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Tomographical aspects of L-convex polyominoes

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Abstract. Our main purpose is to characterize the class of L-convex polyominoes introduced in [3] by means of their horizontal and vertical projections. The achieved results allow an answer to one of the most relevant questions in tomography i.e. the uniqueness of discrete sets, with respect to their horizontal and vertical projections. In this paper, by giving a characterization of L-convex polyominoes, we investigate the connection between uniqueness property and unimodality of vectors of horizontal and vertical projections. In the last section we consider the continuum environment: we extend the definition of L-convex set, and we obtain some results analogous to those holding in the discrete case.

Mathematics Subject Classifications (2000). 68Q25, 68W40

1 Introduction

The aim of *discrete tomography* is the retrieval of geometrical information about a physical structure, regarded as a finite set of points in the integer square lattice $\mathbb{Z} \times \mathbb{Z}$, from measurements, generically known as *projections*, of the number of atoms in the structure that lie on lines with fixed scopes (see [11] for a survey). A common simplification is to represent a finite physical structure as a binary matrix, where an entry is 1 or 0 according as an atom is present or absent in the structure at the corresponding point of the lattice. The challenge is then to reconstruct key features of the structure from a small number of scans of projections.

The space of solutions of this reconstruction problem, however, is really huge and in general quite impossible to control. A good idea may seem to start increasing the number of projections one by one in order to decrease the number of solutions. Unfortunately, the reconstruction problem becomes intractable when the number of projections is greater than two, as proved in [10]. This means that (unless $P = NP$) exact reconstructions require, in general, an exponential amount of time.

In most practical applications we have some a priori information about the images that have to be reconstructed. So, we can tackle the algorithmic challenges by limiting the class of possible solutions using appropriate prior information, such as convexity or connectivity properties.

As an example, the authors of [9] succeeded in characterizing each *convex discrete set* by means of its projections in certain prescribed sets of four directions or in any seven non-parallel coplanar directions. Moreover, there are efficient algorithms for reconstructing subclasses of discrete sets defined by convexity or connectivity properties: for example, there are polynomial time algorithms to reconstruct *hv-convex polyominoes* [1, 6] (i.e., sets which are 4-connected and convex in the horizontal and vertical directions), and special classes of periodical sets [7].

In this paper, we consider the class of *L-convex polyominoes* which has been introduced in [3] as first level of a hierarchy on the hv-convex polyominoes. L-convex polyominoes admit different characterizations, in terms of minimal internal paths connecting two points, and in terms of maximal rectangular elements composing them. In [4, 5], they are analyzed from a combinatorial point of view, giving the enumeration according to the semi-perimeter and the area.

Here, we investigate L-convex polyominoes from a tomographical perspective, and we show a third characterization by means of the horizontal and vertical projections. Furthermore, a strong uniqueness result is also provided; it acquires relevance when considered together with the linear-time reconstruction algorithm for centered polyominoes in [6]. Finally, we extend the notion of L-convexity to the continuum environment, and determine results analogous to those in the discrete case.

2 Definitions and preliminaries

Let our environment be the integer lattice $\mathbb{Z} \times \mathbb{Z}$. A *discrete set* is a finite subset S of $\mathbb{Z} \times \mathbb{Z}$ considered up to translations.

Usually, a discrete set is represented by a binary matrix or by a set of *cells* (unitary squares), as depicted in Fig. 1. In the sequel, we will use the latter representation, and we number the rows and the columns of the set starting from the upper left corner of the minimum rectangle containing it. We denote by (i, j) the cell in the i-th row and j-th column of the rectangle.

In this paper we study a particular class of discrete sets, i.e. the well known class of *polyominoes* [8]. A polyomino is defined as a finite union of cells whose interior is connected. To avoid misunderstandings we assume that a polyomino

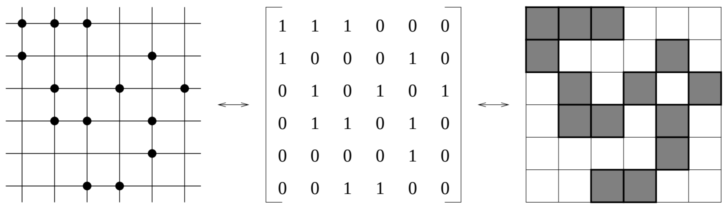


Figure 1: A finite set of $\mathbb{Z}^2$, and its representation in terms of a binary matrix and a set of cells.

is always viewed as the set of discrete points representing its cells.

Given two polyominoes P and P', we say that P is contained in P', and we write $P \subseteq P'$, if the set of discrete points representing P is included in that representing P' (see Fig. 2).

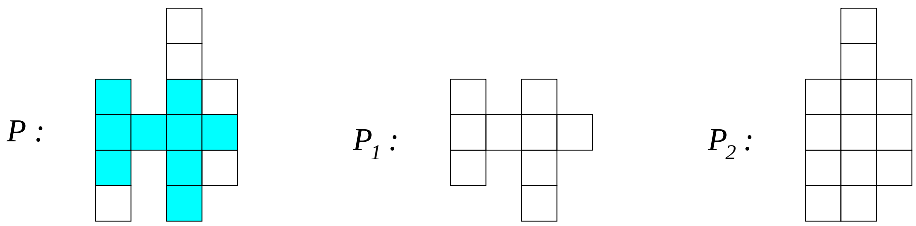


Figure 2: The polyomino P contains the polyomino P_1, but does not contain the polyomino P_2. The shaded cells of P show the inclusion.

