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Groups with finitely many normalizers of subgroups with intransitive normality relation

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Abstract. A group G is called a T-group if all its subnormal subgroups are normal. In this paper groups are investigated with finitely many normalizers of (infinite) subgroups which do not have the property T.

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1 Introduction

In a famous paper of 1955, B.H. Neumann [7] proved that each subgroup of a group G has finitely many conjugates if and only if the centre $Z(G)$ has finite index, and the same conclusion holds if the restriction is imposed only to conjugacy classes of abelian subgroups (see [3]). Thus central-by-finite groups are precisely those groups in which normalizers of (abelian) subgroups have finite index, and this result suggests that the behaviour of normalizers has a strong influence on the structure of a group. In fact, Y.D. Polovickiĭ [8] has shown that if an FC-group G has finitely many normalizers of infinite abelian subgroups, then the factor group $G/Z(G)$ is finite (recall that G is an *FC-group* if it has finite conjugacy classes of elements). Since it is very easy to prove that any group with finitely many normalizers of cyclic subgroups has the property FC, it follows from Polovickiĭ's theorem that a group has finitely many normalizers of abelian subgroups if and only if it is central-by-finite. Moreover, it has recently been proved that any (generalized) soluble group with finitely many normalizers of non-abelian subgroups has finite commutator subgroup (see [2]).

A group G is said to be a *T-group* if all its subnormal subgroups are normal, i.e. if normality in G is a transitive relation. The structure of finite soluble T-groups was described by W. Gaschütz [5], while D.J.S. Robinson [9] investigated infinite soluble groups with the property T; it turns out that soluble T-groups are metabelian and finitely generated infinite soluble groups with the property T are abelian. In the last few years, attention has been given to groups in which many subgroups have a transitive normality relation (see for instance [1], [6], [11]). In particular, in [11] groups were considered in which all non-normal subgroups are T-groups.

The aim of this paper is to study groups with finitely many normalizers of subgroups which do not have the property T. Among other results, it will be proved that any soluble group G with such restriction either has finite commutator subgroup or is a $\bar{T}$-group, i.e. all subgroups of G have the property T. A similar result also holds for soluble groups with finitely many normalizers of infinite non-T subgroups, provided that they do not satisfy the minimal condition on subgroups.

Most of our notation is standard and can for instance be found in [10].

2 Solubility

In this section we shall prove that groups with finitely many normalizers of infinite non-T subgroups are soluble-by-finite, provided that they have no infinite simple sections. Of course, infinite simple groups with such property exist, for instance Tarski groups.

Our first lemma shows that any group with finitely many normalizers of infinite non-T subgroups contains a subgroup of finite index in which all infinite subgroups which do not have the property T are subnormal with defect at most 2.

Lemma 2.1 *Let G be a group with finitely many normalizers of infinite non-T subgroups. Then G contains a characteristic subgroup M of finite index such that $N_M(X)$ is normal in M for each infinite subgroup X of M which is not a T-group.*

Proof. If X is any infinite non-T subgroup of G, its normalizer $N_G(X)$ has obviously finitely many images under automorphisms of G; in particular, the subgroup $N_G(X)$ has finitely many conjugates in G and so the index

$$|G : N_G(N_G(X))|$$

is finite. It follows that also the characteristic subgroup

$$M(X) = \bigcap_{\alpha \in \mathrm{Aut}\, G} N_G(N_G(X))^{\alpha}$$

has finite index in G. Let $\mathcal{H}$ be the set of all infinite subgroups of G which do not have the property T. If X and Y are elements of $\mathcal{H}$ such that $N_G(X) = N_G(Y)$, then $M(X) = M(Y)$, and hence also

$$M = \bigcap_{X \in \mathcal{H}} M(X)$$

is a characteristic subgroup of finite index of G. Let X be any infinite non-T subgroup of M. Then

$$M \leq M(X) \leq N_G(N_G(X)),$$

and so the normalizer $N_M(X) = N_G(X) \cap M$ is a normal subgroup of M. □

Lemma 2.2 *Let n be a positive integer and let G be a soluble group in which every subgroup either is subnormal with defect at most n or has the property T. Then the derived length of G is bounded by a function of n.*

Proof. The statement follows directly from Theorem 2.10 and Lemma 3.2 of [6]. □

Theorem 2.3 *Let G be a group with finitely many normalizers of infinite non-T subgroups. If G has no infinite simple sections, then it is soluble-by-finite.*

