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Characterization and enumeration of some classes of permutominoes

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Abstract. A permutomino of size n is a polyomino whose vertices define a pair of distinct permutations of length n. In this paper we treat various classes of convex permutominoes, including the parallelogram, the directed convex and the stack ones. Using bijective techniques we provide enumeration for each of these classes according to the size, and characterize the permutations which are associated with permutominoes of each class.

Mathematics Subject Classifications (2000). 05A15, 05A05

1 Introduction

In the plane $\mathbb{Z} \times \mathbb{Z}$ a *cell* is a unit square and a *polyomino* is a finite connected union of cells having no cut point. Polyominoes are defined up to translations. A *column* (*row*) of a polyomino is the intersection between the polyomino and an infinite strip of cells whose centers lie on a vertical (horizontal) line. The enumeration problem for general polyominoes is difficult to solve and still open.

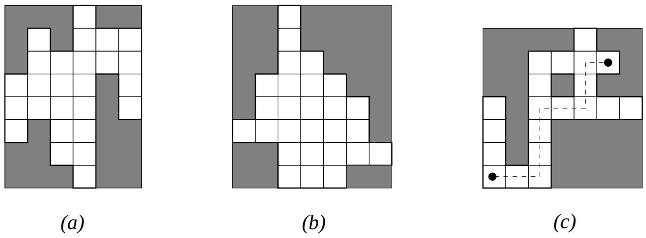


Figure 1: (a) a column-convex polyomino; (b) a convex polyomino; (c) a directed (not convex) polyomino with a hole.

The number a_n of polyominoes with n cells is known up to $n = 56$ [6] and asymptotically, this number satisfies the relation $\lim_{n\to\infty}(a_n)^{1/n} = \mu$, $3.96 < \mu < 4.64$, where the lower bound is a recent improvement of [1].

In order to simplify many problems which are still open on the class of polyominoes, several subclasses were defined by combining two notions: the geometrical notion of *convexity* and the notion of *directed growth*, which comes from statistical physics. A polyomino is said to be *column-convex* [*row-convex*] when its intersection with any vertical [horizontal] line is convex (Figure 1 (a)). A polyomino is *convex* if it is both column and row convex (Figure 1 (b)). In a convex polyomino the *semi-perimeter* is given by the sum of the number of rows and columns, while the area is the number of its cells.

A polyomino P is said to be *directed* when every cell of P can be reached from a distinguished cell, called the *root* (usually the leftmost at the lowest ordinate), by a path which is contained in P and uses only north and east unit steps (Figure 1 (c)). Figure 2 (d) depicts a polyomino which is both directed and convex. Moreover we can define three types of directed and convex polyominoes: the *Ferrers diagrams*, the *parallelogram polyominoes* and the *stack polyominoes* (see Figure 2). Each of these three subsets can be characterized, in the set of convex polyominoes, by the fact that two or three vertices of the minimal bounding rectangle of the polyomino must also belong to the polyomino itself.

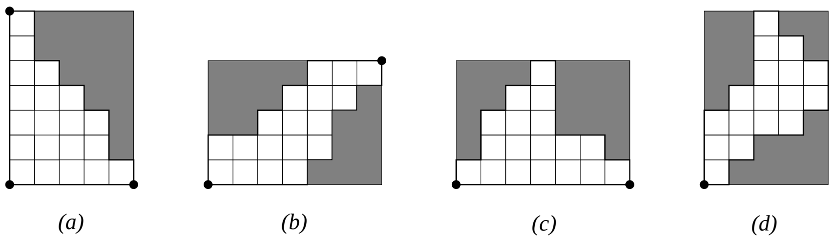


Figure 2: (a) A Ferrers diagram; (c) A parallelogram polyomino; (c) A stack polyomino; (d) A directed convex polyomino which is neither a parallelogram nor a stack one.

The number f_n of convex polyominoes with semi-perimeter $n \geq 2$ was obtained by Delest and Viennot in [3]:

$$f_{n+2} = (2n+11)4^n - 4(2n+1)\binom{2n}{n}, \quad n \geq 0; \quad f_0 = 1, \quad f_1 = 2.$$

Moreover, it is well known [14] that the number of parallelogram polyominoes with semi-perimeter $n \geq 2$ is equal to the $(n-1)$th *Catalan number*, where Catalan numbers are defined by

$$c_n = \frac{1}{n+1}\binom{2n}{n}.$$

Finally, the number of directed convex polyominoes with semi-perimeter $n \geq 2$ is equal to b_{n-2}, where b_n are the *central binomial coefficients*

$$b_n = \binom{2n}{n}.$$

2 Permutominoes

Permutominoes were introduced by F. Incitti in [5] while studying the problem of determining the $\widetilde{R}$-polynomials (related with the Kazhdan–Lusztig R-polynomials) associated with a pair (π_1, π_2) of permutations. Here we recall the basic definitions and the main combinatorial properties of these objects.

2.1 Definitions and basic properties

Let P be a polyomino without holes, having n rows and n columns, $n \geq 1$; we assume without loss of generality that the south-west corner of its minimal bounding square is placed in $(1,1)$. Let $\mathcal{A} = \big(A_1, \ldots, A_{2(r+1)}\big)$ be the list of its vertices (i.e., corners of its boundary) ordered in a clockwise sense starting from the lowest leftmost vertex, and let $\mathcal{P}_1 = (A_1, A_3, \ldots, A_{2r+1})$ and $\mathcal{P}_2 = (A_2, A_4, \ldots, A_{2r+2})$.

We say that P is a *permutomino* if the following conditions hold:

(i) $r = n$;

(ii) there are no two points in $\mathcal{P}_1$ having the same abscissa or having the same ordinate, and the same holds for $\mathcal{P}_2$.

The number $r+1$ is called the *size* of the permutomino. In other words we say that P is a permutomino if on each abscissa and on each ordinate of its minimal bounding rectangle there lies exactly one side of P, or equivalently, exactly one element of $\mathcal{P}_1$, and exactly one element of $\mathcal{P}_2$.

It is straightforward that both the points of $\mathcal{P}_1$ and those of $\mathcal{P}_2$ can be regarded as elements of a permutation matrix, as shown in Fig. 3. We choose to indicate the two permutations defined by them by $\pi_1(P)$ and $\pi_2(P)$, respectively. We say that a permutomino P *is associated with* the pair of permutations $(\pi_1(P), \pi_2(P))$ (briefly, (π_1, π_2)); for simplicity the ith element of π_1 (resp. π_2) will be denoted by $\pi_1(i)$ (resp. $\pi_2(i)$). Figure 3 shows two permutominoes and the associated permutations; we remark that the size of the permutomino – i.e. the length of π_1 and π_2 – is equal to 5, while the side of the bounding square has length equal to 4.

From the definition any permutomino P of size $n \geq 2$ has the property that, for each abscissa (ordinate) between 1 and n there is exactly one vertical (horizontal) side in the boundary of P with that coordinate. It is simple to observe that this property is also a sufficient condition for a polyomino to be a permutomino.

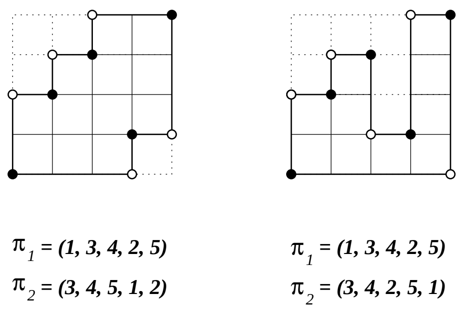


Figure 3: Two permutominoes and the associated permutations. The permutation π_1 (resp. π_2) is represented by black (resp. white) dots.

Recently, some enumeration problems concerning the class of convex permutominoes have been studied by various authors. In particular, convex permutominoes were first enumerated in [10], where the authors proved, using the ECO method, that the number of *convex permutominoes* of size $n+1$ is:

$$2\,(n+3)\,4^{n-2} - \frac{n}{2}\binom{2n}{n} \qquad n \geq 1. \tag{1}$$

The first terms of the sequence are

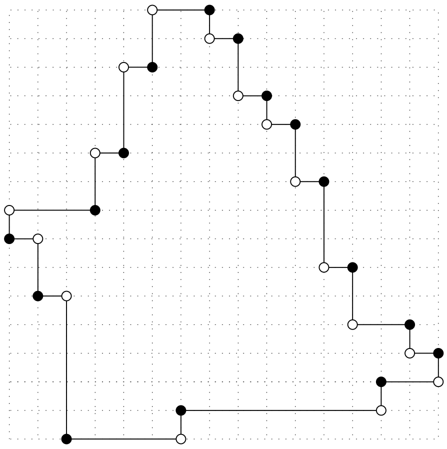
$$1, 4, 18, 84, 394, 1836, 8468, \ldots$$

(sequence A126020) in [13]). The same formula has been obtained successively and independently by Boldi et al. in [2].