A polyomino is said to be *h-convex* (resp. *v-convex*) if every its row (resp. column) is connected. A polyomino is said to be *hv-convex*, or simply *convex*, if it is both h-convex and v-convex (see Fig. 3).

For any two cells A and B in a polyomino, a *path* Π_{AB} from A to B, is a sequence $(i_1, j_1), (i_2, j_2), \ldots, (i_r, j_r)$ of adjacent disjoint cells of the polyomino, with $A = (i_1, j_1)$, and $B = (i_r, j_r)$. For each $1 \leq k < r$, we say that the two consecutive cells $(i_k, j_k), (i_{k+1}, j_{k+1})$ form:

- an *east* step if $i_{k+1} = i_k + 1$ and $j_{k+1} = j_k$;
- a *north* step if $i_{k+1} = h_i$ and $j_{k+1} = j_k + 1$;
- a *west* step if $i_{k+1} = h_i - 1$ and $j_{k+1} = j_k$;
- a *south* step if $i_{k+1} = h_i$ and $j_{k+1} = j_k - 1$.

Finally, we define a path to be *monotone* if it is entirely made of only two of the four types of steps defined above.

The cells in a convex polyomino satisfy a particular connection property that involves the shape of the paths connecting any pair of them [3].

Proposition 1 *A polyomino P is convex iff every pair of cells is connected by a monotone path.*

The property in Proposition 1, allows us to introduce a particular family of convex polyominoes, called *L-convex* polyominoes, defined and studied in [3].

2.1 The class of L-convex polyominoes

Let us consider a polyomino P. A path in P has a *change of direction* in the cell (i_k, j_k), for $2 \leq k \leq r-1$, if

$$i_k \neq i_{k-1} \Longleftrightarrow j_{k+1} \neq j_k.$$

In [3] it is proposed a classification of convex polyominoes based on the number of changes of direction in the paths connecting any two cells of a polyomino. More precisely, we call *k-convex* a convex polyomino such that every pair of its cells can be connected by a monotone path with at most k changes of direction. For $k = 1$ we have the class of *L-convex polyominoes*, i.e. those polyominoes such that every pair of their cells can be connected by a path with at most one change of direction (see Fig. 3).

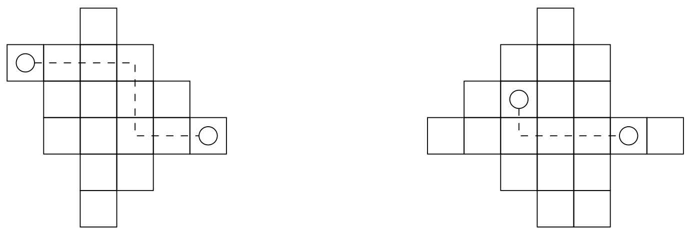


Figure 3: The convex polyomino on the left is not L-convex, while the one on the right is L-convex. For both the polyominoes two cells are highlighted, and a monotone path which connects them and which contains the minimum number of possible changes of direction, is depicted.

In the same paper it is given a nice characterization of L-convex polyominoes that involves the following notion of maximal rectangle.

Let us indicate a rectangular polyomino (briefly *rectangle*) by $[A, B]$, where $A = (i_1, j_1)$ and $B = (i_2, j_2)$ are its upper leftmost and lower rightmost cells, respectively. More precisely, for $i_1 \geq i_2$, and $j_1 \leq j_2$

$$[A, B] = \{(i, j) \,:\, i_2 \leq i \leq i_1 \text{ and } j_1 \leq j \leq j_2\}.$$

We say $[A, B]$ to be *maximal* in P if

$$\forall\, [A', B'],\ [A, B] \subseteq [A', B'] \subseteq P \ \Rightarrow\ [A, B] = [A', B'].$$

Two rectangles $[A, B]$ and $[A', B']$ contained in P have a *crossing intersection* if their intersection is a rectangle having as basis the smallest of the two bases,

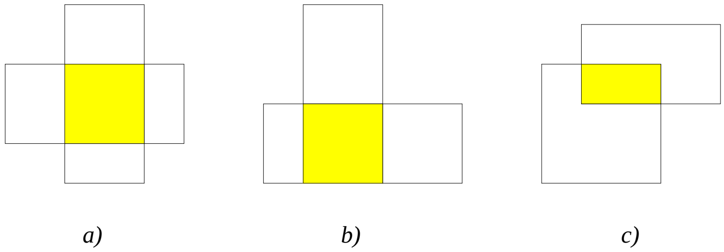


Figure 4: In a) and b) we have crossing intersections among the rectangles, while the intersection in c) is not crossing.

and as height the smallest of the two heights, i.e. $[A, B] \cap [A', B'] = [A'', B'']$ with

$$A'' = (\min\{i_1, i'_1\}, \max\{j_1, j'_1\}), \text{ and } B'' = (\max\{i_2, i'_2\}, \min\{j_2, j'_2\}).$$

Figure 4 shows examples of crossing and non-crossing intersections.

