in a cycle of length greater than 2 or equal to 2 and $\delta(n) \leq k$, or in the set of all pairs of $[n-k-1] \times D_{n-2}^k$ if the integer n lies in a cycle of length equal to 2 and $\delta(n) > k$. So it remains to prove that there exists a map $\tilde{\varphi}$ that

- associates an element D_n^k where the integer n lies in a cycle of length greater than 2 or equal to 2 and the integer n lies in a cycle which contains an integer less or equal to k with every pair of $[n-1] \times D_{n-1}^k$.

- associates an element D_n^k where the integer n lies in a cycle of length equal to 2 and the integer n lies in a cycle which contains an integer greater than k with every pair of $[n-k-1] \times D_{n-2}^k$.
- is the inverse of φ.

We define the permutation $\sigma = \tilde{\varphi}(m, \sigma')$ of the set D_n^k

- either by inserting the integer n in a cycle of the permutation σ' after the integer $m \in [n-1]$ if σ' is an element of the set D_{n-1}^k. In such case, the integer n lies in a cycle of length greater to 2 or in a transposition and $\sigma(n) \leq k$.
- or by creating the transposition (m, n) with $k < m \leq n-2$ and then increasing by 1 all integers between m and $n-2$ in each cycle of the permutation σ' if the permutation σ' is an element of the set D_{n-2}^k. In such case, the integer n is in a transposition and $\sigma(n) > k$.

The map $\tilde{\varphi}$ is the inverse of the map φ. □

Corollary 2.2 *The number d_n^k equals the cardinality of the set of k-fixed-points-permutations in the symmetric group $\mathfrak{S}_n$.*

2.2 Second relation of the numbers d_n^k

We will give another relation satisfied by the numbers d_n^k which is easily deduced from the generating function, but we will give its combinatorial interpretation.

Definition 2.3 Let us consider the map $\vartheta : D_{n-1}^{k-1} \cup D_n^{k-1} \to [k] \times D_n^k$ which associates to a permutation σ a pair $(m, \sigma') = \vartheta(\sigma)$ defined as below

(1) If $\sigma \in D_{n-1}^{k-1}$, then the integer m is equal to k and the permutation σ' is obtained from the permutation σ by creating the cycle (k) and then by increasing by 1 all integers greater or equal to k in each cycle of the permutation σ.

(2) If $\sigma \in D_n^{k-1}$, then the integer m is equal to the smallest integer in the cycle which contains the integer k and the permutation σ' is obtained from the permutation σ by removing the word $k \quad \sigma(k) \cdots \sigma^{-1}(m)$ and then creating the cycle $(k \quad \sigma(k) \cdots \sigma^{-1}(m))$.

Proposition 2.4 *The map ϑ is a bijection.*

Proof. The map ϑ is injective. It suffices to show that ϑ is surjective. Let us look at various cases of the pair (m, σ').

(1) If $m = k$ and $\sigma'(k) = k$, then we define the permutation σ by deleting the cycle (k) and then decreasing by 1 all integers greater than k in each cycle. It follows straightforwardly that the permutation σ is an element of the set D_{n-1}^{k-1}.

(2) If $m = k$ and $\sigma'(k) \neq k$, then $\sigma = \sigma'$ and $\sigma \in D_n^{k-1}$.

(3) If $m \neq k$, then the permutation σ is obtained from the permutation σ' by removing the cycle which contains k and then inserting the word $k\sigma'(k)\sigma'^2(k)\cdots$ in the cycle which contains the integer m just before the integer $\sigma'^{-1}(m)$. The permutation σ is indeed an element of the set D_n^{k-1}.

