

On multi-dimensional patterns

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Abstract. We generalize the concept of pattern occurrence in permutations, words or matrices to that in n -dimensional objects, which are basically sets of $(n + 1)$ -tuples. In the case $n = 3$, we give a possible interpretation of such patterns in terms of bipartite graphs. For *zero-box patterns* we study *vanishing borders* related to bipartite Ramsey problems in the case of two dimensions. Also, we study the maximal number of 1's in binary objects avoiding (in two different senses) a zero-box pattern.

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1 Introduction

In this paper we generalize the concept of pattern occurrence in permutations, words or matrices [4, 5] to pattern occurrence in n -dimensional objects, which are essentially sets of $(n + 1)$ -tuples.

It is well-known (e.g., see [4] and references therein) that the consideration of (1 -dimensional) patterns has proven to be a useful language in a variety of seemingly unrelated problems, from the theory of Kazhdan–Lusztig polynomials, to singularities of Schubert varieties, to Chebyshev polynomials, to planar maps, to rock polynomials for a rectangular board, to various sorting algorithms including sorting stacks and sortable permutations. Increasing the number of dimensions by one, and considering avoidance of certain (2 -dimensional) *numbered polyomino patterns* [5] also led to some interesting connections to other combinatorial problems involving hypercubes, spanning trees, placing of non-attacking kings on certain boards [3], and Ramsey numbers for bipartite graphs [5].

A natural question arises: What happens if we further increase the number of dimensions?

In Section 2, we define the notion of an n -dimensional pattern, which agrees with the cases $n = 1, 2$. In Section 3, we define a coloring problem for bipartite graphs which provides some motivation for studying pattern avoidance

in 3-dimensional objects. In Section 4, we examine vanishing borders of *zero-box patterns*; this is an extension of the 2-dimensional results in [5] to the n -dimensional case. Finally, in Section 5, we study the maximal number of 1's in binary objects avoiding a zero-box pattern. We find an explicit formula for the maximal number of 1's in the case of *segmented-occurrences* of a zero-box pattern, and we obtain a tight recursive lower bound for usual (non-segmented) pattern occurrence.

2 Preliminaries

Note that throughout this paper we use $[m]$ to represent $\{1, 2, \dots, m\}$.

Let $n \in \mathcal{N}$ and $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathcal{N} \cup \{\infty\}$. Then the n -dimensional box bounded by $\alpha_1, \alpha_2, \dots, \alpha_n$ is the subset of \mathcal{N}^n defined by

$$X_{\alpha_1, \alpha_2, \dots, \alpha_n} = \{(x_1, x_2, \dots, x_n) \mid 1 \leq x_i \leq \alpha_i, 1 \leq i \leq n\}.$$

(Note that by $1 \leq x_i \leq \infty$ we mean that $x_i \in \mathcal{N}$; that is, x_i will never assume the value ∞ .)

Let \mathcal{A} be an ordered alphabet on k letters, say $\mathcal{A} = \{0, 1, \dots, k-1\}$, and f a function from $X_{\alpha_1, \alpha_2, \dots, \alpha_n}$ into \mathcal{A} . By an n -dimensional object we mean a set of $(n+1)$ -tuples of the form

$$\{(x_1, x_2, \dots, x_n, f(x_1, x_2, \dots, x_n)) \mid (x_1, x_2, \dots, x_n) \in X_{\alpha_1, \alpha_2, \dots, \alpha_n}\}.$$

Notice that 1-dimensional objects are words, whereas 2-dimensional objects are matrices. We denote an n -dimensional object on the box $X_{\alpha_1, \alpha_2, \dots, \alpha_n}$ by $\Omega(\alpha_1, \alpha_2, \dots, \alpha_n)$.