THEOREM 1 *A convex polyomino is L-convex iff every pair of its maximal rectangles has crossing intersection.*

From Theorem 1, it immediately follows that all the maximal rectangles of an L-convex polyomino are distinct. The same result allows to characterize an L-convex polyomino as one of the overlapping of its maximal rectangles.

Since the set of maximal rectangles can be partially ordered as follows:

$$[A, B] > [A', B'] \text{ if } (i_1 - i_2) > (i'_1 - i'_2) \text{ and } (j_2 - j_1) < (j'_2 - j'_1),$$

then each finite overlapping of comparable rectangles such that any pair of them has a crossing intersection, determines an L-convex polyomino (see Fig. 5 for an example).

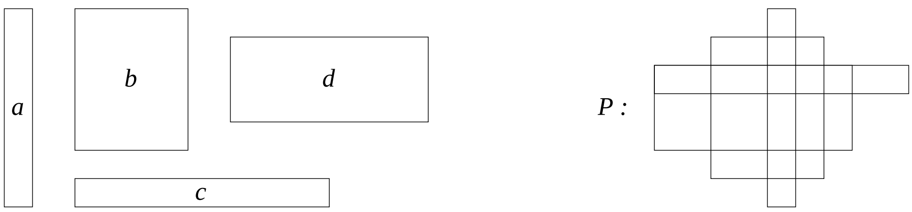


Figure 5: An L-convex polyomino P obtained by one of the overlappings of the four comparable rectangles a, b, c and d having crossing intersection.

2.2 Basic notions of discrete tomography

To each discrete set S, we can associate two integer vectors $H = (h_1, \ldots, h_m)$ and $V = (v_1, \ldots, v_n)$ such that, for each $1 \leq i \leq m$, $1 \leq j \leq n$, h_i and v_j are the number of cells of S which lie on row i and column j, respectively. The vectors H and V are called the horizontal and vertical projections of S, respectively. Given two vectors H and V, we will denote by $\mathcal{U}(H, V)$ the class of discrete sets having H and V as projections.

A discrete set S is *unique* (with respect to H and V) if $\mathcal{U}(H, V) = \{S\}$. In such a case also H and V are said to be *unique*.

Fundamental problems of discrete tomography concern the retrieval of information about some geometrical aspects (cf. [1], [2], [13]) of discrete sets, from the knowledge of their projections (for a survey cf. [11]).

In general, the horizontal and vertical projections of a discrete set are not sufficient to uniquely determine it (see Fig. 7), as it is known from [16], where Ryser pointed out that a discrete set is unique if and only if it does not contain particular configurations of points called *switching components*. Figure 6 shows the two simplest of them, called *elementary switching components*, and defined as follows: a discrete set S contains the elementary switching component $a)$ [resp. $b)$] if there exists two different rows i and i', and two different columns j and j' such that the cells in positions (i, j), and (i', j') [resp. (i', j), and (i, j')] belong to S (represented in the figure by filled squares), while the cells in positions (i', j), and (i, j') [resp. (i, j), and (i', j')] do not belong to S (represented in the figure by dotted squares).

Still in [16], Ryser defined a *composition* on the elementary switchings components, and proved that each switching component either is an elementary switching component, or it is the composition of two or more of them.

Finally, he defined an operator, called *interchange* and successively *switching (operator)*, which modifies a discrete set by changing one of its switching components, if it exists, into the other.

In Fig. 7, the discrete sets P_1 and P_2 are obtained from P by performing the two highlighted switchings.

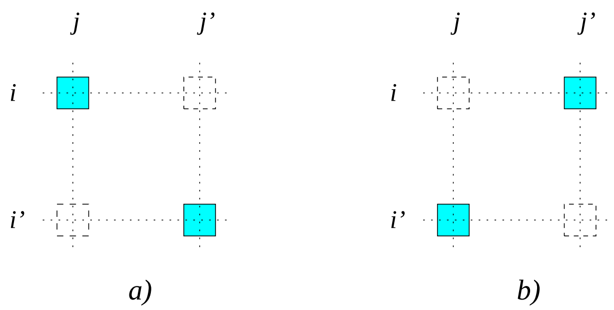


Figure 6: The two elementary switching components. If a set contains $a)$ or $b)$, then it is not unique.

Clearly, switching does not modify the projections of a discrete set, which consequently reveals to be non-unique (cf. [17]). The reverse of this property is also true, as stated in what follows.