A related problem, treated in [9], is the characterization of the pairs of permutations (π_1, π_2) associated with convex permutominoes, i.e. those belonging to the set

$$\{\, (\pi_1(P), \pi_2(P)) \,:\, P \in \mathcal{C}_n \,\},$$

where $\mathcal{C}_n$ is the set of convex permutominoes of size n. Any permutomino of size $n \geq 2$ uniquely determines two permutations π_1 and π_2 of $\mathcal{S}_n$, with



$\pi_1 = (8, 6, 1, 9, 11, 14, 2, 16, 15, 13, 12, 10, 7, 3, 5, 4)$
$\pi_2 = (9, 8, 6, 11, 14, 16, 1, 15, 13, 12, 10, 7, 5, 2, 4, 3)$

Figure 4: A convex permutomino and the associated permutations.

1 $\pi_1(i) \neq \pi_2(i)$, $1 \leq i \leq n$,

2 $\pi_1(1) < \pi_2(1)$, and $\pi_1(n) > \pi_2(n)$,

however, not all the pairs of permutations (π_1, π_2) of n satisfying **1** and **2** define a permutomino: Figure 5 depicts the two problems which may occur.

By studying the sets

$$\widetilde{\mathcal{C}}_n = \{ \pi_1(P) \,:\, P \in \mathcal{C}_n \}, \qquad \widetilde{\mathcal{C}}'_n = \{ \pi_2(P) \,:\, P \in \mathcal{C}_n \},$$

we easily obtain that

1. $\left|\widetilde{\mathcal{C}}_n\right| = \left|\widetilde{\mathcal{C}}'_n\right|$,

2. $\pi \in \widetilde{\mathcal{C}}_n$ if and only if $\pi^R \in \widetilde{\mathcal{C}}'_n$, where as usual, π^R is the *reversal* of π, defined as $\pi^R(i) = \pi(n+1-i)$, for each $i = 1, \ldots, n$.

Therefore, in [9] the authors, without loss of generality, study the combinatorial properties of $\widetilde{\mathcal{C}}_n$. Given a permutation $\pi \in \widetilde{\mathcal{C}}_n$, we say that π is π_1-*associated* with a convex permutomino.

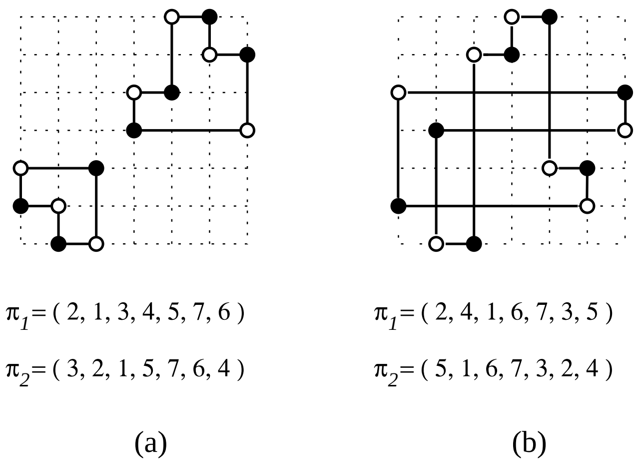


Figure 5: Two permutations π_1 and π_2 of $\mathcal{S}_n$, satisfying 1 and 2, do not necessarily define a permutomino, since two problems may occur: (a) two disconnected sets of cells; (b) the boundary crosses itself.

For small values of n we have that:

$$\begin{aligned}
\widetilde{\mathcal{C}}_1 &= \{1\},\\
\widetilde{\mathcal{C}}_2 &= \{12\},\\
\widetilde{\mathcal{C}}_3 &= \{123, 132, 213\},\\
\widetilde{\mathcal{C}}_4 &= \{1234, 1243, 1324, 1342, 1423, 1432, 2143,\\
&\qquad 2314, 2134, 2413, 3124, 3142, 3214\}.
\end{aligned}$$

Let π be a permutation of $\mathcal{S}_n$, we define $\mu(\pi)$ (briefly, μ) as the maximal upper unimodal sublist of π, and $\sigma(\pi)$ (briefly, σ) as the sublist beginning with $\pi(1)$, ending with $\pi(n)$, and containing the elements not in μ. Both μ and σ retain the indexing of π.

Example 1 Consider the convex permutomino of size 16 represented in Figure 4. We have

$$\pi_1 = (8, 6, 1, 9, 11, 14, 2, 16, 15, 13, 12, 10, 7, 3, 5, 4),$$

and we can determine the decomposition of π_1 into the two subsequences μ and σ:

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
μ	8	-	-	9	11	14	-	16	15	13	12	10	7	-	5	4
σ	8	6	1	-	-	-	2	-	-	-	-	-	-	3	-	4

For the sake of brevity, when there is no possibility of misunderstanding, we use to represent the two sequences omitting the empty spaces, as

$$\mu = (8, 9, 11, 14, 16, 15, 13, 12, 10, 7, 5, 4), \quad \sigma = (8, 6, 1, 2, 3, 4).$$

While μ is upper unimodal by definition, here σ turns out to be lower unimodal. This is a necessary condition for a permutation π to be π_1-associated with a convex permutomino, but it is not sufficient. For instance, if we consider the permutation $\pi = (5, 9, 8, 7, 6, 3, 1, 2, 4)$, then $\mu = (5, 9, 8, 7, 6, 4)$, and $\sigma = (5, 3, 1, 2, 4)$ is lower unimodal, but as shown in Figure 6 (a) there is no convex permutomino π_1-associated with π.

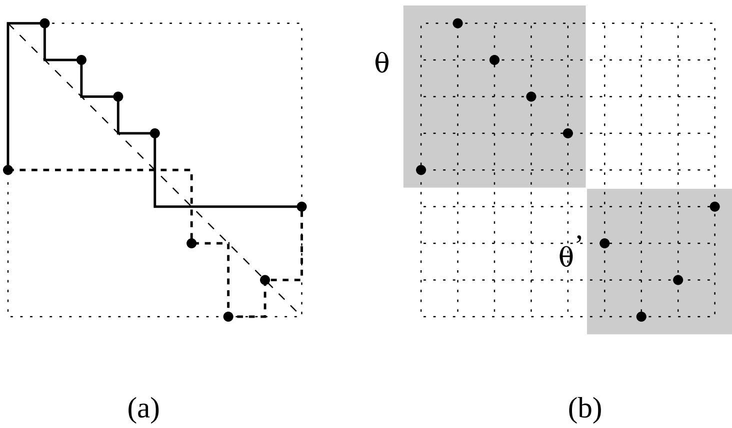


Figure 6: (a) there is no convex permutomino π_1-associated with $\pi = (5, 9, 8, 7, 6, 3, 1, 2, 4)$, since, as shown in (b), the permutation π is the direct difference $\pi = (1, 5, 4, 3, 2) \ominus (3, 2, 1, 4)$.

In order to give a necessary and sufficient condition for a permutation π to be in $\widetilde{\mathcal{C}}_n$, let us recall that, given two permutations $\theta = (\theta_1, \dots, \theta_m) \in \mathcal{S}_m$ and $\theta' = (\theta'_1, \dots, \theta'_{m'}) \in \mathcal{S}_{m'}$, their *direct difference* $\theta \ominus \theta'$ is a permutation of $\mathcal{S}_{m+m'}$ defined as

$$(\theta_1 + m', \dots, \theta_m + m', \theta'_1, \dots, \theta'_{m'}).$$

A pictorial description is given in Figure 6 (b), where $\theta = (1, 5, 4, 3, 2)$, $\theta' = (3, 2, 1, 4)$, and their direct difference is $\theta \ominus \theta' = (5, 9, 8, 7, 6, 3, 1, 2, 4)$.

Finally, the following characterization holds [9].