Proof. By Lemma 2.1 the group G contains a subgroup M of finite index in which every infinite subgroups either is subnormal with defect at most 2 or has the property T, and without loss of generality we may suppose that $M = G$. Assume for a contradiction that G is not soluble-by-finite, so that in particular the subgroup $G^{(n)}$ is infinite for each positive integer n. Thus every subgroup of $G/G^{(n)}$ either is subnormal of defect at most 2 or has the property T, and hence it follows from Lemma 2.2 that there exists a positive integer k, independent on n, such that $G/G^{(n)}$ has derived length at most k. Then the subgroup $N = G^{(k)}$ is the smallest term of the derived series of G. Let H be any infinite proper subgroup of N, and assume that H is not a T-group. Then H is subnormal in G and so every subgroup of G containing H cannot have the property T; it follows that all subgroups of G containing H are subnormal, and hence $G^{(m)} \leq H$ for some m (see [6], Lemma 2.4). This contradiction shows that all infinite proper subgroups of N have the property T, and so N is soluble-by-finite (see [1], Theorem A). Therefore G itself is soluble-by-finite, and this contradiction completes the proof of the lemma. □

The above result has the following consequence for groups whose non-normal subgroups have the property T.

Corollary 2.4 *Let G be an infinite group whose infinite non-normal subgroups are T-groups. If G has no infinite simple sections, then it is soluble.*

Proof. By Theorem 2.3 the group G contains a soluble normal subgroup S such that G/S is finite. As S is infinite, every non-normal subgroup of G/S has the property T, so that G/S is soluble (see [11]) and hence G itself is a soluble group. □

3 Main results

In the first part of this section we will characterize soluble groups whose infinite non-normal subgroups have the property T.

LEMMA 3.1 *Let G be a group whose infinite non-normal subgroups are T-groups, and let H be an infinite soluble non-normal subgroup of G. Then either H is a $\bar{T}$-group or it is an extension of the Prüfer 2-group $Z(2^\infty)$ by a finite T-group.*

Proof. As H is not normal in G, it is a T-group. Then all infinite subgroups of H likewise have the property T, and hence H either has the property $\bar{T}$ or is an extension of $Z(2^\infty)$ by a finite T-group (see [4], p. 579). □

LEMMA 3.2 *Let G be a finitely generated soluble group with finitely many normalizers of infinite non-T subgroups. Then G is nilpotent-by-finite.*

Proof. By Lemma 2.1 the group G contains a characteristic subgroup M of finite index in which every subgroup either is subnormal or has the property T. Assume that M neither is finite nor nilpotent. Then M has a finite non-nilpotent homomorphic image (see [10] Part 2, Theorem 10.51), and hence it contains a non-subnormal subgroup X of finite index. Clearly, X is a finitely generated infinite soluble T-group and so it is abelian. The lemma is proved. □

COROLLARY 3.3 *Let G be a soluble group with finitely many normalizers of infinite non-T subgroups. Then G is locally polycyclic.*

THEOREM 3.4 *Let G be a soluble group whose infinite non-normal subgroups are T-groups. If G is not a Černikov group, then all non-normal subgroups of G have the property T, and in particular either G is a $\bar{T}$-group or its commutator subgroup G' is finite.*

Proof. Let X be any finite non-T subgroup of G, and suppose first that G contains an element a of infinite order. The finitely generated group $H = \langle X, a\rangle$ is polycyclic by Corollary 3.3, and in particular X is intersection of subgroups of finite index of H. On the other hand, if K is a subgroup of finite index of H and $X \leq K$, we have clearly that K is an infinite finitely generated non-abelian group, so that K cannot have the property T and hence it is normal in G. Therefore X itself is a normal subgroup of G in this case.

Suppose now that G is periodic. Since G is not a Černikov group, it contains an abelian subgroup A such that $A^X = A$ and A does not satisfy the minimal condition on subgroups (see [12]). Let S be the socle of A. Then S has finite index in SX and so each subgroup of S has finitely many conjugates in SX. It follows that there exist infinite X-invariant subgroups S_1 and S_2 of S such that

$$S_1 \cap S_2 = \langle S_1, S_2\rangle \cap X = \{1\}.$$

By Lemma 3.1 both subgroups S_1X and S_2X are normal in G, and hence $X = S_1X \cap S_2X$ is likewise a normal subgroup of G. Therefore all non-normal subgroups of G have the property T. In particular either G is a $\bar{T}$-group or G' is finite (see [11], Theorem 5 and Lemma 9). □

The consideration of the locally dihedral 2-group shows that the above theorem cannot be extended to the case of Černikov groups.