THEOREM 2 (RYSER'S THEOREM) *A discrete set is non-unique (with respect to its horizontal and vertical projections) if and only if it has a switching component.*

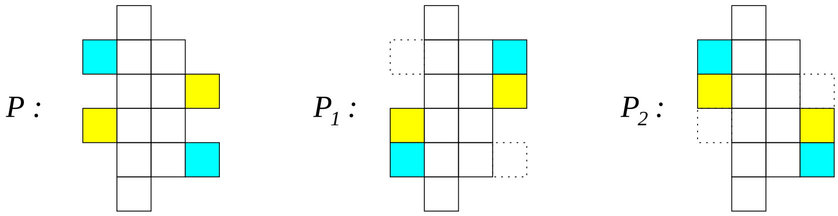


Figure 7: Three polyominoes belonging to the class $\mathcal{U}(H, V)$, with $H = (1, 3, 3, 3, 3, 1)$ and $V = (2, 6, 4, 2)$.

Ryser's Theorem furnishes an invaluable tool for proving uniqueness results on polyominoes: for example, one can easily check that the hv-convex ones can not be characterized by their horizontal and vertical projections, by showing simple and arbitrary large polyominoes containing switching components. On the other hand, in the next paragraph we prove that if we specialize our studies to the subclass of L-convex polyominoes, then no switching components survive.

Such a specialization allows also a fast reconstruction algorithm to be successfully applied: while hv-convex polyominoes admit an $O(mn \min(m^2, n^2))$ reconstruction, on the other hand the subclass of centered polyominoes, to which L-convex polyominoes belong, can be reconstructed in $O(m + n)$ (see [6] for both the results).

3 A characterization theorem for L-convex polyominoes

In this section we furnish a series of lemmas that finally yields to a characterization of L-convex polyominoes in terms of horizontal and vertical projections.

LEMMA 1 *An L-convex polyomino P is uniquely determined by its horizontal and vertical projections.*

Proof. According to Theorem 2 we achieve the uniqueness of P by proving that it does not contain any switching component. So, let us assume that there exists a switching component involving the two cells A and B of P, in positions (i, j) and (i', j'), respectively, where $i \neq i'$ and $j \neq j'$. By definition of switching, the two cells in positions (i, j') and (i', j) do not belong to P, and consequently a monotone path Π_{AB} having at most one change of direction does not exist. This fact contradicts the hypothesis of L-convexity of P. □

LEMMA 2 *Let j and j' be two different columns of an L-convex polyomino P, such that $v_j \leq v_{j'}$. For each row i of P, if $(i, j) \in P$, then $(i, j') \in P$.*

Proof. Let us proceed by contradiction and assume that there exists a row i' of P such that $(i', j) \in P$ and $(i', j') \notin P$. Since $v_j \leq v_{j'}$, there exists a row i'' such that $(i'', j) \notin P$ and $(i'', j') \in P$. These four cells form a switching component, a contradiction by Lemma 1. □

Obviously a result similar to that of Lemma 2 holds if $v_j > v_{j'}$, and, furthermore, if we replace the two different columns of P with two of its rows.

We define an integer vector $X = (x_1, \ldots, x_k)$ to be *unimodal*, if there exists $0 \leq i \leq k$, such that $x_1 \leq x_2 \leq \cdots \leq x_i$ and $x_i \geq x_{i+1} \geq \cdots \geq x_k$.

LEMMA 3 *If P is an L-convex polyomino then its horizontal and vertical projections are unimodal.*

Proof. Let P be an L-convex polyomino belonging to $\mathcal{U}(H, V)$, with $H \in \mathbb{N}^m$ and $V \in \mathbb{N}^n$. By Theorem 1, each element h_i of H is the basis of a maximal rectangle of P. Let us proceed by contradiction and assume H to be non-unimodal, i.e. there exist $1 \leq i < j < k \leq m$ such that $h_j < h_k$ and $h_j < h_i$. The following three cases arise:

$h_i = h_k$: the cells of P lying on row i and row k belong to the same maximal rectangle, so $h_j \geq h_i$, a contradiction;

$h_i < h_k$: the two values h_i and h_j are the bases of two different maximal rectangles. Since each pair of maximal rectangles has crossing intersection, then $h_j \geq h_i$, a contradiction;

$h_i > h_k$: analogous to the previous case.

Since each element v_j of V, with $j = 1, \ldots, m$, is the height of a maximal rectangle of P, a similar reasoning leads to prove that also V is unimodal. □

The properties stated in Lemmas 2 and 3 directly follow from the definition of L-convexity. A less intuitive result is the characterization of L-convex polyominoes by means of the uniqueness and monotonicity of its projections.

THEOREM 3 *Let S be a discrete set in $\mathcal{U}(H, V)$, with $H \in \mathbb{N}^m$ and $V \in \mathbb{N}^n$.*

$$\left.\begin{array}{l} H \text{ and } V \text{ are unimodal} \\ H \text{ and } V \text{ are unique} \end{array}\right\} \Leftrightarrow S \text{ is an L-convex polyomino.}$$

Remind that the two vectors H and V are unique if there exists exactly one discrete set having them as horizontal and vertical projections, respectively.

Proof. ($\Rightarrow$) We prove by contradiction the h-convexity of S: let us assume that there exist three cells $(i,j) \in S$, $(i,j') \notin S$ and $(i,j'') \in S$, with $i < i' < i''$. The unimodality of V allows the following three cases:

$v_j \geq v_{j'} \geq v_{j''}$: Lemma 2 applied to columns j'' and j', implies that $(i,j') \in S$, which is clearly a contradiction;

$v_j \leq v_{j'} \leq v_{j''}$: Lemma 2 applied to columns j and j', implies that $(i,j') \in S$, a contradiction;

$v_j \leq v_{j'}$ and $v_{j'} \geq v_{j''}$: Lemma 2 applied or to columns j and j', or to columns j'' and j' implies that $(i,j') \in S$, again a contradiction.