THEOREM 1 *Let π be a permutation in $\mathcal{S}_n$. Then, $\pi \in \widetilde{\mathcal{C}}_n$ if and only if:*

1. *σ is lower unimodal, and*
2. *there are no two permutations, $\theta \in \mathcal{S}_m$, and $\theta' \in \mathcal{S}'_m$, such that $m + m' = n$, and $\pi = \theta \ominus \theta'$.*

So, for instance $(2,5,6,1,7,3,4)$ satisfies conditions (1) and (2) and it is π_1-associated with at least one convex permutomino (as the one in Figure 3). The reader can check that instead there is no convex permutomino π_1-associated with $(2,5,3,7,4,1,6)$ since $\sigma = (2,3,4,1,6)$ is not lower unimodal, and there is no convex permutomino π_1-associated with $\pi = (5,9,8,7,6,3,1,2,4)$ since $\pi = (1,5,4,3,2) \ominus (3,2,1,4)$.

As a consequence of the characterization of Theorem 1, in [9] it is also proved that the cardinality of $\widetilde{\mathcal{C}}_{n+1}$ is

$$2\,(n+2)\,4^{n-2} \;-\; \frac{n}{4}\left(\frac{3-4n}{1-2n}\right)\binom{2n}{n}, \qquad n \geq 1, \tag{2}$$

defining the sequence $1, 1, 3, 13, 62, 301, 1450, \ldots$, recently added to [13] as A122122.

Another interesting property stated in [9] concerns the number of convex permutominoes P such that $\pi_1(P) = \pi$, for a given $\pi \in \widetilde{\mathcal{C}}_n$. For any $\pi \in \widetilde{\mathcal{C}}_n$, let us consider $[\pi] = \{P \in \mathcal{C}_n : \pi_1(P) = \pi\}$, i.e., the set of convex permutominoes π_1-associated with π. We say that a fixed point i, with $1 < i < n$, is a *free fixed point* if π can be decomposed as the direct sum $\pi = \sigma_1 \oplus (1) \oplus \sigma_2$, where $\sigma_1 \in S_{i-1}$, and $\sigma_2 \in S_{n-i}$. Let $\mathcal{F}(\pi)$ (briefly $\mathcal{F}$) denote the set of free fixed points of π.

Theorem 2 *Let $\pi \in \widetilde{\mathcal{C}}_n$, and let $\mathcal{F}(\pi)$ be the set of free fixed points of π. Then we have:*

$$|\,[\,\pi\,]\,| \;=\; 2^{|\mathcal{F}(\pi)|}.$$

For instance, let $\pi = (2,1,3,4,7,6,5)$ we have $\mu = (2,3,4,7,6,5)$, $\sigma = (2,1,5)$, and $\mathcal{F}(\pi) = \{3,4\}$. Then there are four convex permutominoes π_1-associated with π, as shown in Figure 7.

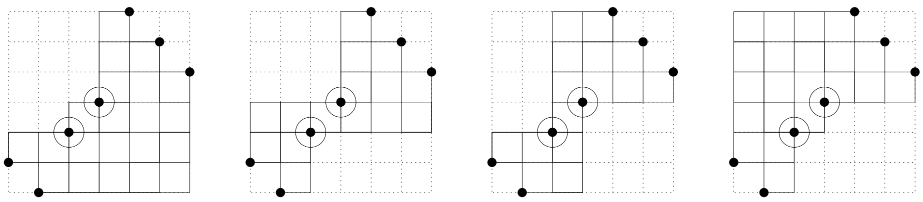


Figure 7: The four convex permutominoes π_1-associated with the permutation $\pi = (2,1,3,4,7,6,5)$. The two free fixed points are encircled.

The aim of this paper is to refine the results obtained in [9] and [10] to several well-known subclasses of convex permutominoes, namely the parallelogram, the directed convex, and the stack ones. We provide the enumeration of each of these classes, and we characterize the sets of permutations π_1-associated with the permutominoes of these classes. All the results will be obtained by using bijective methods rather than analytic techniques, as those employed in [2, 9, 10].

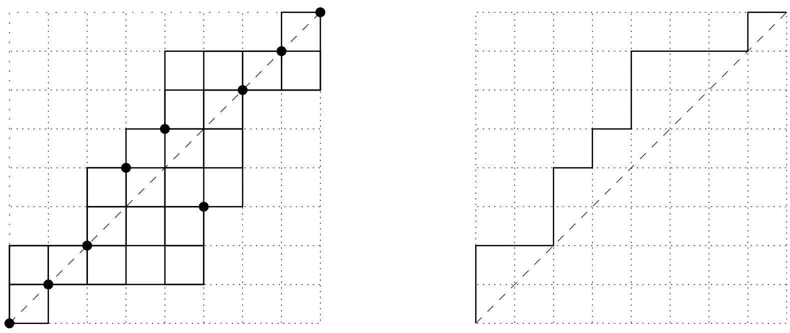
3 Parallelogram permutominoes

Let us denote by $\mathcal{P}_n$ the class of parallelogram permutominoes of size $n \geq 1$. For the enumeration of $\mathcal{P}_n$ we need to recall the definition of a *Dyck path*: it is a path in $\mathbb{N} \times \mathbb{N}$ going from $(0,0)$ to (n,n), using vertical unit steps and horizontal unit steps and remaining weakly above the diagonal $x = y$. It is well-know that the number of Dyck paths made of $2n$ steps is the nth Catalan number c_n (see for instance [14]). A Dyck path is said to be *elevated* if it remains strictly above the diagonal $x = y$.

PROPOSITION 1 *For any $n \geq 1$ we have $|\mathcal{P}_n| = c_{n-1}$.*

Proof. If $n = 1$, then we have the empty permutomino, i.e. $|\mathcal{P}_1| = c_0 = 1$. For $n \geq 2$, we prove that if a permutomino P is parallelogram of size n, then its upper path must remain weakly above the diagonal $x = y$. Let us assume it does not hold and then let us consider the smallest $k < n$ such that the upper path crosses the diagonal $x = y$ in (k,k). Because of the definition, for every abscissa (resp. ordinate) of P between 1 and k there must be exactly one vertical (horizontal) side of P; but this implies, being the permutomino a parallelogram one, that the upper and the lower paths of P need intersect in (k,k), and then P is not a polyomino, against the hypothesis. Hence the upper path of P is a Dyck path of length $2(n-1)$, as shown in Figure 8.

Conversely, each Dyck path of length $2n$ is the upper path of exactly one parallelogram permutomino of size n, $n \geq 2$; hence we have that $|\mathcal{P}_n|$ is equal to the number of Dyck paths of length $2(n-1)$, i.e. c_{n-1}. □



$\pi_1 = (1, 2, 3, 5, 6, 4, 7, 8, 9)$

Figure 8: A parallelogram permutomino π_1-associated with the permutation $\pi_1 = (1,2,3,5,6,4,7,8,9)$, and the corresponding Dyck path.

REMARK. We observe that the previously defined correspondence naturally defines an involution φ on Dyck paths: in fact given a Dyck path D of length $2n$, define $\varphi(D)$ as the lower path of the parallelogram permutomino which is in bijection with D. Clearly, $\varphi(D)$ is the reflection of a Dyck path with respect to the diagonal $x = y$, it has the same length as D, and it is uniquely determined (see Figure 9). It is very simple to check that φ is actually an involution on Dyck paths.

Incidentally, as pointed out by E. Deutsch, it is a new occurrence of an old involution, the so-called "Kreweras involution" on Dyck paths [7, 8].

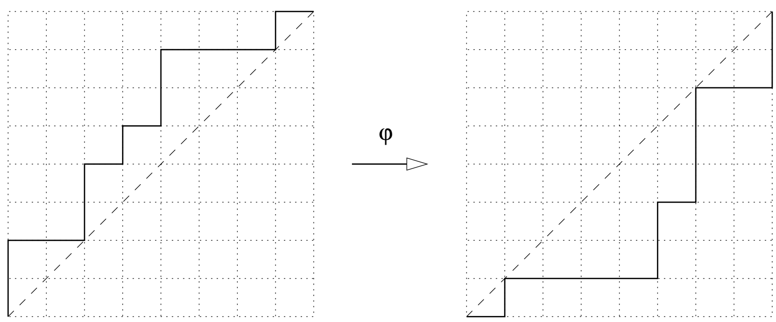


Figure 9: Two Dyck paths put in correspondence by the involution φ.