An n -dimensional pattern is basically an n -dimensional object with two exceptions: we include the symbol “#” in our alphabet to allow for the possibility of “holes” in our patterns and we require that each letter from the alphabet $\mathcal{B} = \{0, 1, 2, \dots, \ell-1\}$ occurs at least once in the pattern for some ℓ . More formally, an n -dimensional pattern on the box $X_{\beta_1, \beta_2, \dots, \beta_n}$, denoted $p = p(\beta_1, \beta_2, \dots, \beta_n)$, is a set of $(n+1)$ -tuples of the form

$$\{(x_1, x_2, \dots, x_n, g(x_1, x_2, \dots, x_n)) \mid (x_1, x_2, \dots, x_n) \in X_{\beta_1, \beta_2, \dots, \beta_n}\},$$

where g is a function from $X_{\beta_1, \beta_2, \dots, \beta_n}$ onto \mathcal{B} or $\mathcal{B} \cup \{\#\}$, depending on whether or not there are holes in the pattern. For instance, the right angled pattern $\begin{smallmatrix} 01 \\ 0 \end{smallmatrix}$ considered in [5] is the set $\{(1, 1, 0), (1, 2, 1), (2, 1, 0), (2, 2, \#)\}$ in our terminology. When $\mathcal{B} = \{0\}$ and p does not contain holes, we call p the $\beta_1 \times \beta_2 \times \dots \times \beta_n$ zero-box. In keeping with the notation of [5], we denote the $\beta_1 \times \beta_2 \times \dots \times \beta_n$ zero-box by $\mathcal{O}_{\beta_1, \beta_2, \dots, \beta_n}$.

We will be given an n -dimensional pattern p on the alphabet $\mathcal{B} = \{0, 1, \dots, \ell-1\}$ and an n -dimensional object Ω on the alphabet $\mathcal{A} = \{0, 1, \dots, k-1\}$. We wish to know if there is an n -dimensional sub-object, Ω' , of Ω which reduces to p . To be more precise, we can easily generalize the definition of pattern occurrence in matrices given in [5]. Let $\Omega = \Omega(\alpha_1, \alpha_2, \dots, \alpha_n)$ be an object on

the alphabet \mathcal{A} and $p = p(\beta_1, \beta_2, \dots, \beta_m)$ a pattern on the alphabet \mathcal{B} . Further, let T be a subset of $[\alpha_1] \times [\alpha_2] \times \dots \times [\alpha_n]$. We say that the pattern p occurs on T in the object Ω if there exists a bijection $\phi : X_{\beta_1, \beta_2, \dots, \beta_n} \rightarrow T$ that satisfies the following condition: Given any two points $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ in $X_{\beta_1, \beta_2, \dots, \beta_n}$ with $g(x), g(y) \neq \#$ and their images $(x'_1, x'_2, \dots, x'_n) = \phi(x)$, $(y'_1, y'_2, \dots, y'_n) = \phi(y)$ we have:

$$x_i < y_i \iff x'_i < y'_i$$

and

$$g(x_1, x_2, \dots, x_n) < g(y_1, y_2, \dots, y_n) \iff f(x'_1, x'_2, \dots, x'_n) < f(y'_1, y'_2, \dots, y'_n).$$

If the pattern p does not occur in Ω , we say that Ω avoids p .

For a given n -dimensional pattern p and a non-negative integer t we can define $a_{\alpha_1, \alpha_2, \dots, \alpha_n}$ to be the number of $\alpha_1 \times \alpha_2 \times \dots \times \alpha_n$ n -dimensional objects on the alphabet $\{0, 1, \dots, t\}$ which avoid p . We can then extend the notion of *vanishing borders* defined in [5]. Specifically, the vanishing border of p , denoted $V(p)$, is defined to be the set of all $(\alpha_1, \alpha_2, \dots, \alpha_n)$ such that $a_{\alpha_1, \alpha_2, \dots, \alpha_n} = 0$ but all of $a_{\alpha_1-1, \alpha_2, \dots, \alpha_n}$, $a_{\alpha_1, \alpha_2-1, \dots, \alpha_n}$, \dots , and $a_{\alpha_1, \alpha_2, \dots, \alpha_n-1}$ are greater than zero. It was shown in [5] that $(2l-1, 2(k-1)\binom{2l-1}{l} + 1)$ is in the vanishing border of the zero-box $\mathcal{O}_{l,k}$ when the alphabet is $\{0, 1\}$. In Section 4 we will study vanishing borders in higher dimensions.

