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On fuzzy ideals of BCI-algebras

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Abstract. The fuzzy p-ideals, fuzzy H-ideals and fuzzy BCI-positive implicative ideals of BCI-algebras are studied, and related properties are investigated. We also give some characterizations of these ideals.

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1 Introduction

BCK/BCI-algebras are two important classes of logical algebras introduced by Iseki in 1966 (see [5,6,14]). Since then, a great deal of literature has been produced on the theory of BCK/BCI-algebras. In particular, emphasis seems to have been put on the ideal theory of BCK/BCI-algebras (see [15]).

Meng [15] introduced the notions of implicative ideals and commutative ideals in BCK-algebras and applied them to characterize implicative BCK-algebras and commutative BCK-algebras, respectively. Meng also showed that a nonempty subset of a BCK-algebra is an implicative ideal if and only if it is both a commutative ideal and a positive implicative ideal. All the above results motivate us to further investigate the relations between algebras and ideals and between ideals and ideals. The notion of quasi-associative BCI-algebras was introduced by Xi [19]. They are all important classes of BCI-algebras. In [21], the notion of p-ideals was introduced and used to characterized-semisimple BCI-algebras.

The concept of fuzzy subset and various operations on it were first introduced by Zadeh in [20]. Since then, fuzzy subsets have been applied to diverse field. The study of fuzzy subsets and their application to mathematical contexts has reached to what is now commonly called fuzzy mathematics. Fuzzy algebra is an important branch of fuzzy mathematics. The study of fuzzy algebraic structures was started with the introduction of the concept of fuzzy sub-groups in 1971 by Rosenfeld [17]. Since then these ideas have been applied to other algebraic structures such as semigroups, rings, ideals, modules and vector spaces. In

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1999, Ougen [16] defined fuzzy subsets in BCK-algebras and investigated some properties. In 1993, Jun [8] applied it in BCI-algebras. Fuzzy p-ideals and fuzzy H-ideals in BCI-algebras is introduced in [9], [10] respectively and several interesting properties of these concepts are studied. Following [9] and [10], we study and give some characterizations of these ideals.

BCK/BCI-algebras are two important classes of logical algebras introduced by Iseki in 1966 (see [5,6]):

An algebra $(X; *, 0)$ of type (2,0) is called a BCI-algebra if it satisfies the following axioms:

(I) $((x * y) * (x * z)) * (z * y) = 0$,

(II) $(x * (x * y)) * y = 0$,

(III) $x * x = 0$,

(IV) $x * y = 0$ and $y * x = 0$ imply $x = y$.

In a BCI-algebra X, a partially ordered relation $\leq$ can be defined by

$$x \leq y \text{ if and only if } x * y = 0.$$

A BCI-Algebra X satisfies the following properties:

(1.1) $(x * y) * z = (x * z) * y$,

(1.2) $x * 0 = x$,

(1.3) $0 * (x * y) = (0 * x) * (0 * y)$,

(1.4) $0 * (0 * (x * y)) = 0 * (y * x)$,

(1.5) $(x * z) * (y * z) \leq x * y$,

(1.6) $x * y = 0$ implies $x * z \leq y * z$ and $z * y \leq z * x$.

Definition 1.1 A subset A of a BCI-algebra X is called an ideal if (i_1) $0 \in A$, (i_2) for any $x, y \in X$, $x * y$, $y \in A$ imply $x \in A$.

Definition 1.2 (Jun [8]) An ideal A in a BCI-algebra X is said to be closed if for all $x \in X$, $0 * x \in A$ implies $x \in A$.

Definition 1.3 (Zadeh [20]) Let X be a set. A fuzzy set μ in X is a function $\mu : X \to [0, 1]$.

Notation. *Let μ and ν be fuzzy sets in a BCI-algebra X. By $\mu \leq \nu$ we mean that $\mu(x) \leq \nu(x)$ for any $x \in X$.*

Definition 1.4 ([8]) Let μ be a fuzzy subset of a BCI-algebra X. For $t \in [0, 1]$, the set $\mu_t = \{x \in X \mid \mu(x) \geq t\}$ is called a level subset of μ.

DEFINITION 1.5 ([8]) A fuzzy set μ in a BCI-algebra X is said to be a fuzzy ideal in X if it satisfies

(F_1) $\mu(0) \geq \mu(x)$,

(F_2) $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$ for all $x, y \in X$.