3.1 Permutations π_1-associated with parallelogram permutominoes

Let us now consider the class $\widetilde{\mathcal{P}}_n = \{\, \pi_1(P) \,:\, P \in \mathcal{P}_n \,\} \subseteq \mathcal{S}_n$. For $n = 1, \dots, 5$ we have

$$\begin{aligned}
\widetilde{\mathcal{P}}_1 &= \{1\};\\
\widetilde{\mathcal{P}}_2 &= \{12\};\\
\widetilde{\mathcal{P}}_3 &= \{123\};\\
\widetilde{\mathcal{P}}_4 &= \{1234, 1324\};\\
\widetilde{\mathcal{P}}_5 &= \{12345, 12435, 13245, 13425, 13245\}.
\end{aligned}$$

Proposition 2 *A permutation π belongs to $\widetilde{\mathcal{P}}_n$ if and only if $\pi(1) = 1$, $\pi(n) = n$, and μ and σ are both increasing.*

For instance, referring to the permutomino in Figure 8 we have that $\mu = (1, 2, 3, 5, 6, 7, 8, 9)$, and $\sigma = (1, 4, 9)$. The following remarkable property is a direct consequence of Proposition 2.

Proposition 3 *If $P \in \mathcal{P}_n$, then $[\,\pi_1(P)\,] \subset \mathcal{P}_n$.*

In practice, all the convex permutominoes π_1-associated with $\pi_1(P)$ are parallelogram ones. For instance, $\pi = (1,4,2,3,5,6) \in \widetilde{\mathcal{P}}_6$ has one free fixed point, $i = 5$, and then according to Proposition 3 the two convex permutominoes π_1-associated with π are both parallelogram ones (see Figure 10 (a), (b)). However, there are several non convex permutominoes which are π_1-associated with π; two of them are depicted in Figure 10 (c), (d). The enumeration for $\widetilde{\mathcal{P}}_n$ is then straightforward.

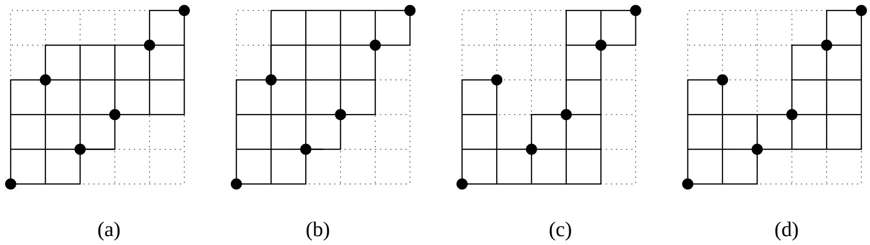


Figure 10: Some permutominoes π_1-associated with $(1,4,2,3,5,6)$.

Proposition 4 *For any $n \geq 1$ we have* $\left|\widetilde{\mathcal{P}}_n\right| = c_{n-2}$.

Proof. The statement is true for $n = 1$. So let Π be a permutation of $\widetilde{\mathcal{P}}_n$, $n \geq 2$. We agree to represent Π by the unique parallelogram permutomino $P(\Pi)$ of $[\,\Pi\,]$ for which the upper path is an *elevated* Dyck path. In practice $P(\Pi)$ is the parallelogram permutomino where the upper path is obtained by connecting the points in Π which remain strictly above the diagonal, i.e. all the fixed points in Π belong to the lower path, except 1 and n. Let $w(\Pi)$ be the upper path of $P(\Pi)$ from which we have removed the first and last step (a vertical and an horizontal one, respectively). Necessarily $w(\Pi)$ is a Dyck path of length $2(n-2)$ (see Figure 11).

Conversely, given any Dyck path w of length $2(n-2)$, with $n \geq 2$, we "elevate" it by adding an initial vertical unit step and a final horizontal unit step, and build up the unique parallelogram permutomino P_w having such obtained path as upper path. Then we have that $\pi_1(P_w) \in \widetilde{\mathcal{P}}_n$.

Finally, the number of permutations of $\widetilde{\mathcal{P}}_n$ is equal to the number of Dyck paths of length $2(n-2)$, i.e. c_{n-2}. $\square$

Now we prove that the permutations π_1-associated with parallelogram permutominoes are indeed a well-known class of permutations, and they are enumerated by the Catalan numbers. To do this we need recall some definitions.

Let n, m be two positive integers with $m \leq n$, and let $\pi = (\pi(1), \ldots, \pi(n)) \in \mathcal{S}_n$ and $\nu = (\nu(1), \ldots, \nu(m)) \in \mathcal{S}_m$. We say that π *contains* the pattern ν if there exist indices $i_1 < i_2 < \ldots < i_m$ such that $(\pi(i_1), \pi(i_2), \ldots, \pi(i_m))$ is in the same relative order as $(\nu(1), \ldots, \nu(m))$. If π does not contain ν we say that π

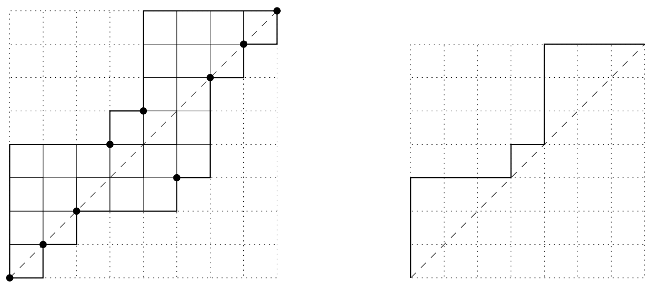


Figure 11: The parallelogram permutomino $P(\Pi)$ representing the permutation $\Pi = (1,2,3,5,6,4,7,8,9)$, and the associated Dyck path $w(\Pi)$.

is ν-*avoiding* [14]. For instance, if $\nu = (1,2,3)$ then $\pi = (5,2,4,3,1,6)$ contains ν, while $\pi = (6,3,2,5,4,1)$ is ν-avoiding.

Let us denote by $\mathcal{S}_n(\nu)$ the set ν-avoiding permutations in $\mathcal{S}_n$. If the pattern has length 3, it is known that for each pattern $\nu \in \mathcal{S}_3$ we have that $|\mathcal{S}_n(\nu)| = c_n$, the nth Catalan number [14].

THEOREM 3 *A permutation π belongs to $\widetilde{\mathcal{P}}_n$ if and only if $\pi(1) = 1$, $\pi(n) = n$, and it avoids the pattern $\nu = (3,2,1)$.*

Proof. ($\Longrightarrow$) If π belongs to $\widetilde{\mathcal{P}}_n$ we have, from Proposition 2, that $\pi(1) = 1$, $\pi(n) = n$ and μ and σ are both increasing. In particular π trivially avoids the pattern $(3,2,1)$.

($\Longleftarrow$) Since $\pi(n) = n$ then, by definition, μ is increasing. Now suppose that σ is not increasing, then there exists an index i and an index $j > i$, such that $\pi(i) > \pi(j)$ with $\pi(i)$ and $\pi(j)$ in σ. Since $\pi(1) = 1$ we have that $i > 1$ and so – since $\pi(i)$ is not in μ – there exists an index $k < i$ with $\pi(k) > \pi(i)$. If we take $(\pi(k), \pi(i), \pi(j))$ we have the pattern $(3,2,1)$. So also σ must be increasing. From Proposition 2 we have the thesis.

4 Directed-convex permutominoes

Let $\mathcal{D}_n$ be the class of directed convex permutominoes of size $n \geq 1$. The study of this class will be carried on using the same approach we have applied to the class of parallelogram permutominoes. On a directed convex permutomino we identify the following points: $O = (1,1)$, $C = (\pi_1^{-1}, n)$, $D = (n, \pi_1(n))$, as shown in Figure 12. Moreover, let $X = (x, \pi_1(n))$, $x \leq \pi_1(n)$, be the point obtained from the intersection of the boundary path from O to C with the line $y = \pi_1(n)$.