3 3-dimensional patterns and bipartite graphs

Let A be an $m \times n$ matrix (that is, a 2-dimensional object). Consider the complete bipartite graph $K_{m,n}$ with vertex classes corresponding to the row and column labels of A . We can view A as an edge-coloring of $K_{m,n}$ by assigning the color $a_{i,j}$ to the edge corresponding to the i^{th} row and the j^{th} column.

For the remainder of this discussion, we assume that the entries of our objects and patterns are in the alphabet $\{0, 1\}$. If A is an $m \times n$ matrix on $\{0, 1\}$, it corresponds to a 2-edge-coloring of $K_{m,n}$. Now let $\mathcal{O}_{m',n'}$ be the $m' \times n'$ zero-box. As discussed in [5], A avoids p if and only if the coloring of $K_{m,n}$ corresponding to A does not contain a monochromatic copy of $K_{m',n'}$.

Let A^1, A^2, \dots, A^h be $m \times n$ matrices with entries in $\{0, 1\}$. We can create a 3-dimensional $m \times n \times h$ object, Ω , by layering these matrices. This object corresponds to a 2^h -edge-coloring of $K_{m,n}$, that is, the color on edge (i, j) is given by the binary number $a_{i,j}^h a_{i,j}^{h-1} \dots a_{i,j}^1$.

Now let $\mathcal{O}_{m',n',h'}$ be the $m' \times n' \times h'$ zero-box pattern. If $h' = h$ and Ω contains $\mathcal{O}_{m',n',h'}$, then the corresponding edge-coloring of $K_{m,n}$ contains a monochromatic copy of $K_{m',n'}$. Unlike the two-dimensional case, the converse of the previous statement is false since a monochromatic coloring of $K_{m',n'}$ by any of the $2^h - 2$ colors not consisting of all zeros or all ones does not result in an occurrence of $\mathcal{O}_{m',n',h'}$.

If $h' < h$ and Ω contains p , then the corresponding edge-coloring of $K_{m,n}$ contains a copy of $K_{m',n'}$ for which the Hamming distance between any two

colors assigned to $K_{m',n'}$ is at most $h - h'$. On the other hand, suppose that Ω avoids p . Consider a 2^h -edge-coloring of $K_{m',n'}$. For $1 \leq i \leq m'$ and $1 \leq j \leq n'$, let $c_{i,j}^h, c_{i,j}^{h-1}, \dots, c_{i,j}^1$ denote the color assigned to edge (i, j) . We say that the coloring of $K_{m',n'}$ is inconsistent in position r ($1 \leq r \leq h$) if there exist $1 \leq i_1, i_2 \leq m'$ and $1 \leq j_1, j_2 \leq n'$ such that $c_{i_1, j_1}^r \neq c_{i_2, j_2}^r$. If Ω avoids p , then the corresponding edge-coloring of each $K_{m',n'}$ is inconsistent in at least $h - h' + 1$ positions.

4 Vanishing borders of zero-box patterns

Recall that the vanishing border of the pattern p , denoted $V(p)$, is defined to be the set of all $(\alpha_1, \alpha_2, \dots, \alpha_n)$ such that $a_{\alpha_1, \alpha_2, \dots, \alpha_n} = 0$ but all of $a_{\alpha_1 - 1, \alpha_2, \dots, \alpha_n}$, $a_{\alpha_1, \alpha_2 - 1, \dots, \alpha_n}, \dots$, and $a_{\alpha_1, \alpha_2, \dots, \alpha_n - 1}$ are greater than zero. In this section we study vanishing borders for the zero-box pattern $\mathcal{O}_{\beta_1, \beta_2, \dots, \beta_n}$. The results in this section are a direct extension of Proposition 7 in [5].