DEFINITION 1.6 (JUN [8]) A fuzzy ideal μ in a BCI-algebra X is said to be closed if for all $x \in X$, $\mu(0 * x) \geq \mu(x)$.

PROPOSITION 1.7 ([13]) *Let μ be a fuzzy ideal in a BCI-algebra X, then $x \leq y$ implies $\mu(y) \leq \mu(x)$.*

PROPOSITION 1.8 ([13]) *A fuzzy set μ satisfying* (F_1) *in a BCI-algebra X is a fuzzy ideal if and only if for all $x, y, z \in X$, $(x * y) * z = 0$ implies $\mu(x) \geq \min\{\mu(y), \mu(z)\}$.*

2 Fuzzy p-ideals of BCI-algebras

In [21], the notion of a p-ideal of a BCI-algebra was introduced and various properties were proved. In this section fuzzy p-ideals of BCI-algebras are studied.

DEFINITION 2.1 ([21, ZHANG]) A nonempty subset A in a BCI-algebra X is called a p-ideal of X if it satisfies

(i_1) $0 \in A$,

(i_3) If for all $x, y, z \in X$, $(x * z) * (y * z) \in A$ and $y \in A$, imply that $x \in A$.

If we put $z = 0$, then it follows that A is an ideal. Thus every p-ideal is an ideal.

PROPOSITION 2.2 ([21]) *An ideal A of a BCI-algebra X is a p-ideal of X if and only if $0 * (0 * x) \in A$ implies $x \in A$, where $x \in X$.*

DEFINITION 2.3 ([9]) Let X be a BCI-algebra. A fuzzy subset μ in X is called a fuzzy p-ideal if it satisfies

(F_1) $\mu(0) \geq \mu(x)$,

(F_2) $\mu(x) \geq \min\{\mu((x * z) * (y * z)), \mu(y)\}$, for all $x, y, z \in X$.

PROPOSITION 2.4 ([11]) *Let X be a BCI-algebra. A fuzzy ideal μ of a BCI-algebra is a fuzzy p-ideal if and only if $\mu(x) \geq \mu(0 * (0 * x))$.*

Now, we give an extension theorem about fuzzy p-ideals of BCI-algebras.

THEOREM 2.5 *Let μ and ν be fuzzy ideals of a BCI-algebra X such that $\mu \leq \nu$ and $\mu(0) = \nu(0)$. If μ is a fuzzy p-ideal of X, then so is ν.*

Proof. By Proposition 2.4, it is enough to show that $\nu(x) \geq \nu(0*(0*x))$ for each $x \in X$. Putting $s = 0*(0*x)$, then $0*(0*(x*s)) = (0*(0*x))*(0*(0*s)) = (0*(0*x))*(0*(0*(0*(0*x)))) = (0*(0*x))*(0*(0*x)) = 0$, by properties (1.3) and (1.4). Hence $\mu(0*(0*(x*s))) = \mu(0) = \nu(0)$. Since μ is a fuzzy p-ideal of X, and using Proposition 2.4, we get $\mu(x*s) \geq \mu(0*(0*(x*s))) = \nu(0)$. Thus $\nu(x*s) \geq \mu(x*s) \geq \nu(0) \geq \nu(s)$. Since ν is a fuzzy ideal, we have $\nu(x) \geq \min\{\nu(x*s), \nu(s)\} = \nu(s) = \nu(0*(0*x))$. So ν is a fuzzy p-ideal of X.

THEOREM 2.6 *A fuzzy set μ of a BCI-algebra X is a fuzzy p-ideal of X if and only if for all $t \in [0,1]$, μ_t is either empty or a p-ideal of X.*