Trivially, a directed convex permutomino P is uniquely determined by its "upper path", i.e. the path that goes from O to D in a clockwise orientation. We encode such a path (hence the permutomino P) in terms of a second path, denoted $\phi(P)$, running from $(1,1)$ to (n,n), where the sequences of maximal

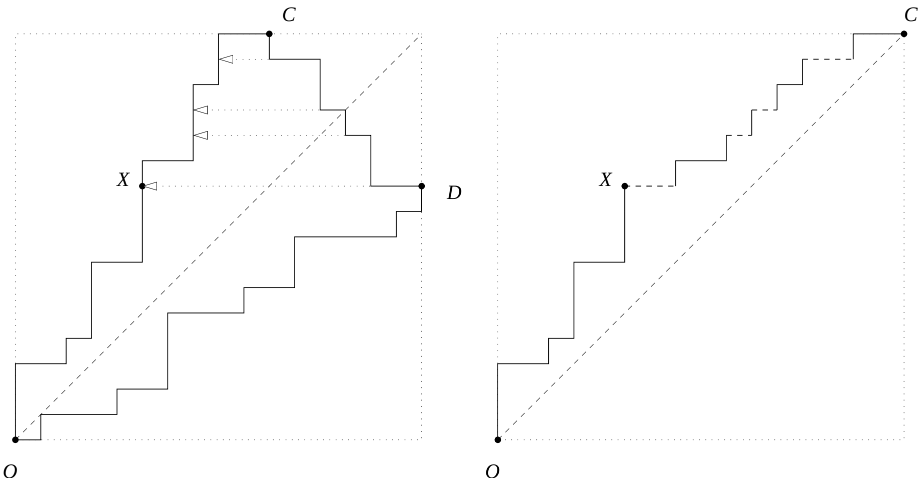


Figure 12: A directed convex permutomino and its associated path: $111001011100111\overline{00}1001\overline{0}1\overline{0}101\overline{00}100$. In order to simplify the reading, the cells of the polyomino have not been drawn.

length of consecutive horizontal steps can be of two different types, which we represent for simplicity by means *solid* or *dotted* steps. More precisely, to P we associate a path $\phi(P)$ obtained as the concatenation, from left to right, of the following paths:

- the boundary path of P, from O to X,

- a path encoding the upper part of boundary of the permutomino, i.e. the part above the line $y = \pi_1(n)$; clearly, for each ordinate greater than or equal to $\pi_1(n)$, we have a set of horizontal edges of the boundary of P, placed on the left or on the right of C. Thus the path $\phi(P)$ is obtained by concatenating:

  i. a sequence of solid horizontal steps for each sequence of edges on the left of C;

  ii. a sequence of dotted horizontal steps for each sequence of edges on the right of C,

  where each group of steps is separated by a unit vertical step, as shown in Figure 12.

Roughly speaking, the path $\phi(P)$ is obtained by projecting, over the path from X to C, all the horizontal steps of the boundary path from C to D, whereas projected horizontal steps are denoted by means of dotted steps.

Clearly, if P is a parallelogram permutomino then the points C and D coincide, and $\phi(P)$ by definition is exactly the Dyck path determined by P in the bijection of Proposition 1, not containing dotted steps. Otherwise we need the following characterization.

LEMMA 1 *A directed convex (non parallelogram) permutomino P of size n is encoded by a Dyck path $\phi(P)$ of length $2(n-1)$ running from $(1,1)$ to (n,n), using horizontal steps of two types, say solid and dotted, and which can be uniquely decomposed into two parts:*

1. *a non-empty path of the form:* $1^+ \; (0^+1^+)^*$;
2. *a non-empty path of the form:* $\overline{0}^+1 \left(0^+1 \vee \overline{0}^+1 \right)^* 0^+$,

where 0, $\overline{0}$, *and* 1 *denote respectively the solid horizontal, dotted horizontal, and vertical unit steps.*

Proof. By construction the path $\phi(P)$ starts in $(1,1)$ with a sequence of 1's and ends in (n,n) with a sequence of 0's. Moreover, the path on the right of X must have a sequence of horizontal steps, all solid or all dotted, for any ordinate between $\pi_1(n)$ and n, hence it can be uniquely decomposed as a sequence of "blocks" of the form 0^h1 or $\overline{0}^h1$, $h \geq 1$, and it must begin with a block $\overline{0}^h1$ and end with a block 0^h1.

Finally, the path $\phi(P)$ cannot cross the diagonal $x = y$. In fact if it happened, the crossing point should be on the right of X; but, due to the decomposition previously determined, the subpath of $\phi(P)$ on the right of X cannot contain two consecutive 1's, then $\phi(P)$ must necessarily remain below the diagonal $x = y$ until it reaches the point (n,n), thus ending with a sequence of vertical steps, which is against our hypothesis. □

Vice versa it can be easily proved that each Dyck path from $(1,1)$ to (n,n) which can be decomposed into two parts satisfying properties 1. and 2. represents a unique directed convex (non parallelogram) permutomino. Let $U_{n-1} = \phi(\mathcal{D}_n)$ be the set of the paths – of length $2(n-1)$ – encoding directed convex permutominoes of size n through ϕ.

Now we would like to use the coding ϕ in order to determine the number of directed convex permutominoes in a bijective way, and to provide some interesting combinatorial information on the class of directed convex permutominoes. To reach this goal we need to introduce some further definitions.

It is well known that every non-empty Dyck path Z can be uniquely decomposed as

$$Z \;=\; Y\, 10^{k_1}10^{k_2}\ldots 10^{k_r},$$

where $r \geq 1$, $k_i \geq 1$, $i = 1,\ldots,r$ and Y is the empty path or is a path remaining weakly above the x-axis and ending with 1. According to such a decomposition we say that the Dyck path Z is r-*graded*. In practice, r is the number of peaks in the path having no doublerises on their right. For instance the Dyck path depicted in Figure 8 is 2-graded, since it has a unique decomposition as:

$$Z \;=\; 1100110101 \quad 1\,000 \;\; 1\,0.$$

Let $Z_{n,k}$ be the set of k-graded Dyck paths having length $2n$. Clearly, for $n \geq 1$ the whole set Z_n of Dyck paths of length $2n$ is given by the disjoint union of the sets $Z_{n,k}$:

$$| Z_n | = \sum_{k=1}^{n} | Z_{n,k} | . \tag{3}$$

To our knowledge the statistic "number of k-graded Dyck paths" is not studied in literature yet, and the meaning and the reasons of our investigation will be clarified later. Table 1 shows the numbers $| Z_{n,k} |$ for small values of n and k: one immediately recognizes the well known *Catalan triangle* [12] (sequence A009766 in [13]), whose elements are also known with the name of *ballot numbers.*

n/k	1	2	3	4	5	6	$\dots$
1	1						
2	1	1					
3	2	2	1				
4	5	5	3	1			
5	14	14	9	4	1		
6	42	42	28	14	5	1	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 1: The numbers $| Z_{n,k} |$ for $n, k = 1, \dots, 6$.

In fact we can state the following

THEOREM 4 *The class $Z_{n,k}$ is enumerated by the* ballot numbers, *i.e.*

$$| Z_{n+1,k+1} | = \frac{1+k}{n+1}\binom{2n-k}{n-k}, \qquad n, k \geq 0. \tag{4}$$

Proof. We provide a bijection between $Z_{n,k}$ and the class $\hat{Z}_{n,k}$ of Dyck paths ending with exactly k horizontal steps; the latter one is a very well-known class of objects enumerated by the ballot numbers (see for instance [13, 14]). For any fixed $n \geq 1$ let us define the function $\varphi : Z_{n,k} \to \hat{Z}_{n,k}$ by induction on $k \geq 1$:

Base. We set $\varphi(10) = 10$. Any other 1-graded Dyck path Z can be decomposed as $Z = Y\,110^h$, with $h \geq 2$, then we set:

$$\varphi(Z) = Y\,10^{h-1}\,10,$$

which ends exactly with an horizontal step.

Inductive step. Let us assume that the path $Z \in Z_{n,k}$, with $k > 1$. We need to distinguish among the two cases:

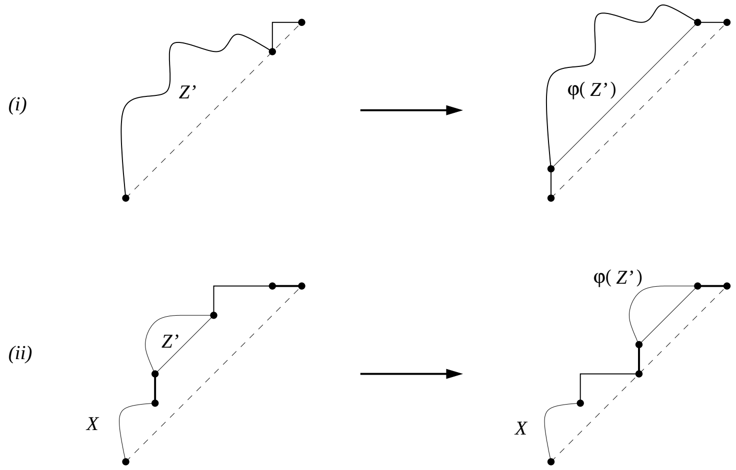


Figure 13: A pictorial description of the bijection φ.