A similar reasoning leads to the v-convexity of S.

Finally, for any pair of cells (i,j) and (i',j') belonging to S, the uniqueness of S implies that $(i',j) \in S$ or $(i,j') \in S$, so the cells (i,j) and (i',j') can be connected by a path having at most one change of direction. This determines the connectedness of S (and consequently that S is a polyomino), together with its L-convexity.

($\Leftarrow$) The result follows from Lemmas 1 and 3. □

The following remark is a direct consequence of the proof of Theorem 3:

Remark 1 A discrete set is convex and unique if and only if it is L-convex.

3.1 2-convex polyominoes: a quick excursus

In this section we consider the class of 2-*convex polyominoes*, which constitutes the second level in the hierarchy of convex polyominoes defined in Section 2, and we observe that a characterization analogous to that of Theorem 3 does not hold.

Let us consider the 2-convex polyomino depicted in Fig. 8 (a); we notice that the application of the highlighted switching operator produces the polyomino in Fig. 8 (b) which is not 2-convex.

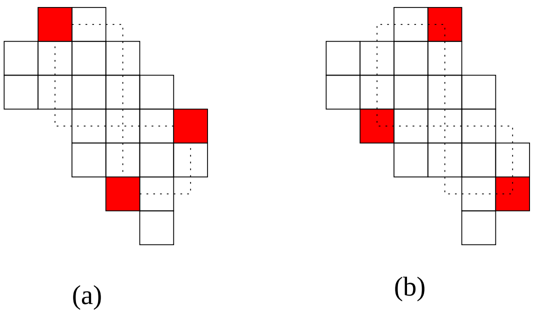


Figure 8: The highlighted switching transforms the 2-convex set (a) into the non 2-convex set (b).

This observation prevents an analogue of Lemma 1 to hold since the very beginning of the hierarchy, i.e. for any k-convex polyomino, with $k > 1$.

Furthermore, the same picture shows a stronger negative result: given two projections H and V, the set $\mathcal{U}(H, V)$ may contain both 2-convex and non 2-convex polyominoes.

Finally, Fig. 9 shows a 2-convex polyominoes whose vertical projections are not unimodal.

It is worthwhile to prove the existence of 2-convex polyominoes having an arbitrary number of peak values in the projections, consequently also Lemma 3 is strictly connected with L-convexity. Hence,

up to now, no tomographical characterization of 2*-convex polyominoes is known.*

3.2 Centered convex polyominoes.

An interesting subclass of 2-convex polyominoes which includes the L-convex polyominoes and which partially maintains the properties stated in Lemmas 2 and 3 is that of *centered convex polyominoes.*

Following [6], a convex polyomino is said to be *horizontally [vertically] centered* if there is at least one row [column] touching both the left [bottom] and right [top] side of its minimal bounding rectangle (Figure 9 depicts an horizontally centered polyomino which is not L-convex).

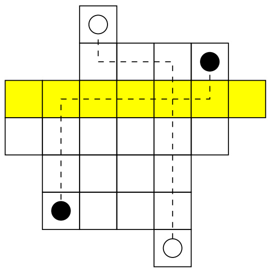


Figure 9: A centered convex polyomino: the central row is highlighted. Two paths of the form $(north)^3(east)^4(north)^1$ and $(north)^5(west)^2(north)^1$ are also pointed out.

It can be easily verified that horizontally centered polyominoes, say C_h, can be characterized as the class of convex polyominoes such that each couple of cells can be connected by a path either of the form

$$(north)^{i_1}(east)^{i_2}(north)^{i_2} \quad \text{or} \quad (north)^{i_1}(west)^{i_2}(north)^{i_3},$$

with $i_1, i_2, i_3 \geq 0$ (see again Fig. 9).

A symmetrical characterization can be furnished for the class of vertically centered polyominoes C_v.

This property implies that the class of 2-convex polyominoes includes both C_h and C_v, and that their union, on its turn, properly includes the class of L-convex polyominoes.

Concerning what stated in Lemmas 2 and 3, we give two positive results:

Proposition 2 *Let P and P' be two convex polyominoes having the same horizontal and vertical projections. If P belongs to $\mathcal{C}_h$ [$\mathcal{C}_v$], then also P' belongs to $\mathcal{C}_h$ [$\mathcal{C}_v$].*

Proposition 3 *If $P \in \mathcal{C}_h$ [$P \in \mathcal{C}_v$], then the vector of its horizontal [vertical] projections is unimodal.*

Property 2 allows the authors of [6] to define a linear time algorithm for reconstructing centered convex polyominoes which specializes a slowest one for the class of convex polyominoes.

4 Extension to measurable plane sets

In this section, we introduce the concept of L-convex plane set in order to extend to the continuum environment the uniqueness results stated in Section 3.