Proof. Let μ be a fuzzy p-ideal in X and $\mu_t \neq \emptyset$ for $t \in [0,1]$. Since $\mu(0) \geq \mu(x) \geq t$ for any $x \in \mu_t$, we get $0 \in \mu_t$. If $(x*z)*(y*z) \in \mu_t$ and $y \in \mu_t$, then $\mu((x*z)*(y*z)) \geq t$ and $\mu(y) \geq t$. By (F_3), we have $\mu(x) \geq \min\{\mu((x*z)*(y*z)), \mu(y)\} \geq t$. Hence $x \in \mu_t$. This means that μ_t is a p-ideal of X. Conversely, suppose that for each $t \in [0,1]$, μ_t is either empty or a p-ideal of X. For any $x \in X$, setting $\mu(x) = t$, then $x \in \mu_t$. Since $\mu_t (\neq \emptyset)$ is a p-ideal of X, we have $0 \in \mu_t$ and hence $\mu(0) \geq t = \mu(x)$. Thus $\mu(0) \geq \mu(x)$ for all $x \in X$. Now we prove that μ satisfies (F_3). If not, then there exist $x_0, y_0, z_0 \in X$ such that $\mu(x_0) < \min\{\mu((x_0*z_0)*(y_0*z_0)), \mu(y_0)\}$. Put $t_0 = 1/2[\mu(x_0) + \min\{\mu((x_0*z_0)*(y_0*z_0)), \mu(y_0)\}]$, then $\mu(x_0) < t_0 < \min\{\mu((x_0*z_0)*(y_0*z_0)), \mu(y_0)\}$. Hence $(x_0*z_0)*(y_0*z_0) \in \mu_{t_0}$ and $y_0 \in \mu_{t_0}$. But $x_0 \notin \mu_{t_0}$, thus μ_{t_0} is not a p-ideal of X. This contradicts the hypothesis. Therefore μ is a fuzzy p-ideal of X.

THEOREM 2.7 *Let I be a p-ideal of a BCI-algebra X. Then there exists a fuzzy p-ideal μ of X such that $\mu_t = I$, for some $t \in (0,1]$.*

Proof. Define $\mu : X \longrightarrow [0,1]$ by

$$\mu(x) = \begin{cases} t; & x \in I \\ 0; & x \notin I, \end{cases}$$

where t is a fixed number in $(0,1]$. We show that μ is a fuzzy p-ideal of X. Since I is a p-ideal of X, if $0*(0*x) \in I$ then $x \in I$. Hence $\mu(0*(0*x)) = \mu(x) = t$. If $0*(0*x) \notin I$, $x \in I$ then $\mu(x) = t > 0 = \mu(0*(0*x))$ and if $0*(0*x) \notin I$, $x \notin I$ then $\mu(x) = 0 = \mu(0*(0*x))$. Therefore, $\mu(x) \geq \mu(0*(0*x))$. This means that μ satisfies (F_3). Since $0 \in I$, $\mu(0) = t \geq \mu(x)$ for all $x \in X$ and so μ satisfies F_1. Then μ is a fuzzy p-ideal of X. It is clear that $\mu_t = I$ and so the result follows.

3 Fuzzy H-ideals of BCI-algebras

In [10], the notion of a H-ideal of a BCI-algebra was introduced and various properties were proved. The aim of this section is to investigate the properties of fuzzy H-ideals of BCI-algebras.

Definition 3.1 ([10]) A nonempty subset A of a BCI-algebra X is called an H-ideal of X if it satisfies

(i_1) $0 \in A$,

(i_4) $x * (y * z) \in A$, $y \in A$ imply $x * z \in A$.

If we put $z = 0$, then it follows that A is an ideal. Thus, every H-ideal is an ideal.

Definition 3.2 ([10]) A fuzzy subset μ of a BCI-algebra X is called a fuzzy H-ideal if it satisfies

(F_1) $\mu(0) \geq \mu(x)$,

(F_5) $\mu(x * z) \geq \min\{\mu(x * (y * z)), \mu(y)\}$, for all $x, y, z \in X$.

Clearly $z = 0$ gives μ is a fuzzy ideal of X.

Proposition 3.3 ([10]) *A fuzzy set μ of a BCI-algebra X is a H-ideal of X if and only if for any $t \in [0, 1]$, μ_t is either empty or an H-ideal of X.*

Theorem 3.4 *Let μ be a fuzzy ideal of a BCI-algebra X. Then the following are equivalent:*

(i) *μ is a fuzzy H-ideal of X,*

(ii) *$\mu((x * y) * z) \geq \mu(x * (y * z))$ for all $x, y, z \in X$,*

(iii) *$\mu(x * y) \geq \mu(x * (0 * y))$.*

Proof. (i)→(ii) Since μ is a fuzzy H-ideal of X, we have $\mu((x * y) * z) \geq \min\{\mu((x*y)*(0*z)), \mu(0)\} = \mu((x*y)*(0*z))$. On the other hand $(x*y)*(0*z) = (x * y) * ((y * z) * y) \leq x * (y * z)$, thus $\mu(x * (y * z)) \leq \mu((x * y) * (0 * z))$.