- If Z ends with a peak, i.e. $Z = Z'\,10$, then Z' is a $(k-1)$-graded Dyck path, hence:

$$\varphi(Z) = 1\,\varphi(Z')\,0,$$

i.e. is an elevated Dyck path (see Figure 13 (i)).

- Otherwise $Z = X\,1\,Z'\,10^h$, where Z' is a $(k-1)$-graded Dyck path and $h > 1$. Then we set:

$$\varphi(Z) = X\,1\,0^{h-1}\,1\,\varphi(Z')\,0,$$

as it is graphically explained in Figure 13 (ii).

It should be clear that φ is a bijection, and it maps a k-graded path into one of the same length and ending with exactly k horizontal steps (see Figure 14 for an example). □

Remark. To the authors' knowledge φ is a new bijection on Dyck paths, and it should be further investigated independently from this paper, since it shows interesting combinatorial properties. For instance it happens that φ is not an involution, it has no fixed point different from 10, and the period of φ is equal to n if and only if n is a prime number.

Passing to generating functions, for any $k \geq 1$, let $Z_k(x) = \sum_{n\geq 0} |\, Z_{n,k} \,|\; x^n$. A well-known property of the Catalan triangle is that its kth column is defined by the kth power of the Catalan generating function $Z(x)$,

$$Z(x) = \frac{1 - \sqrt{1-4x}}{2x} = 1 + x + 2x^2 + 5x^3 + 14x^4 + 42x^5 + \ldots.$$

More precisely, the series $Z_k(x)$ can be expressed as follows:

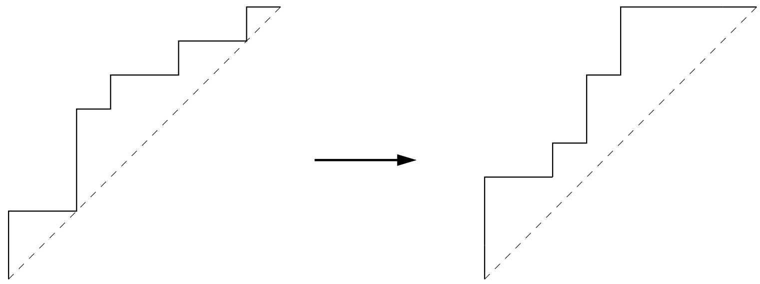


Figure 14: A 4-graded Dyck path is mapped though φ into a Dyck path of the same length and ending with exactly 4 horizontal steps.

i. $Z_1(x) = x\,Z(x)$,

ii. $Z_k(x) = x\,Z_{k-1}(x)\,Z(x) = x^k\,Z^k(x)$.

Moreover from (3) we have that:

$$Z(x) = 1 + \sum_{k\geq 1} Z_k(x) = \sum_{k\geq 0} x^k\,Z^k(x) = \frac{1}{1 - xZ(x)},$$

which is the well known functional equation for Catalan numbers.

Going back to our initial problem of counting the paths in U_n, we observe that, for any $k \geq 1$, each path of $Z_{n,k}$ uniquely determines 2^{k-1} paths of U_n (the $k-1$ exponent is due to the fact that the horizontal steps of the kth sequence need be solid, while the other sequences of horizontal steps may be indifferently made all of dotted steps or all of solid steps), as sketched in Figure 15.

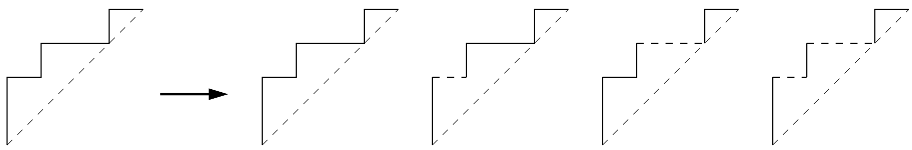


Figure 15: The 2^2 paths in U_4 obtained from a 3-graded path in $Z_{4,3}$.

Hence, for all $n \geq 1$ we have $U_n = \sum_{k=1}^{n} |\,Z_{n,k}\,|\,2^{k-1}$, and passing to generating functions,

$$U(x) = \sum_{k\geq 0} 2^k\,x^{k+1}\,Z^{k+1}(x) = x\,Z(x) + 2x^2Z^2(x) + 4x^3Z^3(x) + \ldots, \quad (5)$$

hence:

$$U(x) = \frac{x\,Z(x)}{1 - 2x\,Z(x)} = \frac{1}{2\sqrt{1-4x}} - 1. \quad (6)$$

From (6) can state the following:

Proposition 5 *For any $n \geq 2$ we have that:*

$$|\,\mathcal{D}_n\,| \;=\; |\,U_{n-1}\,| \;=\; \frac{1}{2}\,b_{n-1}.$$

Remark. There is a simpler bijective proof of Proposition 5, recently suggested by S. Elizalde [11]. The proof consists in a direct transformation of the path $\phi(P)$, encoding a directed convex permutomino P, into an unrestricted path from $(1,1)$ to (n,n), starting with a vertical step. The number of such paths is clearly $\frac{1}{2}\,b_{n-1}$. Such a bijection, however, does not preserve much of the combinatorial structure of the permutomino P.

4.1 Permutations π_1-associated with directed convex permutominoes

Now we turn out to study the class of permutations π_1-associated with directed convex permutominoes, i.e. the set

$$\widetilde{\mathcal{D}}_n \;=\; \{\,\pi_1(P)\;:\;P \in \mathcal{D}_n\,\}\,.$$

Every permutation π_1-associated with a directed convex permutominoes is characterized by the following property:

Proposition 6 *A permutation π belongs to $\widetilde{\mathcal{D}}_n$ if and only if $\pi(1) = 1$ and the sequence σ is increasing.*

For $n = 1, \ldots, 4$ we have:

$$\begin{aligned}
&\widetilde{\mathcal{D}}_1 = \{1\};\\
&\widetilde{\mathcal{D}}_2 = \{12\};\\
&\widetilde{\mathcal{D}}_3 = \{123, 132\};\\
&\widetilde{\mathcal{D}}_4 = \{1234, 1243, 1324, 1342, 1423, 1432\}.
\end{aligned}$$

Analogously to Proposition 3 we can state that:

Proposition 7 *If $P \in \mathcal{D}_n$, then $[\pi_1(P)] \subset \mathcal{D}_n$.*

In a word, Proposition 7 states that if P is a directed convex permutomino, then all the convex permutominoes π_1-associated with $\pi_1(P)$ are directed convex ones.

Let us now turn to study the cardinality of $\widetilde{\mathcal{D}}_n$, again using bijective arguments. For any $\Pi \in \widetilde{\mathcal{D}}_n$, we choose to represent it by means of the unique permutomino $P\,(\,\Pi\,) \in [\,\Pi\,]$ (denoted for brevity by $\widetilde{P}$) such that the path from O to C remains strictly above the diagonal $x = y$ (see Figure 16 (i)).
Moreover let $\phi(\widetilde{P}) \in U_{n-1}$ be the image of $\widetilde{P}$ through ϕ. Clearly, $\phi(\widetilde{P})$ is a path which can be decomposed in two sub-paths, as explained in Lemma 1. But here

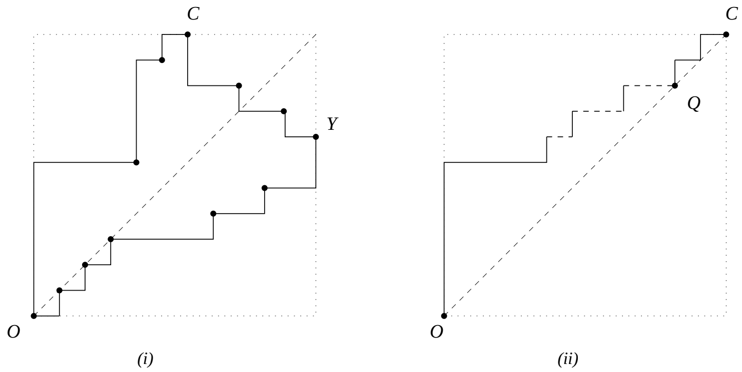


Figure 16: (i) the directed convex permutomino $\widetilde{P}\,(\,\Pi\,)$ π_1-associated with the permutation $\Pi\,=\,(1,2,3,4,7,11,12,5,10,6,9,8)$; (ii) the path $\phi(\widetilde{P})$.

we will present a different decomposition of $\phi(\widetilde{P})$, determined by the leftmost point, namely Q, where $\phi(\widetilde{P})$ returns on the diagonal $x=y$. By the definition of ϕ, and the choice of $\widetilde{P}$, $\phi(\widetilde{P})$ can be uniquely decomposed into two parts:

1. on the left of Q we have an elevated path;
2. on the right of Q we simply have a (possibly empty) concatenation of *peaks* of the form $1\,0$ or $1\,\overline{0}$, necessarily ending with $1\,0$ (see Figure 16 (ii)).