In the case of generic measurable plane sets, G.G. Lorentz gave in [14] necessary and sufficient conditions for a pair of projections to be respectively unique, non-unique and consistent. These results were obtained by using analytic transformations of the projection functions. Further studies considered the same problem from a geometrical point of view, with the aim of defining a switching theory which translates in the continuum what was introduced for discrete sets. In particular in [12], the authors introduced the notion of switching components in the continuum, and stated a result similar to Theorem 2. Furthermore, they furnished other nice characterizations of plane sets related to their geometrical properties. In this section we will often rely on these works in order to support our results.

So, let us start by recalling the following standard definitions: a set S of $\mathbb{R}^2$ is called *h-convex* (resp. *v-convex*) if, for each pair of points $(x, y), (u, v) \in S$, with $y = v$ (resp. $x = u$), the horizontal (resp. vertical) line segment which joins them is entirely contained in S. We call *hv-convex* the plane sets that are both h-convex and v-convex.

Furthermore, a *step polygon* is a self-avoiding polygonal curve containing only horizontal and vertical line segments. A step polygon joining two distinct points $(x, y), (u, v) \in \mathbb{R}^2$ can be represented as a finite sequence of vertices $(x_0, y_0), (x_1, y_1), \ldots, (x_k, y_k)$ such that each vertex is connected by a line segment to the next one, $(x_0, y_0) = (x, y)$, and $(x_k, y_k) = (u, v)$. To our purpose, line segments are the continuum counterpart of the four kinds of steps defined in Section 2 for the discrete lattice, and so they can be classified as *north*, *south*, *east* and *west segments* as well. A step polygon is called *monotone* if it is composed of at most two different kinds of these segments.

Hence, we have the following natural translation of Proposition 1 to *hv*-convex plane sets:

Proposition 4 *A plane set S is hv-convex iff every pair of points in S can be joined by a monotone step polygon lying in S.*

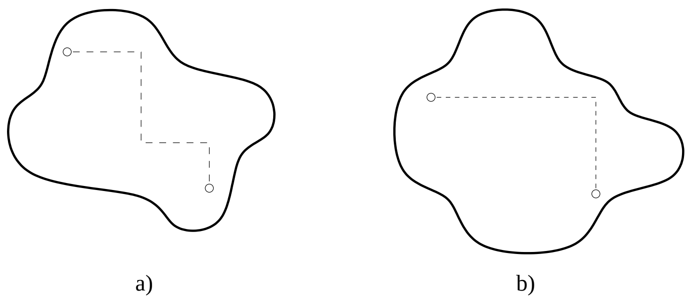


Figure 10: Set a): an hv-convex plane set and a monotone step polygon lying inside it. Set b): an L-convex plane set, and two of its cells joined with a three vertex monotone step polygon.

Now we can finally define a plane set S to be *L-convex* if each pair of its points can be joined by a monotone step polygon with at most three vertices, and entirely contained in S (see Fig. 10).

4.1 A characterization theorem for L-convex plane sets

A function $f(x)$, defined in the interval $[a,b] \subset \mathbb{R}$, is *unimodal* if there exists $\overline{x} \in [a,b]$ (called *mode*) such that $f(x)$ increases from a to $\overline{x}$ and decreases from $\overline{x}$ to b.

Let $S \subseteq \mathbb{R}^2$ be a measurable set such that $\lambda_2(S) < \infty$ (λ_2 being the two dimensional Lebesgue measure), and let $f(x,y)$ be its characteristic function. Using notations and definitions from [12], we call *horizontal projection* of S the function

$$f_x(y) = \int_{-\infty}^{\infty} f(x,y)\,\mathrm{d}x \tag{1}$$

and *vertical projection* of S the function

$$f_y(x) = \int_{-\infty}^{\infty} f(x,y)\,\mathrm{d}y. \tag{2}$$

These functions exist almost everywhere on $\mathbb{R}$ and they are integrable (Fubini's theorem).

In [12], it is introduced a notion of switching components in the continuum which naturally extends the one for discrete sets.

In particular, let t,u be two real numbers. The sets

$$S(t,0) = \{(x,y) \mid (x-t,y) \in S\}$$
$$S(0,u) = \{(x,y) \mid (x,y-u) \in S\}$$

are called horizontal and vertical translation of S, respectively.

We say that S admits a *switching component* if there exist four sets A, B, C, D and two real numbers t and u such that $B \cup C \subseteq S$, $A \cup D \cap S = \emptyset$, and such that $B = A_{(t,0)}$, $C = A_{(0,u)}$ and $D = A_{(t,u)}$.

We have that if S admits a switching component, then S is not uniquely determined by its projections, in fact the set

$$S' = (S - (B \cup C)) \cup (A \cup D)$$

is different from S, and it has its same horizontal and vertical projections.

