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# The weakly pullback flat properties of right Rees factor $S$-acts

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**Abstract.** In this paper, we investigate the characterization of the monoids over which all weakly pullback flat right Rees factor $S$-acts are pullback flat (projective) and the monoids for which all right Rees factor $S$-acts having some flatness properties are weakly pullback flat.

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## 1 Introduction

Throughout this paper, $S$ always stands for a monoid with identity element 1. A nonempty set $A$ is called a right $S$-act, if $S$ acts on $A$ unitarily from the right; that is, there exists a mapping $A \times S \to A, (a, s) \longmapsto as$, satisfying the condition $(as)t = a(st)$ and $a1 = a$, for all $a \in A$ and all $s, t \in S$.

Let $S$ be a monoid and $B$ be a right $S$-act, $B$ is called *flat* if the functor $B \bigotimes -$ (from the category of left $S$-acts to the category of sets) preserves embeddings of left $S$-acts. If this functor preserves embeddings of [principal] left ideals of $S$ into $S$ (considered as a left $S$-act), then $B$ is called [*principally*] *weakly flat*. A right $S$-act $A$ is called *torsion free* if $ac = bc$, with $a, b \in A$ and $c$ right cancellable element of $S$, implies $a = b$.

In [12] Stenström called a right $S$-act $A$ *strongly flat*, if the functor $A \otimes -$ preserves pullbacks and equalizers. In fact he proved that $A$ is strongly flat if and only if it satisfies the following condition (P) and condition (E):

(P) $\quad (\forall\ a, a' \in A)(\forall\ s, s' \in S)(as = a's'$
$\quad\quad \Rightarrow (\exists\ a'' \in A)(\exists\ u, v \in S)(a = a''u \wedge a' = a''v \wedge us = vs'))$.

(E) $\quad (\forall\ a \in A)(\forall\ s, s' \in S)(as = as'$
$\quad\quad \Rightarrow (\exists a' \in A)(\exists\ u \in S)(a = a'u \wedge us = us'))$.

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A right $S$-act $A$ is called *pullback flat*, if the functor $A \otimes -$ preserves pullbacks. It was shown in [2] that, in fact, pullback flatness and strong flatness coincide.

Normak [10] was the first to consider condition (P) and condition (E) on its own. In [8] Valdis Laan defined a new condition (O) and shown that the class of $S$-acts satisfying this condition lies strictly between the classes of strongly flat $S$-acts and $S$-acts satisfying condition (P). And he proved that a right $S$-act $A$ satisfies the condition (O) if and only if it satisfies condition (P) and the following condition (E′):

$$(\mathrm{E}') \qquad (\forall\, a \in A)(\forall\, s, s', z \in S)(as = as' \wedge sz = s'z \\ \Rightarrow (\exists\, a' \in A)(\exists\, u \in S)(a = a'u \wedge us = us')).$$

In [9] Valdis Laan called a right $S$-act $A$ *weakly pullback flat*, if it satisfies condition (P) and condition (E′). For a more complete discussion of flatness of $S$-acts, the reader is referred to [9]. And in [9] the authors introduce the relations of these flatness properties, where the relations of these flatness properties are described as follows

$$\mathrm{FR} \Rightarrow \mathrm{PR} \Rightarrow \mathrm{PF} \Rightarrow \mathrm{WPF} \Rightarrow (\mathrm{P}) \Rightarrow \mathrm{F} \Rightarrow \mathrm{WF} \Rightarrow \mathrm{PWF} \Rightarrow \mathrm{TF}.$$

The abbreviations stand for the following properties of $S$-acts:

| | |
|---|---|
| FR | — free |
| PR | — projective |
| PF | — pullback flatness |
| WPF | — weak pullback flatness |
| F | — flatness |
| WF | — weak flatness |
| PWF | — principally weak flatness |
| TF | — torsion freeness |

A monoid $S$ is said to be *right reversible* if for any $p, q \in S$, there exist $u, v \in S$ such that $up = vq$. A monoid $S$ is said to be *left collapsible* if for any $p, q \in S$, there exists $r \in S$ such that $rp = rq$.

In [11] we characterize the monoids for which all torsion free right Rees factor $S$-acts have some properties that follow from projectivity, such as (weak) flatness, strong flatness, condition (P), etc. In this paper, we give the characterizations of monoids over which all weakly pullback flat right Rees factor $S$-acts are pullback flat (projective) and the monoids for which all right Rees factor $S$-acts having some flatness properties are weakly pullback flat.

# 2 The weakly pullback flat properties of right Rees factor $S$-acts

Definition 2.1 A monoid $S$ is called weakly left collapsible if

$$(\forall s, t \in S)(\forall z \in S)(sz = tz \Rightarrow (\exists u \in S)(us = ut)).$$

LEMMA 2.2 ([6]) *Let $I$ be a proper right ideal of $S$. The right Rees factor $S$-act $S/I$ satisfies condition* (P) *if and only if* $|I| = 1$.