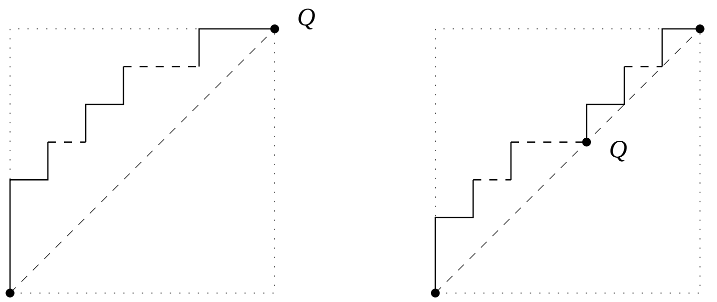
We remark that P is a parallelogram permutomino if and only if $\phi(\widetilde{P})$ is an elevated Dyck path – which is equivalent to say that $Q=(n,n)$ – as in Proposition 4. On the contrary, if the sub-path on the right of Q is non empty, then the sub-path on the left must contain at least one block $1\overline{0}^h$. If not, the sub-path on the left would be a Dyck path (with no dotted steps) of semi-length less than $n-1$. The sub-path of $\phi(\widetilde{P})$ containing no dotted steps and having maximal length is necessarily contained in the image of the sub-path from O to C. But, (i) in $\widetilde{P}$ the path from O to C remains strictly above the diagonal $x=y$, hence (ii) $\phi(\widetilde{P})$ maps the sub-path from O to C into a prefix of a Dyck path remaining strictly above the diagonal $x=y$. Then the sub-path on the left of Q must contain at least a sequence of dotted steps. These simple properties help us to prove the following enumeration result.

THEOREM 5 *For any $n\geq 2$ we have that* $|\,\widetilde{\mathcal{D}}_n\,|\;=\;b_{n-2}$.

Proof. Let us consider the set:

$$\widetilde{U}_{n-1}=\left\{\,\phi(\widetilde{P})\;:\;\widetilde{P}=P(\Pi),\;\Pi\in\widetilde{\mathcal{D}}_n\,\right\}\subseteq U_{n-1}.$$

Clearly, for all $n\geq 2$, $|\,\widetilde{U}_{n-1}\,|\;=\;|\,\widetilde{\mathcal{D}}_n\,|$. Easily, $\widetilde{U}_1=1$; for $n>1$ we divide the set $\widetilde{U}_n$ into two disjoint sets:



(i) *(ii)*

Figure 17: (i) a path in $\widetilde{U'}_n$; (ii) a path in $\widetilde{U''}_n$.

1. $\widetilde{U'}_n$ is the set of elevated paths in $\widetilde{U}_n$ (i.e. where the point Q coincides with (n, n)). In this case the path $\widetilde{P}$ ends with a sequence of solid steps (see Figure 17 (i));
2. $\widetilde{U''}_n$ is the remaining set of paths in $\widetilde{U}_n$ (i.e. those such that the path on the right of Q is not empty, and end with a sequence of peaks, see Figure 17 (ii)).

We study separately the two classes:

1. $|\widetilde{U'}_n| = |U_{n-1}| = \frac{1}{2} b_{n-1}$, it trivially follows from the definition of $\widetilde{U'}_n$;
2. $|\widetilde{U''}_n| = |U_{n-1}| = \frac{1}{2} b_{n-1}$. Every path in $\widetilde{U''}_n$ can be unambiguously represented as:
$$Y \left(10 \vee 1\overline{0}\right)^* 10,$$
where Y is an elevated Dyck path containing at least one sequence of dotted steps. Each of these elevated paths Y has length equal to $2(h+1)$ with $h \leq n-2$ and it can be obtained from the Dyck paths in U_h. In fact since Y's last sequence of horizontal steps can be solid or dotted, we duplicate each path in U_h by turning its last sequence of horizontal steps from solid to dotted ones. Then we subtract the elevated Dyck paths that do not have dotted steps in U_h, since we want that each path Y contains at least one sequence of dotted steps. We recall that the number of paths in U_h without dotted steps is equal to the number of parallelogram permutominoes of size $h+1$, i.e. the h-th Catalan number. So we have:
$$2\,|U_h| - |\mathcal{P}_{h+1}| = \binom{2h}{h} - \frac{1}{h+1}\binom{2h}{h} = \binom{2h}{h-1}.$$
From this computation we have excluded the case where Y is the path $1\,\overline{0}$, then the whole path has the form
$$1\,\overline{0}\left(10 \vee 1\overline{0}\right)^* 10.$$

As a consequence, we have that, for $n \geq 1$:

$$|\widetilde{U''}_n| = 2^{n-2} + \sum_{k=0}^{n-2} \binom{2k}{k-1} \cdot 2^{n-k-2} = \frac{1}{2} b_{n-1}. \tag{7}$$

Finally, we have that for $n \geq 1$:

$$|\widetilde{\mathcal{D}}_n| = |\widetilde{U}_{n-1}| = |\widetilde{U'}_{n-1}| + |\widetilde{U''}_{n-1}| = b_{n-2}. \qquad \square$$

As a generalization of Theorem 3, we can also state that the permutations of $\widetilde{\mathcal{D}}_n$ are characterized by the avoidance of four patterns of length 4.

Theorem 6 *A permutation π belongs to $\widetilde{D}_n$ if and only if $\pi(1) = 1$, and it avoids the patterns $\nu_1 = (3,2,1,4)$, $\nu_2 = (3,2,4,1)$, $\nu_3 = (4,2,1,3)$, $\nu_4 = (4,2,3,1)$.*

Proof.

($\Rightarrow$) One only has to check that if a permutation π begins with $\pi(1) = 1$ and contains one of the four patterns ν_1, ν_2, ν_3, ν_4 then its σ sequence cannot be increasing.

($\Leftarrow$) Let us assume by contradiction that $\pi \notin \widetilde{D}_n$, i.e. that the sequence σ is not increasing. Then there are two indices i_1 and i_2 such that $1 < i_1 < i_2 < n$, and such that $\sigma(i_1) > \sigma(i_2)$. Moreover, there should be two indices j, k belonging to μ, such that:

$j < i_1$, and $\pi(j) > \sigma(i_1)$; in fact, without such an element, i_1 would belong to the increasing part of μ and not to σ;

$k > i_1$, and $\pi(k) > \sigma(i_1)$; similarly, without such an element, i_2 would belong to the decreasing part of μ and not to σ.

This leads to four possible configurations which are sketched in Figure 18, and each of them defines one of the four patterns $\nu_1, \nu_2, \nu_3, \nu_4$. $\square$

5 Stack permutominoes

The case of stack permutominoes is quite simple to study. Indeed, if a permutomino is a stack, then all its columns must have different lengths from 1 to n (see Figure 19). Then all the stack permutominoes of size $n+1$ can be produced from those of size n in a unique way by performing the following two operations: (i) add a row of length n in the lowest leftmost position, (ii) add a row of length n in the lowest rightmost position.