It is impossible by construction of the map ϑ that $m = k$ and the integer k is in the same cycle as an integer smaller than k. □

Theorem 2.5 *For all integers* $1 \leq k \leq n$*, we have*

$$kd_n^k = d_{n-1}^{k-1} + d_n^{k-1}.$$

Proof. By the bijection ϑ, we have

$$\#D_{n-1}^{k-1} + \#D_n^{k-1} = \#[k] \times D_n^k,$$

that is,

$$kd_n^k = d_{n-1}^{k-1} + d_n^{k-1}.$$ □

2.3 Third relation for the numbers d_n^k

The following unexpected relation is a generalization of the famous relation on derangement numbers and a bijective proof will be given.

Theorem 2.6 *For all integers* $0 \leq k \leq n-1$*, one has*

$$nd_{n-1}^k = d_n^k + d_{n-2}^{k-1}.$$

Proof. Let us consider the map $\varsigma : [n] \times D_{n-1}^k \to D_n^k \cup D_{n-2}^{k-1}$ which associates to a pair (m, σ) a permutation $\sigma' = \varsigma((m, \sigma))$ defined in the following ways

(1) If $m < n$, then the permutation σ' is obtained from the permutation σ by inserting the integer n in the cycle which contains m just before the integer m itself. The permutation σ' is indeed an element of the set D_n^k.

(2) If $m = n$ and $\sigma(1) \neq 1$, then the permutation $\sigma' = \varsigma((n, \sigma))$ is obtained from the permutation σ by removing the integer $\sigma(1)$ and then creating the cycle $(n \quad \sigma(1))$. The permutation σ' is indeed an element of the set D_n^k and $\sigma'(n) > k$.

(3) If $m = n$ and $\sigma(1) = 1$, then the permutation $\sigma' = \varsigma((n, \sigma))$ is obtained from the permutation σ by removing the cycle (1) and then by decreasing by 1 all integers in each cycle. It follows straightforwardly that the permutation σ' is an element of the set D_{n-2}^k.

The map ς is a bijection. The map ς is injective. It suffices to show that ς is surjective. Let us look at various cases of the permutation σ'.

(1) If the permutation σ' is an element of the set D_n^k and the cycle which contains n is different of the transposition $(n \quad \sigma'(n))$ where $\sigma'(n) > k$, then the couple (m, σ) is defined by $m = \sigma'^{-1}(n)$ and the permutation σ is obtained by removing the integer n from the cycle containing it.

(2) If the permutation σ' is an element of the set D_n^k and the cycle which contains n is a transposition $(n \quad \sigma'(n))$ where $\sigma'(n) > k$, then the couple (m, σ) is defined by $m = n$ and the permutation σ is obtained by removing the cycle $(n \quad \sigma'(n))$ and inserting the integer $\sigma'(n)$ in the cycle which contains the integer 1 just after 1.

(3) If the permutation σ' is an element of the set D_{n-2}^{k-1}, then the couple (m, σ) is defined by $m = n$ and the permutation σ is obtained by increasing by 1 all the integers in each cycle of the permutation σ' and then creating the new cycle (1). □

Remark 2.7 If we set $d_{-1}^{-1} = 1$ and $d_{n-1}^{-1} + d_n^{-1} = 0d_n^0$, that is, $d_{n-1}^{-1} + d_n^{-1} = 0$, then we obtain

$$d_n^0 + d_{n-2}^{-1} = nd_{n-1}^0.$$

We will give a combinatorial interpretation of this relation directly on derangements in the following section.

3 The famous $d_n = nd_{n-1} + (-1)^n$

Notice that the set D_n of derangements or permutations without fixed points is equal to the set D_n^0.

Definition 3.1 Let us define the *critical derangement* $\Delta_n = (1\,2)(3\,4)\cdots(n-1 \quad n)$ if the integer n is even and the sets

- $E_n = \{\Delta_n\}$ if the integer n is even and $E_n = \emptyset$ otherwise,
- $F_n = \{(n, \Delta_{n-1})\}$ if the integer n is odd and $F_n = \emptyset$ otherwise.