(ii)→(iii) Letting $y = 0$ and $z = y$ in (F_5).

(iii)→(i) Since $(x * (0 * y)) * (x * (z * y)) \leq (z * y) * (0 * y) \leq z$, by Proposition 2.4 we have $\mu(x * (0 * y)) \geq \min\{\mu(x * (z * y)), \mu(z)\}$. Hence by hypothesis $\mu(x * y) \geq \min\{\mu(x * (z * y)), \mu(z)\}$. Therefore μ is a fuzzy H-ideal of X.

Theorem 3.5 *Let μ be an fuzzy ideal of BCI-algebra X. If $\mu(x * y) \geq \mu(x)$ for all $x, y \in X$, then μ is a H-ideal of X.*

Proof. Since μ is an ideal of X, by hypothesis we have $\min\{\mu(x*(y*z)), \mu(y)\} \leq \min\{\mu((x * z) * (y * z)), \mu(y * z)\} \leq \mu(x * z)$, for all $x, y, z \in X$.

Definition 3.6 ([11]) Let X be a BCI-algebra. A fuzzy set μ in X is said to be a fuzzy subalgebra of X if $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$.

THEOREM 3.7 *A fuzzy H-ideal μ of a BCI-algebra X is a fuzzy subalgebra of X.*

Proof. If μ is a fuzzy H-ideal, then by (F_5), we have $\mu(x * z) \geq \min\{\mu(x * (y * z)), \mu(y)\}$. Putting $z = y$, then $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$. This shows that μ is a subalgebra of X.

Now, we give an extension theorem about fuzzy H-ideals of BCI-algebras.

THEOREM 3.8 *Let μ and ν be fuzzy ideals of a BCI-algebra X such that $\mu \leq \nu$ and $\mu(0) = \nu(0)$. If μ is a fuzzy H-ideal of X, then so is ν.*

Proof. By Theorem 3.4 (iii) it is enough to show that $\nu(x * y) \geq \nu(x * (0 * y))$ for each $x, y \in X$. Putting $s = x * (0 * y)$, we have $(x * s) * (0 * y) = 0$. Hence $\mu((x * s) * (0 * y)) = \mu(0) = \nu(0)$. By Theorem 3.4 (iii), since μ is a fuzzy H-ideal of X, $\mu((x * s) * y) \geq \mu((x * s) * (0 * y)) = \nu(0)$. Thus, $\nu((x * y) * s) \geq \mu((x * y) * s) \geq \nu(0) \geq \nu(s)$. Since ν is a fuzzy ideal, we have $\nu(x * y) \geq \min\{\nu((x * y) * s), \nu(s)\} = \nu(s) = \nu(x * (0 * y))$ and the result follows.

PROPOSITION 3.9 ([8]) *A fuzzy set μ of a BCI-algebra X is a closed fuzzy ideal of X if and only if for every $t \in [0, 1]$, μ_t is either empty or a closed ideal of X.*

THEOREM 3.10 *Let I be a H-ideal of a BCI-algebra X. Then there exists a fuzzy H-ideal μ of X such that $\mu_t = I$ for some $t \in (0, 1]$.*

Proof. The proof is similar to that of Theorem 2.6.

THEOREM 3.11 *Let X be a BCI-algebra. Then the following are equivalent:*

(i) *every H-ideal of X is closed,*

(ii) *every fuzzy H-ideal of X is a closed fuzzy ideal of X.*

Proof. Let μ be a fuzzy H-ideal of X, then by Proposition 3.3, μ_t is a H-ideal of X. Thus μ_t is a closed ideal of X. Hence by Proposition 3.9, μ is a closed fuzzy ideal of X. Conversely, assume that I is a H-ideal of X. By Theorem 3.10, there exists a fuzzy H-ideal μ of X such that $\mu_t = I$ for some $t \in (0, 1]$. Since μ is a fuzzy H-ideal of X, μ is a closed fuzzy ideal of X. Hence by Proposition 3.9, μ_t is a closed ideal of X. So I is a closed ideal of X.