THEOREM 1 *Consider the zero-box pattern $\mathcal{O}_{\beta_1, \beta_2, \dots, \beta_n}$, and set*

$$x_1 = 2(\beta_1 - 1) + 1$$

and

$$x_j = 2(\beta_j - 1) \prod_{i=1}^{j-1} \binom{x_i}{\beta_i} + 1$$

for $2 \leq j \leq n$. Then $(x_1, x_2, \dots, x_n) \in V(\mathcal{O}_{\beta_1, \beta_2, \dots, \beta_n})$.

Proof. The proof is by induction on n . The base case was established in Proposition 7 of [5]. So we may suppose that $(x_1, x_2, \dots, x_{n-1})$ is in the vanishing border of $V(\mathcal{O}_{\beta_1, \beta_2, \dots, \beta_{n-1}})$.

Now consider the n -dimensional object $\Omega = \Omega(\alpha_1, \alpha_2, \dots, \alpha_n)$. We can view Ω as a sequence $\Psi_1, \Psi_2, \dots, \Psi_{\alpha_n}$ of $\alpha_1 \times \alpha_2 \times \dots \times \alpha_{n-1}$ ($n - 1$)-dimensional objects, which we call *layers* of Ω . The zero-box pattern $\mathcal{O}_{\beta_1, \beta_2, \dots, \beta_n}$ occurs in Ω if and only if there is a sub-sequence $\Psi_{i_1}, \Psi_{i_2}, \dots, \Psi_{i_{\beta_n}}$ and a subset T of $[\alpha_1] \times [\alpha_2] \times \dots \times [\alpha_{n-1}]$ such that $\mathcal{O}_{\beta_1, \beta_2, \dots, \beta_{n-1}}$ occurs on T in Ψ_i for all $i \in \{i_1, i_2, \dots, i_{\beta_n}\}$. Therefore, if (y_1, y_2, \dots, y_n) is in the vanishing border for $\mathcal{O}_{\beta_1, \beta_2, \dots, \beta_n}$, then $y_i \geq x_i$ for $1 \leq i \leq n - 1$ by the induction hypothesis.

Suppose $\Omega = \Omega(x_1, x_2, \dots, x_n)$. By the induction hypothesis, the zero-box pattern $\mathcal{O}_{\beta_1, \beta_2, \dots, \beta_{n-1}}$ occurs in each layer of Ω . In each layer, $\mathcal{O}_{\beta_1, \beta_2, \dots, \beta_{n-1}}$ may occur in $\prod_{i=1}^{j-1} \binom{x_i}{\beta_i}$ different positions. Moreover, it may occur as a box of zeros or a box of ones. It follows from the Dirichlet Principle that $\mathcal{O}_{\beta_1, \beta_2, \dots, \beta_n}$ occurs in Ω .

It remains to show that there is an object $\Omega = \Omega(x_1, x_2, \dots, x_n - 1)$ which does not contain an occurrence of $\mathcal{O}_{\beta_1, \beta_2, \dots, \beta_n}$. We need the following lemma.

LEMMA 2 *There exists an object $\Omega = \Omega(x_1, x_2, \dots, x_n)$ that contains exactly one occurrence of $\mathcal{O}_{\beta_1, \beta_2, \dots, \beta_n}$.*

Proof. Once again, the proof is by induction on n and the base case was established in the proof of Proposition 7 in [5]. By the induction hypothesis, we can construct an $(n - 1)$ -dimensional object $\Psi = \Psi(x_1, x_2, \dots, x_{n-1})$ that contains exactly one occurrence of $\mathcal{O}_{\beta_1, \beta_2, \dots, \beta_{n-1}}$. By permuting $[x_1], [x_2], \dots,$ and $[x_{n-1}]$ we may position the occurrence of $\mathcal{O}_{\beta_1, \beta_2, \dots, \beta_{n-1}}$ in any of the $k = \prod_{i=1}^{j-1} \binom{x_i}{\beta_i}$ possible positions. Place an order on these positions. Let Ψ_{q, p_i} be the $(n - 1)$ -dimensional object that contains exactly one occurrence of $\mathcal{O}_{\beta_1, \beta_2, \dots, \beta_{n-1}}$ as a box of q 's ($q \in \{0, 1\}$) in position p_i . We can obtain the desired Ω by layering each of $\Psi_{0, p_1}, \Psi_{0, p_2}, \dots, \Psi_{0, p_k}, \Psi_{1, p_1}, \Psi_{1, p_2}, \dots,$ and Ψ_{1, p_k} exactly $\beta_n - 1$ times and adding an additional layer Ψ where Ψ is any $(n - 1)$ -dimensional object that contains exactly one occurrence of $\mathcal{O}_{\beta_1, \beta_2, \dots, \beta_{n-1}}$. \square