Let $\tau : [n] \times D_{n-1} \setminus F_n \to D_n \setminus E_n$ be the map which associates to a pair (i, δ) a permutation $\delta' = \tau((i, \delta))$ defined as below:

(1) If the integer $i < n$, then the permutation $\delta' = \delta(i \quad n)$. In other words, the permutation δ' is obtained from the permutation δ by inserting the integer n in the cycle which contains the integer i just after the integer i.

(2) If the integer $i = n$, then let p be the smallest integer such that the transpositions $(1\,2), (3\,4), \ldots, (2p-1 \quad 2p)$ are cycles of the permutation δ and the transposition $(2p+1 \quad 2p+2)$ is not.

   (a) If $\delta(2p+1) = 2p+2$, then the permutation δ' is obtained from the permutation δ by removing the integer $2p+1$ from the cycle which contains it, and then creating the new cycle $(2p+1 \quad n)$.

(b) If $\delta(2p+1) \neq 2p+2$, then we have to distinguish the following two cases:

(i) If the length of the cycle which contains the integer $2p+1$ is equal to 2, then the permutation δ' is obtained from the permutation δ by removing the cycle $(2p+1\,\delta(2p+1))$, and then inserting the integer $2p+1$ in the cycle which contains the integer $2p+2$ just before the integer $2p+2$ and creating the new cycle $(\delta(2p+1) \quad n)$.

(ii) If the length of the cycle which contains the integer $2p+1$ is greater than 2, then then the permutation δ' is obtained from the permutation δ by removing the integer $\delta(2p+1)$ and then creating the new cycle $(\delta(2p+1) \quad n)$.

PROPOSITION 3.2 *The map τ is bijective.*

Proof. Notice that the only pair (i, δ) which is not defined by the map τ is the pair (n, Δ_{n-1}) if the integer $n-1$ is even. Notice also that the image $\tau([n-1] \times D_{n-1})$ is contained in the set of all derangements D_n where the integer n lies in a cycle of length greater or equal to 3 and the image $\tau(\{n\} \times D_{n-1} \setminus F_n)$ is contained in the set of all derangements D_n where the integer n lies in a cycle of length equal to 2. So we only need to show that there exists a map ζ that

- associates an element of $[n-1] \times D_{n-1}$ with every derangement of D_n in which the integer n lies in a cycle of length greater or equal to 3.
- associates an element of $\{n\} \times D_{n-1} \setminus F_n$ with every derangement of D_n in which the integer n lies in a cycle of length 2.
- is the inverse of τ.

It is straightforward to verify that the map ζ is defined as follows.

(1) If the integer n lies in a cycle of length greater or equal to 3, then $\zeta(\delta)$ is the pair (i, δ') where $i = \delta^{-1}(n)$ and the permutation δ' is obtained by removing the integer n from the derangement δ. The permutation δ' is a derangement of D_{n-1} and the integer i is smaller than n.

(2) If the integer n lies in a cycle of length equal to 2, then let p the smaller nonnegative integer such that $(1\,2), (3\,4), \ldots, (2p-1 \quad 2p)$ are cycles of the derangement δ and the transposition $(2p+1 \quad 2p+2)$ is not.

(a) If $\delta(n) = 2p+1$, then $\zeta(\delta)$ is the pair (n, δ') where the permutation δ' is obtained from the derangement δ by deleting the cycle $(n \quad 2p+1)$ and then inserting the integer $2p+1$ in the cycle which contains the integer $2p+2$ just before the integer $2p+2$.
In other words, we have

$$\delta = (12)(34)\cdots(2p-1 \quad 2p)(2p+1 \quad n)(2p+2\ldots)\cdots$$

and

$$\delta' = (12)(34)\cdots(2p-1 \quad 2p)(2p+1 \quad 2p+2\ldots)\cdots.$$

(b) If $\delta(2p+1) \neq n$, then we have to distinguish the following two cases:

(i) If $\delta(2p+1) \neq 2p+2$, then $\zeta(\delta)$ is the pair (n, δ') where the permutation δ' is obtained from the derangement δ by deleting the cycle $(n\, \delta(n))$ and then inserting the integer $\delta(n)$ in the cycle which contains the integer $2p+1$ just before the integer $2p+1$. In other words, we have

$$\delta = (12)(34)\cdots(2p-1 \quad 2p)(2p+1\ldots)\cdots(\delta(n) \quad n)\cdots$$

and

$$\delta' = (12)(34)\cdots(2p-1 \quad 2p)(2p+1\ldots\delta(n))\cdots.$$

(ii) If $\delta(2p+1) = 2p+2$, then $\zeta(\delta)$ is the pair (n, δ') where the permutation δ' is obtained from the derangement δ by deleting the cycle $(n \quad \delta(n))$ and the integer $2p+1$ and then creating the new cycle $(2p+1 \quad \delta(n))$.
In other words, we have

$$\delta = (12)(34)\cdots(2p-1 \quad 2p)(2p+1 \quad 2p+2\ldots)\cdots(\delta(n) \quad n)\cdots$$

and

$$\delta' = (12)(34)\cdots(2p-1 \quad 2p)(2p+1 \quad \delta(n))(2p+2\ldots)\cdots.$$

Notice that the derangement Δ_n, if the integer n is even, is the only derangement which is not defined by the map ζ. □

Corollary 3.3 *If the integer n is even, then we have*

$$d_n = nd_{n-1} + 1.$$

If the integer n is odd, then we have

$$d_n + 1 = nd_{n-1}.$$

Acknowledgements. The author is very grateful to the referrees for their suggestions and their comments on these results.

References

[1] R.J. Clarke, G.N. Han and J. Zeng, *A combinatorial interpretation of the Seidel generation of q-derangement numbers*, Annals of Combinatorics, **1** (1997), 313–327.

[2] J. Désarmenien, *Une autre interprétation du nombre de dérangements,* Actes 8e Sém. Lothar. Comb., IRMA Strasbourg (1984), 11–16.

[3] J. Désarménien and M. Wachs, *Descentes des dérangements et Mots circulaires,* Actes 19e Sém. Lothar. Comb., IRMA Strasbourg (1988), 3–21.

[4] D. Dumont and A. Randrianarivony, *Dérangements et nombres de Genocchi*, Discrete Math., **132** (1990), 37–49.

[5] D. Foata and M.P. Schützenberger, *Théorie Géométrique des Polynômes Eulériens*, Lect. Notes in Math., **138**, Springer-Verlag, Berlin, (1970).

[6] R. Mantaci and F. Rakotondrajao, *A permutation representation that knows what "Eulerian" means,* Discrete Mathematics and Theoretical Computer Science, **4** (2001), 101–108.

[7] R. Mantaci and F. Rakotondrajao, *Exceedingly deranging!*, Advances in Applied Mathematics, **30** (1/2) (January 2003), 177–188.

[8] F. Rakotondrajao, Ph.D thesis, Université d' Antananarivo, Antananarivo, Madagascar, (1999).

[9] F. Rakotondrajao, *Magic squares, rook polynomials and permutations,* Séminaire Lotharingien de Combinatoire, **B54A** (2006). *http://www.mat.univie.ac.at*

[10] F. Rakotondrajao, *Permutations by numbers of anti-excedances and fixed points,* Proc. FPSAC'02, University of Melbourne, Australia (July 2002).

[11] F. Rakotondrajao, *On Euler's difference table*, Proc. FPSAC'07, Nankai University, Tianjin, China (July 2007).

[12] C. Remmel, *A note on a recursion for the number of derangements*, Europ. J. Combin., **4** (4) (1983), 371–373.

[13] J. Riordan, *An Introduction to Combinatorial Analysis*, John Wiley & Sons, New York, (1958).

[14] R.P. Stanley, *Enumerative Combinatorics*, **1**, Cambridge, (1997).

[15] M. Wachs, *On q-derangement numbers,* Proc. Amer. Math. Soc., **106** (1) (1989), 273–278.