EXAMPLE 3.12 Let $X = \{0, 1, 2, 3\}$. Define $*$ on X as

$*$	0	1	2	3
0	0	0	3	2
1	1	0	3	2
2	2	2	0	3
3	3	3	2	0

Then X is a BCI-algebra. Define $\mu : X \longrightarrow [0, 1]$ as $\mu(0) = \mu(1) = 1$ and $\mu(2) = \mu(3) = 1/2$. Then routine calculations show that μ is a fuzzy p-ideal of

X but it is not an H-ideal of X. Since $\mu(3) = \mu(2 * 3) < \mu(2 * (0 * 3)) = \mu(0)$ this contradicts Theorem 3.4 (iii).

Example 3.13 ([10]) Let $X = \{0, l, m, n, p, q\}$. Define $*$ on X as

$*$	0	l	m	n	p	q
0	0	0	0	n	n	n
l	l	0	l	p	n	p
m	m	m	0	q	q	n
n	n	n	n	0	0	0
p	p	n	p	l	0	l
q	q	q	n	m	m	0

Then X is a BCI-algebra. Define $\mu : X \to [0,1]$ as $\mu(0) = t_1$, $\mu(l) = t_2$, $\mu(m) = \mu(q) = \mu(p) = \mu(n) = t_3$, for $t_1 > t_2 > t_3$. Then routine calculations show that μ is a fuzzy H-ideal of X but it is not a fuzzy p-ideal of X, since $\mu(m) < \min\{\mu((m * m) * (0 * m)), \mu(0)\}$. This contradicts Definition 2.3.

4 Fuzzy positive implicative BCI-algebra

In this section we obtain some characterizations of fuzzy BCI-positive implicative ideals of BCI-algebras and investigate their properties.

Definition 4.1 ([11]) Let X be a BCI-algebra. A fuzzy set μ in X is called a fuzzy BCI-positive implicative ideal of X if it satisfies (F_1) and

(F_4) $\mu(x * z) \geq \min\{\mu(((x * z) * z) * (y * z)), \mu(y)\}$ for all $x, y, z \in X$.

Proposition 4.2 ([11]) *Let μ be a fuzzy ideal of a BCI-algebra X. Then the following are equivalent:*

(i) *μ a fuzzy BCI-positive implicative ideal of X,*

(ii) *$\mu(x * y) = \mu(((x * y) * y) * (0 * y))$ for all $x, y \in X$,*

(iii) *$\mu(x * y) \geq \mu(((x * y) * y) * (0 * y))$ for all $x, y \in X$.*

Corollary 4.3 ([11]) *Any fuzzy p-ideal is a fuzzy BCI-positive implicative ideal, but the converse is not true.*

Corollary 4.4 *Let I be a p-ideal of a BCI-algebra X. Then there exists a fuzzy BCI-positive implicative ideal μ of X such that $\mu_t = I$, for some $t \in (0, 1]$.*

Proof. Combining Theorem 2.6 and Corollary 4.3 implies the result.

Recall that for each x, y in a BCI-algebra X, $x \wedge y$ is defined to be $x \wedge y = y * (y * x)$.

Corollary 4.5 *Let X be a BCI-algebra and μ a fuzzy BCI-positive implicative ideal of X such that $\mu(x \wedge y) \leq \mu(y)$. Then μ is a H-ideal of X.*

Proof. Since μ is fuzzy BCI-positive implicative by Proposition 4.2, we have $\mu(x*y) \geq \mu(((x*y)*y)*(0*y)) \geq \min\{\mu((((x*y)*y)*(0*y))*x), \mu(x)\} = \min\{\mu(0*y), \mu(x)\} \geq \min\{\mu(x \wedge (0*y)), \mu(x)\} = \mu(x)$. Thus by Theorem 3.5, μ is a H-ideal of X.

Theorem 4.6 *Let μ be a fuzzy ideal of a BCI-algebra X. If for any $x, y \in X$, $\mu(x*(x*(y*(y*x)))) \geq \mu((x*(x*y))*(y*x))$ then μ is a fuzzy BCI-positive implicative ideal.*

Proof. Putting $u = x*y$, we have $\mu((u*(u*x))*(x*u)) = \mu(((x*y)*((x*y)*x))*(x*(x*y))) = \mu(((x*y)*(0*y))*(x*(x*y))) = \mu(((x*(x*(x*y)))*y)*(0*y)) = \mu(((x*y)*y)*(0*y))$. On the other hand $\mu(u*(u*(x*(x*u)))) = \mu((x*y)*((x*y)*((x*y)*(x*(x*(x*y)))))) = \mu((x*y)*((x*y)*(x*y))) = \mu(x*y)$. By hypothesis we have $\mu(u*(u*(x*(x*u)))) \geq \mu((u*(u*x))*(x*u))$. Therefore $\mu(x*y) \geq \mu(((x*y)*y)*(0*y))$. Hence by Proposition 4.2 (iii), μ is a fuzzy BCI-positive implicative ideal.