Since the object $\Omega = \Omega(x_1, x_2, \dots, x_n - 1)$ obtained by layering each of $\Psi_{0, p_1}, \Psi_{0, p_2}, \dots, \Psi_{0, p_k}, \Psi_{1, p_1}, \Psi_{1, p_2}, \dots,$ and Ψ_{1, p_k} exactly $\beta_n - 1$ times does not contain an occurrence of $\mathcal{O}_{\beta_1, \beta_2, \dots, \beta_n}$, the theorem follows. \square

5 Maximal number of 1's in binary objects avoiding zero-box patterns

Let p be a pattern. We define $f(\alpha_1, \alpha_2, \dots, \alpha_k; p)$ to be the maximal number of 1's in a binary object of size $\alpha_1 \times \alpha_2 \times \dots \times \alpha_k$ not containing an occurrence of p .

Studying $f(\alpha_1, \alpha_2; p)$ is related to Turán's theory in extremal graph theory where we have the following question: Given a graph G , what is the maximum number of edges in an n -vertex graph which does not contain G as a subgraph? If we restrict our attention to the universe of bipartite graphs, then we may view the bipartite graphs as matrices (2-dimensional objects). As was mentioned in [2], the key difference between Turán's theory and our question is order; that is, the vertices in our case (the rows and columns) are ordered. However, in some special cases the restriction on the order is insignificant.

Another example more closely related to our work is the problem of determining the maximal number of 1's in a matrix that avoids a specified *configuration* [2]. A configuration, $C = (c_{ij})$ ($1 \leq i \leq u, 1 \leq j \leq v$), is a partial matrix with 1's and blanks (holes) as the entries (rather than 1's, 0's and holes as in our case). A binary matrix $M = (m_{ij})$ has the configuration C if there exist u rows $i_1 < i_2 < \dots < i_u$ and v columns $j_1 < j_2 < \dots < j_v$ in M such that the corresponding sub-matrix contains C , i.e. $m_{i_\alpha, j_\beta} = 1$ whenever $c_{\alpha, \beta} = 1$. For a configuration C , one usually considers the problem of determining the asymptotics of $f(n_1, n_2; C)$. If $n_1 = n_2 = n$, then the asymptotic behavior of $f(n; C) = f(n, n; C)$ for several forbidden C 's is determined in [2]. In [7], a conjecture of Füredi and Hajnal was proven which states that $f(n, C)$ grows linearly if C is a permutation matrix. Intriguingly, the proof of that result settled at once the well-known conjecture of Stanley–Wilf on the asymptotics of the number of permutations avoiding a given pattern (e.g., see [4, 6]).

For the sake of simplicity, we assume that the patterns and the objects under our consideration have no holes. If a pattern p contains both 0's and 1's, then the $\alpha_1 \times \alpha_2 \times \dots \times \alpha_k$ object containing only 1's obviously avoids p , so $f(\alpha_1, \alpha_2, \dots, \alpha_k; p) = \alpha_1 \cdot \alpha_2 \cdot \dots \cdot \alpha_k$. Thus, we restrict our attention to zero-box patterns.