LEMMA 2.3 ([9]) *The one-element right $S$-act* $\Theta$

- *is (weakly) flat if and only if $S$ is right reversible.*
- *satisfies condition* (P) *if and only if $S$ is right reversible.*
- *is always principally weakly flat.*

LEMMA 2.4 *The one-element right $S$-act $\Theta$ satisfies condition* (E$'$) *if and only if $S$ is a weakly left collapsible monoid.*

Proof. It is clear. □

LEMMA 2.5 *The one-element right $S$-act $\Theta$ satisfies condition* (E) *if and only if $S$ is a left collapsible monoid.*

Proof. It is clear. □

THEOREM 2.6 *Let $S$ be a monoid, then the following conditions on monoids are equivalent:*

(1) *All right Rees factor $S$-acts satisfying condition* (P) *are weakly pullback flat.*

(2) *If $S$ is a right reversible monoid, then it is weakly left collapsible.*

Proof. (1)$\Longrightarrow$(2) If $S$ is a right reversible monoid, then by Lemma 2.3 the right Rees factor $S$-act $S/S = \Theta$ satisfies condition (P). By assumption $\Theta$ is weakly pullback flat, it satisfies condition (E$'$) and by Lemma 2.4 $S$ is weakly left collapsible.

(2)$\Longrightarrow$(1) Assume $I$ is a right ideal of $S$ and the right Rees factor $S$-act $S/I$ satisfies condition (P). We have the following two cases to consider:

*Case* 1: $I = S$. Now $S/S = \Theta$ satisfies condition (P), by Lemma 2.3 $S$ is right reversible and by assumption $S$ is weakly left collapsible. Thus by Lemma 2.4 $S/S = \Theta$ is weakly pullback flat.

*Case* 2: $I$ is a proper right ideal of $S$. Since $S/I$ satisfies condition (P), by Lemma 2.2 $|I| = 1$ and $S/I = S$, it is clear that $S/I = S$ is weakly pullback flat. □

THEOREM 2.7 *Let $S$ be a monoid, then the following conditions on monoids are equivalent:*

(1) *All weakly pullback flat right Rees factor $S$-acts are pullback flat.*

(2) *If $S$ is a right reversible and weakly left collapsible monoid, then it is left collapsible.*

Proof. (1)$\Longrightarrow$(2) If $S$ is a right reversible and weakly left collapsible monoid, then by Lemma 2.3 and Lemma 2.4 the right Rees factor $S$-act $S/S = \Theta$ is weakly pullback flat. By assumption $\Theta$ is pullback flat, it satisfies condition (E) and by Lemma 2.5 $S$ is left collapsible.

(2)$\Longrightarrow$(1) Assume $I$ is a right ideal of $S$ and the right Rees factor $S$-act $S/I$ is weakly pullback flat. We have the following two cases to consider:

*Case* 1: $I = S$. Now $S/S = \Theta$ is weakly pullback flat, by Lemma 2.3 and Lemma 2.4 $S$ is right reversible and weakly left collapsible, by assumption $S$ is left collapsible. Thus by Lemma 2.5 $S/S = \Theta$ is pullback flat.

*Case* 2: $I$ is a proper right ideal of $S$. Since $S/I$ is weakly pullback flat, $S/I$ satisfies condition (P), by Lemma 2.2 $|I| = 1$ and $S/I = S$, it is clear that $S/I = S$ is pullback flat. □

Lemma 2.8 ([7]) *Let $S$ be a monoid and $\rho$ be a right congruence on $S$. Then $S/\rho$ is projective if and only if there exists $e^2 = e \in S$ such that $e\rho 1$ and $u\rho v$ imply $eu = ev$ for all $u, v \in S$.*

Lemma 2.9 *Let $S$ be a monoid, then the following conditions on monoids are equivalent:*

(1) *The one-element right $S$-act $\Theta$ is projective.*

(2) *$S$ contains left zero element.*

Proof. (1) $\Longrightarrow$(2) Since $\Theta$ is projective, by Lemma 2.8 there exists $e^2 = e \in S$ such that $e\rho 1$ and $u\rho v$ imply $eu = ev$ for all $u, v \in S$. Especially for every $s \in S$, $s\rho 1$ imply $es = e$. Thus $e$ is a left zero element of $S$.