The existence of a switching component is also a necessary condition to guarantee the non-uniqueness of the set S (see [12] for a proof), so we have the following result, analogous to Lemma 1:

THEOREM 4 *A measurable plane set having finite measure is non-uniquely determined by its projections iff it has a switching component.*

Finally, L-convexity of a plane set causes the existence of a mode both in its horizontal and in its vertical projections:

LEMMA 4 *If a plane set is L-convex, then both its horizontal and its vertical projections are unimodal.*

The proof can be simple inferred from that of Lemma 3. As a consequence we can obtain, for the continuous case, the same characterization result as for discrete sets:

THEOREM 5 *Let f_x and f_y be projection functions defined in $\mathbb{R}^2$ of a plane set S. It holds that*

$$\left.\begin{array}{l} f_x \textit{ and } f_y \textit{ are unimodal} \\ f_x \textit{ and } f_y \textit{ are unique} \end{array}\right\} \Leftrightarrow \quad S \textit{ is L-convex.}$$

A last remark is needed: in [12], a different and interesting characterization of unique plane sets is provided. Let S be a measurable plane set of finite measure. The rectangle $X \times Y$ is *measurably inscribed* (briefly m-inscribed) in S if

$$X \times Y \subseteq S \quad \text{and} \quad \overline{X} \times \overline{Y} \subseteq \overline{S}\,.$$

The set S is *m-inscribable* if it is the union of m-inscribed rectangles. We can immediately argue that the presence of m-inscribed rectangles inside the set S is similar to the presence of maximal rectangles inside an L-convex polyomino.

This idea is strengthened by the fact that, using Theorem 4, in [12] it is proved the following

THEOREM 6 *A measurable plane set having finite measure is uniquely determined by its projection functions iff it is m-inscribable.*

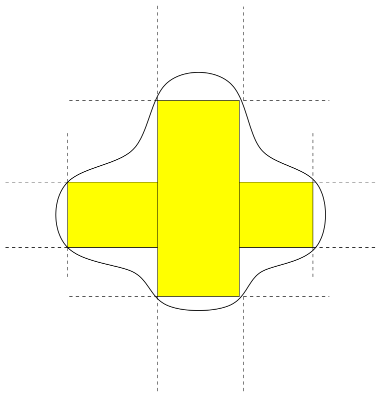


Figure 11: Two m-inscribed rectangles inside an L-convex plane set.

We observe that the notion of crossing intersection in the continuum environment leads to the equivalence between L-convex plane sets and m-inscribable plane sets. In fact, at the same time, we obtain a nice generalization of Theorem 1 and an uniqueness result.

THEOREM 7 *Let S be a measurable plane set. It holds that S is L-convex iff S is m-inscribable by rectangles with crossing intersection.*

5 Generalizations and further work

In this work we have proposed a characterization of L-convex sets in terms of features relevant to discrete tomography. As a main result, we proved that each L-convex set is unique with respect to its horizontal and vertical projections, and that both the projections have a unimodal behavior. The characterization of the class is then achieved after showing that these two properties are also sufficient to obtain an L-convex set.

The second part of the paper concerns the natural extension of our main result to the continuum environment.

Finally we would like to point out some open problems: first, one can ask whether similar tomographical characterizations can be determined when the definition of L-convexity is slightly changed, for example by considering two or more directions different from (or possibly strictly including) the horizontal and the vertical ones.

A different line of research could be the extension of the notion of L-convexity to the three dimensional lattice. In this case the main problem is to choose a good definition of three dimensional L-convex polyominoes. In fact, while in the two-dimensional environment, as pointed out in Section 2.1, the definition of L-convex polyominoes given in terms of monotone path coincides with the one that uses maximal rectangles, this property is lost in three dimensions.

5.1 L-convexity along more directions

A natural generalization of the notion of L-convexity can be obtained by adding to the horizontal and vertical steps, others along different directions. As an example, we can consider the two new directions $v_1 = (1, 1)$ and $v_2 = (-1, 1)$, and then enlarge the set of the admissible steps using also diagonal unitary steps, as shown in Fig. 12.

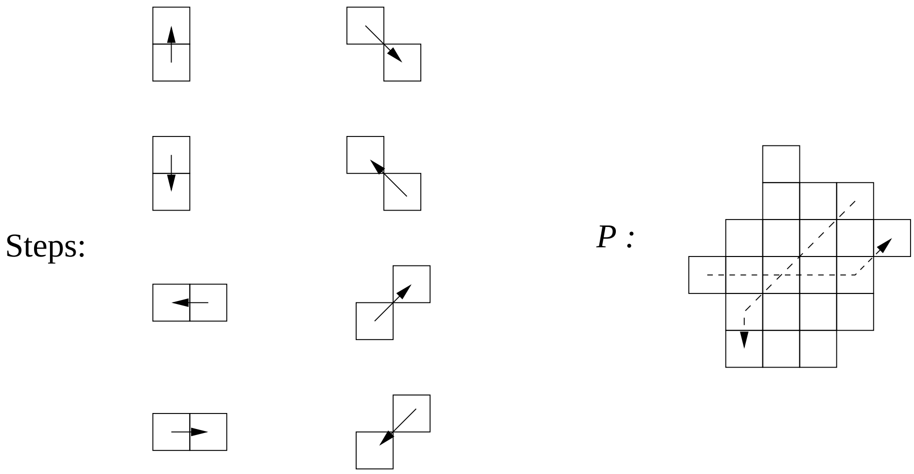


Figure 12: The polyomino P is L-convex with respect to the four directions $v = (0, 1)$, $h = (1, 0)$, $v_1 = (1, 1)$, and $v_2 = (-1, 1)$. In the polyomino are pointed out two of its paths.