It is worth mentioning that determining the maximal number of 1's in a binary object which avoids a given zero-box pattern can be formulated as a 0/1-integer linear programming problem (0/1-IP). Unfortunately, this IP has a large number of constraints. Specifically, if we set

$$S = \{I_1 \times I_2 \times \dots \times I_n \mid I_j \subseteq [\alpha_j], |I_j| = \beta_j\}$$

then $f(\alpha_1, \alpha_2, \dots, \alpha_n; \mathcal{O}_{\beta_1, \beta_2, \dots, \beta_n})$ is equal to the objective function value of the following 0/1-IP:

$$\max z = \sum_{(i_1, i_2, \dots, i_n) \in X_{\alpha_1, \alpha_2, \dots, \alpha_n}} x_{i_1 i_2 \dots i_n}$$

subject to

$$\sum_{(i_1, i_2, \dots, i_n) \in S} x_{i_1 i_2 \dots i_n} \geq 1 \tag{1}$$

$$\sum_{(i_1, i_2, \dots, i_n) \in S} x_{i_1 i_2 \dots i_n} \leq \beta_1 \cdot \beta_2 \cdot \dots \cdot \beta_n - 1 \tag{2}$$

$$x_{j_1 j_2 \dots j_n} \in \{0, 1\} \quad \forall (j_1, j_2, \dots, j_n) \in X_{\alpha_1, \alpha_2, \dots, \alpha_n}.$$

Here (1) prohibits the occurrence of $\mathcal{O}_{\beta_1, \beta_2, \dots, \beta_n}$ as a box of zeros and (2) prohibits its occurrence as a box of ones. Since this IP contains $\alpha_1 \cdot \alpha_2 \cdot \dots \cdot \alpha_n$ variables and $2 \prod_{i=1}^n \binom{\alpha_i}{\beta_i}$ equations, we note that it is not practical to solve even moderate instances of this problem with the integer programming approach. Hence we seek combinatorial methods for solving this problem.

Let us first consider *segmented-occurrence* of patterns which is more structured than the pattern occurrence we have considered thus far. We will call $S \subseteq [k]$ a *consecutive subset* of $[k]$ if there exist $1 \leq i \leq j \leq k$ such that $S = \{i, i + 1, \dots, j\}$ and $T \subseteq X_{\alpha_1, \alpha_2, \dots, \alpha_n}$ a *consecutive block* of $X_{\alpha_1, \alpha_2, \dots, \alpha_n}$ if there are consecutive subsets T_1, T_2, \dots , and T_n of $[\alpha_1], [\alpha_2], \dots$, and $[\alpha_n]$, respectively, such that $T = T_1 \times T_2 \times \dots \times T_n$. We then say that the object $\Omega = \Omega(\alpha_1, \alpha_2, \dots, \alpha_n)$ has a *segmented-occurrence* of the pattern $p = p(\beta_1, \beta_2, \dots, \beta_n)$ if there exists a consecutive block T of $X_{\alpha_1, \alpha_2, \dots, \alpha_n}$ such that p occurs on T in Ω . For instance, in the case of matrices and $p = p(\beta_1, \beta_2)$, a segmented-occurrence of p corresponds to the intersection of β_1 consecutive rows and β_2 consecutive columns.

For a given pattern p , we define $f^s(\alpha_1, \alpha_2, \dots, \alpha_n; p)$ to be the maximal number of 1's in a binary object of size $\alpha_1 \times \alpha_2 \times \dots \times \alpha_n$ avoiding a segmented-occurrence of p . Observe, that if we formulate a 0-1 IP in the case of segmented-occurrence of patterns, the number of constraints is significantly reduced to $2 \prod_{i=1}^n (\alpha_i - \beta_i + 1)$ for the general case of n dimensional patterns and objects.