Hence we have that, being $\mathcal{S}t_n$ the class of stack permutominoes of size n,

Proposition 8 *For any $n \geq 2$, $|\mathcal{S}t_n| = 2^{n-2}$.*

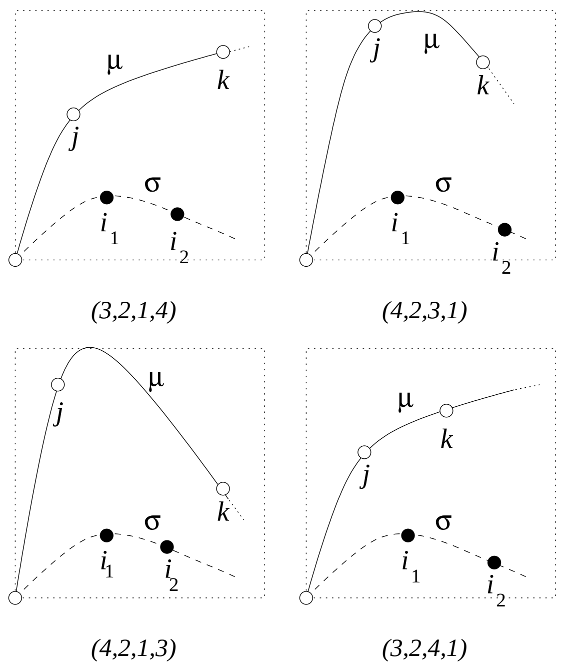


Figure 18: The four forbidden patterns in the permutations of $\widetilde{\mathcal{D}}_n$.

The permutations π_1-associated with a stack permutomino have a very simple structure.

Proposition 9 *A pair (π_1, π_2) of permutations of $\mathcal{S}_n$ defines a stack permutomino if and only if π_1 and π_2 are upper unimodal and $\pi_1(1) = \pi_2(n) = 1$.*

In particular the previous statement implies that, for any stack permutomino, the μ sequence coincides with the whole permutation π_1, and then $\sigma = (1, \pi(n))$. On the other way, if a permutation π is upper unimodal with $\pi(1) = 1$, then $\pi \in \widetilde{\mathcal{D}}_n$, and $[\pi] \subseteq \mathcal{D}_n$; but, differently from the other classes of permutominoes we have studied in the paper, $[\pi] \not\subseteq \mathcal{S}t_n$. In fact we have the following.

Proposition 10 *For any stack permutomino P we have that $[\,\pi_1(P)\,] = \{\,P\,\}$.*

Moreover, the number of unimodal permutations of n with $\pi(1) = 1$ is $|\widetilde{St}_n| = 2^{n-2}$, hence we have again the result of Proposition 8.

Example 2 Let us consider $\pi = (1, 2, 3, 5, 4)$; we have $\pi(1) = 1$, $\mu = \pi$, and $\sigma = (1, 4)$. Hence, according to Propositions 6 and 7 we have that $[\,\pi\,] \subseteq \mathcal{D}_5$. From Theorem 2 we have that $|[\,\pi\,]| = 4$ (see Figure 20). But Proposition 10

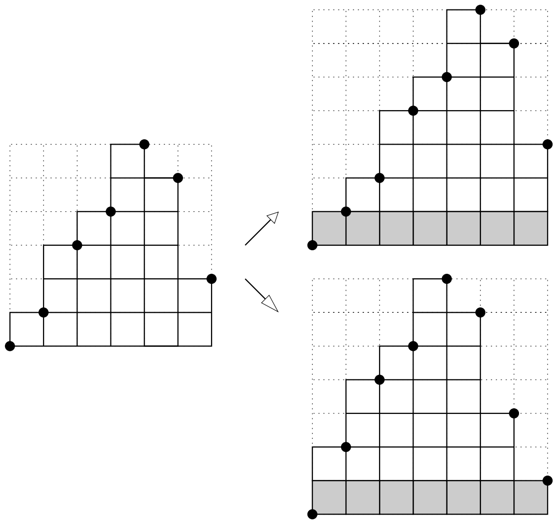


Figure 19: Every stack permutomino of size n uniquely determines two stack permutominoes of size $n+1$.

ensures that only one of the four directed convex permutominoes π_1-associated with π is a stack (precisely, the leftmost in Figure 20).

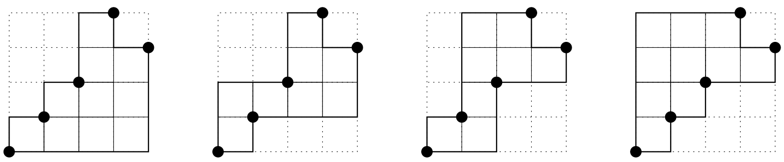


Figure 20: The four directed convex polyominoes π_1-associated with $\pi = (1,2,3,5,4)$.

6 Concluding remarks

In this paper we have considered some classes of convex permutominoes: the parallelogram, the directed convex, and the stack permutominoes. For each of these classes we have determined the enumeration according to the size, and the characterization of the permutations π_1-associated with them.

These enumeration results, obtained throughout bijections and simple computations involving binomial coefficients and Catalan numbers, are shown in Table 2. However, some problems regarding classes of convex permutomnoes remain open, and below we list some of the most interesting to the authors'.

Class of convex permutominoes	*Number of convex permutominoes with size n*	*Associated class of permutations*
Parallelogram	$\vert\mathcal{P}_n\vert = c_{n-1}$	$\vert\widetilde{\mathcal{P}}_n\vert = c_{n-2}$
Directed convex	$\vert\mathcal{D}_n\vert = \frac{1}{2}b_{n-1}$	$\vert\widetilde{\mathcal{D}}_n\vert = b_{n-2}$
Stack	$\vert\mathcal{S}t_n\vert = 2^{n-2}$	$\vert\widetilde{\mathcal{S}}t_n\vert = 2^{n-2}$

Table 2: Enumeration results.

On directed convex permutominoes. By Proposition 5 we have that the number of directed convex permutominoes of size n is half the number of directed convex polyominoes of semi-perimeter $n+1$. It would be interesting to explain this enumeration result by a direct combinatorial proof.

A bijection for the number of centered permutominoes. Let us consider the set of *horizontally centered* (or simply *centered*) convex polyominoes. A convex polyomino is said to be centered if it contains at least one row touching both the left and the right side of its minimal bounding rectangle. The number Q_n of centered polyominoes having semi-perimeter equal to $n+2$ satisfies the recurrence relation

$$Q_n = 6\,Q_{n-1} - 8\,Q_{n-2}\,, \quad n > 2,$$

with $Q_0 = 1$, $Q_1 = 2$, $Q_2 = 7$ [4]. In some recent studies we have considered the class of convex permutominoes which are also centered. Figure 20 depicts four convex permutominoes: the first, the second and the forth are centered, while the third is not centered. Using standard analytical techniques it is possible to prove that the number of centered permutominoes of size $n \geq 2$ is equal to 4^{n-2}. We believe that this result should have a simple combinatorial explanation.

Moreover, some characterization and enumeration problems on permutominoes are still open.

The number of column convex permutominoes. We would like to point out that the property stated in Theorem 2 does not hold for column convex permutominoes. For instance, referring to Figure 10, we observe that the number of column convex permutominoes which are π_1-associated with a certain permutation does not depend only on the number of its free fixed points. Then, it would be interesting to determine an extension of Theorem 2 for column convex permutominoes, and provide a way to construct, for all $\pi \in \mathcal{S}_n$, all the permutominoes of the set

$$\{\, P \,:\, P \text{ is a column convex permutomino, and } \pi_1(P) = \pi \,\}\,.$$

Permutations π_1-associated with permutominoes. Let us consider permutominoes with no restriction. The problem of their enumeration seems to be as difficult as the enumeration of polyominoes, thus it is more convenient to start studying the problem of characterizing the permutations associated with permutominoes. The following result can be proved by writing a rather simple algorithm that builds up the boundary of a column convex permutomino starting from any $\pi \in \mathcal{S}_n$.

PROPOSITION 11 *If $\pi \in \mathcal{S}_n$, $n \geq 2$, then there is at least one column convex permutomino P such that $\pi = \pi_1(P)$ or $\pi = \pi_2(P)$.*

Hence another interesting question regards the characterization of the permutations which are π_1-associated with permutominoes, i.e. those of the class

$$\widetilde{P} = \{ \pi_1(P) \, : \, P \text{ is a permutomino} \} .$$

Motivated by experimental evidence, and by Proposition 1, we are lead to conjecture the following.

CONJECTURE: *A permutation $\pi \in \widetilde{P}$ if and only if there are no two permutations, θ and θ', such that $\pi = \theta \ominus \theta'$.*

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