In this case, similarly to what proved for 2-convex polyominoes, horizontal and vertical projections do not provide enough information to characterize a polyomino (see Fig. 13). Further studies are addressed to minimize the number of projections needed to assure uniqueness with respect to the set of steps allowed in the L-convexity definition.

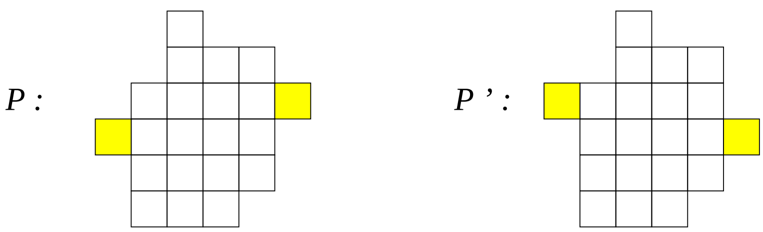


Figure 13: Two L-convex polyominoes with respect to the four directions $v = (0, 1)$, $h = (1, 0)$, $v_1 = (1, 1)$, and $v_2 = (-1, 1)$, having the same horizontal and vertical projections.

5.2 Higher dimensional cases

What follows is one of the possible attempts to generalize two dimensional tomographical definitions and results to higher dimension lattices.

The definition of projection in $\mathbb{Z}^2$ can be easily generalized to $\mathbb{Z}^t$: let $1 \leq p \leq t-1$, and L be a p-dimensional subspace of $\mathbb{Z}^t$. The p-dimensional projections of the set S parallel to L are defined as the cardinalities of the intersections of the set S with each p-dimensional plane parallel to L. The horizontal and vertical projections studied in this paper are special cases when $p = 1$. Keeping $p = 1$, it should be worth studying the projections of a k-convex polycube (the analog of a polyomino when the dimension t of the space is greater than two) along the canonical directions of $\mathbb{Z}^t$, with the aim of obtaining a characterization result similar to that of Theorem 3 in the case of $\mathbb{Z}^2$.

Let S be a subset of $\mathbb{Z}^t$, with $t \geq 3$. We define S to be a polycube if it is composed by connected t-dimensional cubes of unitary side. The notions of unitary steps and path naturally extend those in $\mathbb{Z}^2$, and let us define a path to be monotone if its projection on each hyperplane generated by the canonical directions of $\mathbb{Z}^t$ is a (two dimensional) monotone path. In the same way, the orthotope, that we denote by $[x_1, x_2, \ldots, x_t]$, with $x_i \in \mathbb{N}$, extends the concept of rectangular polyomino. We say that two orthotopes $[x_1, x_2, \ldots, x_t]$ and $[y_1, y_2, \ldots, y_t]$ has crossing intersection if

$$[x_1, x_2, \ldots, x_t] \cap [y_1, y_2, \ldots, y_t] = [\min\{x_1, y_1\}, \min\{x_2, y_2\}, \ldots, \min\{x_t, y_t\}].$$

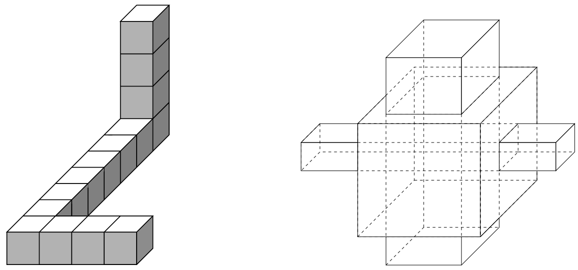
The definition of k-convexity requires a little bit more attention: below two different versions are proposed.

In the first approach we define a polycube to be k-convex if each pair of its cubes can be connected by a monotone path having at most k changes of direction for each one of the t canonical hyperplanes of $\mathbb{Z}^t$.

According to this definition, a 1-convex polycube cannot be characterized as union of maximal orthotopes having pairwise crossing intersection. As an example, in Fig. 14, a), it is depicted a 1-path in $\mathbb{Z}^3$ which is a 1-convex polycube, and which contains two maximal non intersecting orthotopes. On the other hand, each planar projection on the canonical hyperplanes of a 1-convex polycube in $\mathbb{Z}^3$ in an L-convex polyomino in $\mathbb{Z}^2$, so, composed by rectangles having crossing intersection.

A second possibility is to explicitly define a k-convex polycube as the union of maximal orthotopes having pairwise crossing intersection (see Fig. 14, b)). In this case we obtain a subclass of the k-convex polycubes previously defined.

We finally point out that both the approaches beg for deeper investigations about the uniqueness and the reconstruction of the defined classes of polyominoes from two or more projections in higher dimensional environments.



a) *b*)

Figure 14: An 1-path in $\mathbb{Z}^3$ which is an 1-convex polycube, *a*), and a polycube which is the union of orthotopes having pairwise crossing intersection, *b*).

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