THEOREM 3 Let $\mathcal{O}_{\beta_1, \beta_2, \dots, \beta_n}$ be a zero-box pattern and $\alpha_1, \alpha_2, \dots, \alpha_n$ be integers with $\alpha_i \geq \beta_i$ for all $i \in [n]$. Then

$$f^s(\alpha_1, \alpha_2, \dots, \alpha_n; \mathcal{O}_{\beta_1, \beta_2, \dots, \beta_n}) = \prod_{i=1}^n \alpha_i - \prod_{i=1}^n \left\lfloor \frac{\alpha_i}{\beta_i} \right\rfloor.$$

Proof. Partition the object $\Omega = \Omega(\alpha_1, \alpha_2, \dots, \alpha_n)$ into the consecutive blocks $B(i_1, i_2, \dots, i_n) = \{(x_1, x_2, \dots, x_n, f(x_1, x_2, \dots, x_n)) \in \Omega \mid 1 + \beta_j i_j \leq x_j \leq 1 + \beta_j(i_j + 1)\}$ where $i_j \in \{0, 1, \dots, \lfloor \frac{\alpha_j}{\beta_j} \rfloor\}$ for all $j \in [n]$. Clearly, the object contains $\prod_{i=1}^n \lfloor \frac{\alpha_i}{\beta_i} \rfloor$ consecutive blocks of size $\beta_1 \times \beta_2 \times \dots \times \beta_n$. In order to avoid an occurrence of $\mathcal{O}_{\beta_1, \beta_2, \dots, \beta_n}$, each of these blocks must contain at least one zero. Hence $f^s(\alpha_1, \alpha_2, \dots, \alpha_n; \mathcal{O}_{\beta_1, \beta_2, \dots, \beta_n}) \leq \prod_{i=1}^n \alpha_i - \prod_{i=1}^n \lfloor \frac{\alpha_i}{\beta_i} \rfloor$.

On the other hand, suppose that for all $(x_1, x_2, \dots, x_n) \in X_{\alpha_1, \alpha_2, \dots, \alpha_n}$

$$f(x_1, x_2, \dots, x_n) = \begin{cases} 0, & \text{if } x_j \text{ is a multiple of } \beta_j \forall j \in [n] \\ 1, & \text{otherwise.} \end{cases}$$

In the two-dimensional case we have the object in Figure 1. In this case, any consecutive block of Ω of size $\beta_1 \times \beta_2 \times \dots \times \beta_n$ contains a zero and, thus, avoids $\mathcal{O}_{\beta_1, \beta_2, \dots, \beta_n}$. Hence, $f^s(\alpha_1, \alpha_2, \dots, \alpha_n; \mathcal{O}_{\beta_1, \beta_2, \dots, \beta_n}) \geq \prod_{i=1}^n \alpha_i - \prod_{i=1}^n \lfloor \frac{\alpha_i}{\beta_i} \rfloor$. \square

1	1	1	1
0	0	0	0
1	1	1	1
0	0	0	0
1	1	1	1

Figure 1: Partition of a matrix into the blocks.

Let us now consider our usual (non-segmented) pattern occurrence. The crucial difference between segmented- and non-segmented pattern occurrence is the existence of vanishing borders (see Section 4). For the zero-box pattern $\mathcal{O} = \mathcal{O}_{\beta_1, \beta_2, \dots, \beta_n}$, we restrict our attention to objects of dimensions $\alpha_1 \times \alpha_2 \times \dots \times \alpha_n$ where $a_{\alpha_1, \alpha_2, \dots, \alpha_n} > 0$.

THEOREM 4 Suppose $\mathcal{O} = \mathcal{O}_{\beta_1, \beta_2, \dots, \beta_n}$ is a zero-box pattern, and for all $i \in [n]$, $a_{\alpha_1, \alpha_2, \dots, \alpha_i} > 0$. Then, $f(\alpha_1; \mathcal{O}_{\beta_1}) = \beta_1 - 1$, and for $n \geq 2$,

$$f(\alpha_1, \alpha_2, \dots, \alpha_n; \mathcal{O}) \geq \alpha_1 \cdot \alpha_2 \cdot \dots \cdot \alpha_{n-1} \cdot (\beta_n - 1) + (\alpha_n - \beta_n + 1)f(\alpha_1, \alpha_2, \dots, \alpha_{n-1}; \mathcal{O}'),$$

where $\mathcal{O}' = \mathcal{O}_{\beta_1, \beta_2, \dots, \beta_{n-1}}$.

Proof. The case $n = 1$ is easy to see.