(2)$\Longrightarrow$(1) Suppose $e$ is the left zero element of $S$. It is clear that $e^2 = e \in S$ such that $\Theta \cdot e = \Theta$ and $\Theta \cdot u = \Theta \cdot v$ implies $eu = ev = e$ for all $u, v \in S$. □

Theorem 2.10 *Let $S$ be a monoid, then the following conditions on monoids are equivalent:*

(1) *All weakly pullback flat right Rees factor $S$-acts are projective.*

(2) *If $S$ is a right reversible and weakly left collapsible monoid, then $S$ contains left zero element.*

Proof. (1) $\Longrightarrow$(2) If $S$ is a right reversible and weakly left collapsible monoid, then by Lemma 2.3 and Lemma 2.4 the right Rees factor $S$-act $S/S = \Theta$ is weakly pullback flat. By assumption $\Theta$ is projective, by Lemma 2.9 $S$ contains left zero element.

(2)$\Longrightarrow$(1) Assume $I$ is a right ideal of $S$ and the right Rees factor $S$-act $S/I$ is weakly pullback flat. We have the following two cases to consider:

*Case* 1: $I = S$. Now $S/S = \Theta$ is weakly pullback flat, by Lemma 2.3 and Lemma 2.4 $S$ is right reversible and weakly left collapsible, by assumption $S$ contains left zero element. Thus by Lemma 2.9 $S/S = \Theta$ is projective.

*Case* 2: $I$ is a proper right ideal of $S$. Since $S/I$ is weakly pullback flat, $S/I$ satisfies condition (P), by Lemma 2.2 $|I| = 1$ and $S/I = S$, it is clear that $S/I = S$ is projective. □

LEMMA 2.11 ([3]) *Let $S$ be a monoid, then the following conditions on monoids are equivalent:*

(1) *All projective right Rees factor $S$-acts are free.*

(2) *If $S$ contains left zero element, then $S = \{1\}$.*

THEOREM 2.12 *Let $S$ be a monoid, then the following conditions on monoids are equivalent:*

(1) *All weakly pullback flat right Rees factor $S$-acts are free.*

(2) *If $S$ is a right reversible and weakly left collapsible monoid, then $S = \{1\}$.*

Proof. (1) $\Longrightarrow$(2) All weakly pullback flat right Rees factor $S$-acts are free, then all weakly pullback flat right Rees factor $S$-acts are projective. If $S$ is a right reversible and weakly left collapsible monoid, by Theorem 2.10 $S$ contains a left zero element. All weakly pullback flat right Rees factor $S$-acts are free, then all projective right Rees factor $S$-acts are free, by Lemma 2.11 $S = \{1\}$.

(2)$\Longrightarrow$(1) Assume $I$ is a right ideal of $S$ and the right Rees factor $S$-act $S/I$ is weakly pullback flat. We have the following two cases to consider:

*Case* 1: $I = S$. Now $S/S = \Theta$ is weakly pullback flat, by Lemma 2.3 and Lemma 2.4 $S$ is right reversible and weakly left collapsible, by assumption $S = \{1\}$. It is clear that $\Theta$ is free.

*Case* 2: $I$ is a proper right ideal of $S$. Since $S/I$ is weakly pullback flat, $S/I$ satisfies condition (P), by Lemma 2.2 $|I| = 1$ and $S/I = S$, it is clear that $S/I = S$ is free. □

DEFINITION 2.13 A right ideal $J$ of $S$ is called left stabilizing if $j \in Jj$ for every $j \in J$.

Let $S$ be a monoid, we say $S$ satisfies *condition* $(*)$, it means $S$ has no proper and left stabilizing right ideal $J$ with $|J| > 1$.

LEMMA 2.14 ([5]) *Let $I$ be a right ideal of $S$. The Rees factor $S$-act $S/I$ is flat (weakly flat) if and only if $S$ is reversible and for every $i \in I$ there exists $j \in I$ such that $ji = i$.*

LEMMA 2.15 ([3]) *Let $S$ be a monoid, then the following conditions on monoids are equivalent:*

(1) *All flat (weakly flat) right Rees factor $S$-acts satisfy condition* (P).

(2) *If $S$ is a right reversible monoid, then $S$ satisfies condition* $(*)$.

THEOREM 2.16 *Let $S$ be a monoid, then the following conditions on monoids are equivalent:*

(1) *All weakly flat right Rees factor $S$-acts are weakly pullback flat.*

(2) *All flat right Rees factor $S$-acts are weakly pullback flat.*

(3) *If $S$ is a right reversible monoid, then it is weakly left collapsible and satisfies condition $(*)$.*

Proof. The implication (1)$\Longrightarrow$(2) is clear.