Lemma 3.5 *Let G be a group with finitely many normalizers of infinite non-T subgroups, and let A be a torsion-free abelian normal subgroup of G. Then A is contained in the centre of G.*

Proof. Assume by contradiction that the statement is false, and choose a counterexample G with a minimal number k of proper normalizers of infinite non-T subgroups. Let x be an element of G such that $[A, x] \neq \{1\}$. Consider an element a of A such that $ax \neq xa$, and put $H = \langle a, x\rangle$ and $B = A \cap H$. The soluble group $\langle A, x\rangle$ is polycyclic by Corollary 3.3. Suppose first that $B \cap \langle x\rangle = \{1\}$. Clearly, there exists a prime number p such that $[B^{p^n}, x] \neq \{1\}$ for all positive integers n, and the infinite subgroup $\langle B^{p^n}, x\rangle$ does not have the property T. It follows that $\langle B^{p^n}, x\rangle$ must be normal in G, since otherwise its normalizer would be a counterexample with less than k proper normalizers of infinite non-T subgroups. Thus

$$\langle x\rangle = \bigcap_{n\in\mathbb{N}} \langle B^{p^n}, x\rangle$$

is also a normal subgroup of G and so $[B, x] = \{1\}$, a contradiction. Suppose now that $B \cap \langle x\rangle = \langle x^r\rangle \neq \{1\}$, and write

$$B/B \cap \langle x\rangle = E/B \cap \langle x\rangle \times C/B \cap \langle x\rangle,$$

where $E/B \cap \langle x\rangle$ is finite and $C/B \cap \langle x\rangle$ is torsion-free. As $B \cap \langle x\rangle$ is contained in $Z(H)$ and $\langle E, x\rangle/B \cap \langle x\rangle$ is finite, by Schur's theorem we have that $[E, x]$ is a finite subgroup of B and so $[E, x] = \{1\}$. On the other hand, B/E is a torsion-free abelian normal subgroup of H/E and $\langle xE\rangle \cap B/E = \{1\}$, so that it follows from the first part of the proof that $[B, x] \leq E$ and so $[B, x, x] = \{1\}$. Therefore

$$[B, x]^r = [B, x^r] = \{1\}$$

and hence $[B, x] = \{1\}$. This last contradiction completes the proof of the lemma. □

If G is any group with periodic commutator subgroup and A is a torsion-free abelian normal subgroup of G, we have obviously that A is contained in the centre of G. Our next lemma shows that the imposition of this property to all factor groups characterizes groups with periodic commutator subgroup, at least within the universe of soluble groups.

Lemma 3.6 *A soluble group G has periodic commutator subgroup if and only if every torsion-free abelian normal section of G is central.*

Proof. Assume that the statement is false and choose a soluble group G with minimal derived length such that all torsion-free abelian normal sections of G are central and the commutator subgroup G' is not periodic. If T is the largest periodic normal subgroup of G, replacing G by the factor group G/T it can be assumed without loss of generality that G has no periodic non-trivial normal subgroups. Let H be the smallest non-trivial term of the derived series of G and let A be a maximal abelian normal subgroup of G containing H. By the minimal choice of G, the factor group G/A has periodic commutator subgroup and in particular its elements of finite order form a subgroup K/A. Clearly, A is torsion-free, and so it is contained in $Z(G)$. Thus $K/Z(K)$ is periodic and hence K' is likewise periodic by Schur's theorem. It follows that K is abelian, so that $K = A$ and G/A is torsion-free. Then G/A is abelian and G is nilpotent. Therefore $C_G(A) = A$ and $G = A$ is abelian. This contradiction proves the statement. □

COROLLARY 3.7 *Let G be a soluble group with finitely many normalizers of infinite non-T subgroups. Then the commutator subgroup G' of G is periodic.*

Proof. As the hypotheses are inherited by homomorphic images, the statement follows directly from Lemma 3.5 and Lemma 3.6. □

It was proved by R. Baer that any group covered by finitely many abelian subgroups is central-by-finite (see [10] Part 1, Theorem 4.16). As a consequence of this result, we can prove the following lemma on groups covered by subgroups with finite commutator subgroup.