Now consider the case $n \geq 2$. As in the proof of Theorem 1, we view our object Ω as a sequence containing α_n elements where each element is an $\alpha_1 \times \alpha_2 \times \dots \times \alpha_{n-1}$ layer. Construct an object Ω as follows: fill any $\beta_n - 1$ of these layers with 1's (this gives $\alpha_1 \cdot \alpha_2 \cdot \dots \cdot \alpha_{n-1} \cdot (\beta_n - 1)$ 1's) and fill the other layers in such way that each of them avoids the pattern \mathcal{O}' . The maximum number of ones we can include in these layers is given by $(\alpha_n - \beta_n + 1)f(\alpha_1, \alpha_2, \dots, \alpha_{n-1}; \mathcal{O}')$. It is easy to see that the object Ω constructed in this way avoids \mathcal{O} , and thus we obtained a lower bound for the maximum number of 1's. \square

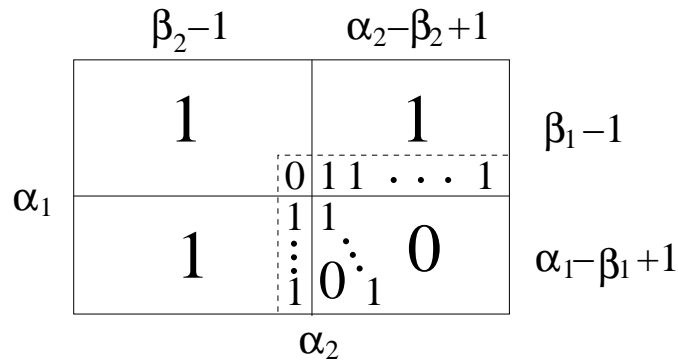


Figure 2: A matrix avoiding $\mathcal{O}_{\beta_1, \beta_2}$ with a large number of 1's.

REMARK 5 The lower bound in the statement of Theorem 4 is tight. For example, it is not hard to see that we have equality if $n = 2$, $\alpha_1 = \beta_1$, and $\alpha_2 = \beta_2$. So, in general we can not get a better lower bound. However, for some particular cases, the lower bound can be improved. For instance, suppose that $n = 2$, $\beta_1 \geq \alpha_1/2$ and $\beta_2 \geq \alpha_2/2$. The following statement holds in this case: $f(\alpha_1, \alpha_2; \mathcal{O})$ is greater than or equal to

$$\alpha_1 \cdot \alpha_2 - (\alpha_2 - \beta_2 + 1)(\alpha_1 - \beta_1 + 1) + \alpha_1 - \beta_1,$$

if $\alpha_1 - \beta_1 \leq \alpha_2 - \beta_2$, and

$$\alpha_1 \cdot \alpha_2 - (\alpha_2 - \beta_2 + 1)(\alpha_1 - \beta_1 + 1) + \alpha_2 - \beta_2,$$

otherwise.

To prove this, we may assume that $\alpha_1 - \beta_1 \leq \alpha_2 - \beta_2$ (the second case is similar). Construct a matrix M as shown schematically in Figure 2, where $M(\beta_1 - 1, \beta_2 - 1) = 0$, $M(\beta_1, \beta_2) = 1$, $M(\beta_1 + 1, \beta_2 + 1) = 1$, and so on up to $M(\alpha_1, \beta_2 + \alpha_1 - \beta_1) = 1$; all other elements in the three blocks are 1s and in the lower-right block are 0's. It is easy to see that the lower-right block indicated

by a dashed line does not contain an occurrence of $\mathcal{O}_{2,2}$ as a box of 1's, and thus M does not contain an occurrence of $\mathcal{O}_{\beta_1, \beta_2}$ as a box of 1's.

Moreover, since $\beta_1 \geq \alpha_1/2$ and $\beta_2 \geq \alpha_2/2$, M does not contain an occurrence of $\mathcal{O}_{\beta_1, \beta_2}$ as a box of 0's. Hence M avoids $\mathcal{O}_{\beta_1, \beta_2}$, and the number of 1's in M is as stated.

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