Definition 4.7 (see also [2]) Let μ be a fuzzy ideal of a BCI-algebra X and $x \in X$. Then the fuzzy subset μ_x which is defined by $\mu_x(y) = \min\{\mu(x*y), \mu(y*x)\}$ is called a fuzzy coset of μ. The set of all fuzzy cosets of μ is denoted by X/μ.

Proposition 4.8 ([1]) *A BCI-algebra is positive implicative (i.e., weakly positive implicative) if and only if it satisfies $x*y = ((x*y)*y)*(0*y)$.*

In [11], it is proved that, if μ is a fuzzy BCI-positive implicative ideal, then $(X/\mu, *, \mu_0)$ is a positive implicative BCI-algebra, where for each $\mu_x, \mu_y \in X/\mu$, $\mu_x * \mu_y = \mu_{x*y}$. Now we show that the converse is valid for closed fuzzy ideals.

Theorem 4.9 *Let μ be a closed fuzzy ideal of a BCI-algebra X. Then μ is a fuzzy BCI-positive implicative ideal if and only if $(X/\mu, *, \mu_0)$ is a positive implicative BCI-algebra.*

Proof. Let $(X/\mu, *, \mu_0)$ be a positive implicative BCI-algebra. By Proposition 4.8 we have, $\mu_x * \mu_y = ((\mu_x * \mu_y) * \mu_y) * (\mu_0 * \mu_y)$. Therefore $\mu_{x*y} = \mu_{((x*y)*y)*(0*y)}$. Hence $\mu(0) = \mu((x*y)*(((x*y)*y)*(0*y)))$ and $\mu(0) = \mu((((x*y)*y)*(0*y))*(x*y))$. Since μ is a fuzzy ideal, therefore $\mu(x*y) \geq \min\{\mu((x*y)*(((x*y)*y)*(0*y))), \mu(((x*y)*y)*(0*y))\} = \mu(((x*y)*y)*(0*y))$ and $\mu(((x*y)*y)*(0*y)) \geq \min\{\mu((((x*y)*y)*(0*y))*(x*y)), \mu(x*y)\} = \mu(x*y)$. Hence $\mu(x*y) = \mu(((x*y)*y)*(0*y))$. By Proposition 4.2 (iii), μ is a fuzzy BCI-positive implicative ideal and the result follows by [11, Theorem 5.8].

Theorem 4.10 *Suppose that μ is a closed fuzzy ideal of a BCI-algebra X. Then the following are equivalent:*

(i) *μ is fuzzy BCI-positive implicative.*

(ii) *For all $x, y \in X$, $\mu((x * (x * y)) * (y * x)) = \mu(x * (x * (y * (y * x))))$.*

Proof. (i)→(ii) Assume that μ is fuzzy BCI-positive implicative. By hypothesis μ is a closed fuzzy ideal of X, hence $(X/\mu, *, \mu_0)$ is positive implicative. Therefore we have $(\mu_x * (\mu_x * \mu_y)) * (\mu_y * \mu_x) = \mu_x * (\mu_x * (\mu_y * (\mu_y * \mu_x)))$. Then $\mu_{(x*(x*y))*(y*x)} = \mu_{x*(x*(y*(y*x)))}$, thus $\mu(0) = \mu(((x*(x*y))*(y*x))*(x*(x*(y*(y*x)))))$ and $\mu(0) = \mu((x*(x*(y*(y*x))))*((x*(x*y))*(y*x)))$ for all $x, y \in X$. Since μ is a fuzzy ideal, we have $\mu((x*(x*y))*(y*x)) \geq \mu(x*(x*(y*(y*x))))$ and $\mu(x*(x*(y*(y*x)))) \geq \mu((x*(x*y))*(y*x))$. Hence $\mu((x*(x*y))*(y*x)) = \mu(x*(x*(y*(y*x))))$.
(ii)→(i) By Theorem 4.6 it is clear.

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