LEMMA 3.8 *Let the group G have a finite covering consisting of finite-by-abelian subgroups. Then G is finite-by-abelian.*

Proof. By a result of B.H. Neumann (see [10] Part 1, Lemma 4.17) the group G is covered by finitely many finite-by-abelian subgroups $X_1, \dots, X_t$ of finite index. Then for each i the commutator subgroup X_i' is finite and has finitely many conjugates, so that its normal closure $(X_i')^G$ is finite by Dietzmann's Lemma. It follows that

$$E = \langle (X_1')^G, \dots, (X_t')^G \rangle$$

is a finite normal subgroup of G and the factor group G/E has a finite covering consisting of abelian subgroups, so that G/E is central-by-finite by the theorem of Baer. Application of Schur's theorem yields that G' is finite. □

We can now prove the main result of the paper.

THEOREM 3.9 *Let G be a soluble group with finitely many normalizers of infinite non-T subgroups. If G is not a Černikov group, then either G has the property $\bar{T}$ or its commutator subgroup G' is finite.*

Proof. By Theorem 3.4 the statement is true if all infinite non-normal subgroups of G have the property T. Thus it can be assumed that G contains infinite subgroups which neither are normal nor T-groups. Let

$$N_G(X_1), \dots, N_G(X_k)$$

be the proper normalizers of infinite non-T subgroups of G and suppose first that

$$G = N_G(X_1) \cup \dots \cup N_G(X_k).$$

Then we have also

$$G = N_G(X_{i_1}) \cup \dots \cup N_G(X_{i_t}),$$

where each $N_G(X_{i_j})$ has finite index in G and in particular is not a Černikov group (see [10] Part 1, Lemma 4.17). Clearly, every $N_G(X_{i_j})$ is not a $\bar{T}$-group and has less than k proper normalizers of infinite non-T subgroups, so that by induction on k the commutator subgroup $N_G(X_{i_j})'$ is finite for each j. It follows that G is covered by finitely many finite-by-abelian subgroups, and hence G' is finite by Lemma 3.8.

Suppose now that $N_G(X_1) \cup \dots \cup N_G(X_k)$ is a proper subset of G, and let x be an element of the set

$$G \setminus \big(N_G(X_1) \cup \dots \cup N_G(X_k)\big),$$

so that $xy \neq yx$ for some $y \in G$. Assume that G is periodic. Since G is not a $\bar{T}$-group, it contains a finite subgroup E which is not a T-group, and each finite subgroup H of G containing $X = \langle E, x\rangle$ is not a T-group, as finite soluble T-groups have the property $\bar{T}$. The group G contains an abelian subgroup A such that $A^H = A$ and A does not satisfy the minimal condition on subgroups (see [12]). Let S be the socle of A. Then S has finite index in SH and so it contains infinite H-invariant subgroups S_1 and S_2 such that

$$S_1 \cap S_2 = \langle S_1, S_2\rangle \cap H = \{1\}.$$

Clearly, the subgroup S_1H is residually finite, and so it contains a normal subgroup K of finite index such that $H \cap K = \{1\}$. It follows that $HK/K \simeq H$ is not a T-group and hence S_1H/K likewise does not have the property T. Thus S_1H is a normal subgroup of G. The same argument shows that S_2H is normal in G so that also the subgroup $H = S_1H \cap S_2H$ is normal in G. Therefore X is normal in G and G/X is a Dedekind group, so that G' is finite.

Assume finally that G contains an element a of infinite order. If g is any element of G, the finitely generated subgroup $\langle x, y, a, g\rangle$ is not a T-group and so it is normal in G. In particular, the subgroup $[G, g]$ is contained in $\langle x, y, a, g\rangle$ and hence it is finite by Corollary 3.3 and Corollary 3.7. This shows that G is an FC-group; in particular, the centralizer $C_G(x)$ has finite index in G and so it is not periodic. Let W be any infinite non-T subgroup of $C_G(x)$; as x belongs to $N_G(W)$, we have that W is a normal subgroup of G. It follows that all infinite non-normal subgroups of $C_G(x)$ have the property T, and so the commutator subgroup of $C_G(x)$ is finite by Theorem 3.4. Therefore G is a finite-by-abelian-by-finite FC-group, and hence G' is finite. The theorem is proved. □

It is known that if G is a soluble group whose non-normal subgroups have the property T, then either G is a $\bar{T}$-group or its commutator subgroup is finite (see [11], Theorem 5 and Lemma 9). Thus arguments similar to those used in the proof of Theorem 3.9 give the following result.

THEOREM 3.10 *Let G be a soluble group with finitely many normalizers of non-T subgroups. Then either G is a $\bar{T}$-group or its commutator subgroup G' is finite.*

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