(2)$\Longrightarrow$(3) If $S$ is a right reversible monoid, then by Lemma 2.3 the right Rees factor $S$-act $S/S = \Theta$ is flat, by assumption $\Theta$ is weakly pullback flat, then $\Theta$ satisfies condition $(E')$, so by Lemma 2.4 $S$ is weakly left collapsible. All weakly flat right Rees factor $S$-acts are weakly pullback flat, then all weakly flat right Rees factor $S$-acts satisfy condition (P), by Lemma 2.15 $S$ satisfies condition $(*)$.

(3)$\Longrightarrow$(1) Assume $I$ is a right ideal of $S$ and the right Rees factor $S$-act $S/I$ is weakly flat. We have the following two cases to consider:

*Case* 1: $I = S$. Then $S/I = \Theta$ is weakly flat, by Lemma 2.3 $S$ is right reversible, by assumption $S$ is weakly left collapsible. Thus by Lemma 2.3 and Lemma 2.4 $S/S = \Theta$ is weakly pullback flat.

*Case* 2: $I$ is a proper right ideal of $S$. Let $|I| > 1$, then since $S/I$ is weakly flat, $S$ is right reversible by Lemma 2.14 and $I$ is a proper left stabilizing right ideal of $S$, against the assumption. So $|I| = 1$, then $S/I = S$ is weakly pullback flat. □

LEMMA 2.17 ([5]) *Let $I$ be a right ideal of $S$. The right Rees factor $S$-act $S/I$ is principally weakly flat if and only if $I$ is a left stabilizing right ideal of $S$.*

THEOREM 2.18 *Let $S$ be a monoid, then the following conditions on monoids are equivalent:*

(1) *All principally weakly flat right Rees factor $S$-acts are weakly pullback flat.*

(2) *$S$ is weakly left collapsible and right reversible monoid, and $S$ satisfies condition $(*)$.*

Proof. (1)$\Longrightarrow$(2) By Lemma 2.3 $\Theta$ is always principally weakly flat, by assumption $\Theta$ is weakly pullback flat, so $\Theta$ satisfies condition (P) and condition $(E')$, by Lemma 2.3 and Lemma 2.4 $S$ is right reversible and weakly left collapsible. On the other hand, all principally weakly flat right Rees factor $S$-acts are weakly pullback flat, then all weakly flat right Rees factor $S$-acts satisfy condition (P), now since $S$ is right reversible, by Lemma 2.15 $S$ satisfies condition $(*)$.

(2)$\Longrightarrow$(1) Assume $I$ is a right ideal of $S$ and the right Rees factor $S$-act $S/I$ is principally weakly flat. We have the following two cases to consider:

*Case* 1: $I = S$. Since $S$ is weakly left collapsible and right reversible, by Lemma 2.3 and Lemma 2.4 $S/I = \Theta$ is weakly pullback flat.

*Case* 2: $I$ is a proper right ideal of $S$. We must have $|I| = 1$, then $S/I = S$ is weakly pullback flat. Otherwise if $|I| > 1$, since $S/I$ is principally weakly flat, by 2.17 $I$ is a proper left stabilizing right ideal of $S$, this contradicts condition $(*)$. □

Lemma 2.19 ([11]) *The following conditions on monoids are equivalent:*

(1) *All torsion free right Rees factor $S$-acts satisfy condition* (P).

(2) *$S$ is a right reversible and right cancellative monoid, or $S$ is a right cancellative monoid with* 0 *adjoint.*

Lemma 2.20 ([11]) *Let $S$ be a right cancellative monoid with* 0 *adjoint and $I$ be a right ideal of $S$. If the Rees factor $S$-act $S/I$ is torsion free, then $I = S$ or $|I| = 1$.*

Theorem 2.21 *Let $S$ be a monoid, then the following conditions on monoids are equivalent:*

(1) *All torsion free right Rees factor $S$-acts are weakly pullback flat.*

(2) *$S$ is a right reversible and right cancellative monoid, or $S$ is a right cancellative monoid with* 0 *adjoint.*

Proof. (1)$\Longrightarrow$(2) By Lemma 2.19 it is clear.

(2)$\Longrightarrow$(1) If $S$ is a right reversible and right cancellative monoid, by Lemma 2.19 all torsion free right Rees factor $S$-acts satisfy condition (P). And if $S$ is a right cancellative monoid, it is obvious that all $S$-acts satisfy condition (E$'$). If $S$ is a right cancellative monoid with 0 adjoint, assume $I$ is a right ideal of $S$ and the right Rees factor $S$-act $S/I$ is torsion free, by Lemma 2.20 $I = S$ or $|I| = 1$, and the result follows. □

By [3] all right Rees factor $S$-acts are torsion free if, and only if, every right cancellable element of $S$ is right invertible. Then using Theorem 2.21 we have the following Corollary 2.22.

Corollary 2.22 ([4]) *All right Rees factor acts of $S$ are weakly pullback flat if and only if $S$ is a group or a group with zero adjointed.*

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