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On the number of return words in infinite words constructed by interval exchange transformations

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Abstract. In this article, we count the number of return words in some infinite words with complexity $2n + 1$. We also consider some infinite words given by codings of rotation and interval exchange transformations on k intervals. We prove that the number of return words over a given word w for these infinite words is exactly k.

Mathematics Subject Classifications (2000). 37E05, 68R15

1 Introduction

The starting point of this article is the following question: *Does there exist a way to characterize by return words the infinite word with complexity* $2n+1$*?* We first give the definition of return words for a recurrent infinite word in a finite alphabet (a recurrent word is an infinite word such that each word appears infinitely many times). Considering each occurrence of a word w in a recurrent infinite word U, we define the set of return words over w to be the set of all distinct words beginning with an occurrence of w and ending exactly before the next occurrence of w in the infinite sequence. This mathematical tool was introduced independently by Durand, Holton and Zamboni in order to study primitive substitutive sequences (see [10, 11, 16]). This notion is quite natural and can be seen as a discrete version of the usual notion of first return map for a dynamical system. Many developments of the notion of return words have been given. For example, Allouche, Davinson, Queffelec and Zamboni study the transcendence of Sturmian or morphic continued fractions and the main tool is to show, using return words, that arbitrary long prefixes are "almost squares" (see [2]). We can also use return words to study low complexity infinite words. For example, the author shows that an infinite word is Sturmian if and only if for each word appearing in the infinite word, the cardinality of the set of return words over w is exactly two (see [21]). Recall that Sturmian infinite words are aperiodic words with complexity $p(n) = n + 1$ for all n (the complexity function $p(n)$ counts the number of distinct factors of length n in the infinite word) (see [4, 5, 15]). Fagnot and Vuillon show a generalization of the notion of balanced property for Sturmian words and the proof is based on return words

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and combinatorics on words (see [13]). Cassaigne uses this tool to investigate a Rauzy's conjecture (see [7]).

Our propose is to compute the set of return words for a class of infinite words with complexity $p(n) = 2n + 1$ for all n (see [3, 6, 8, 9, 12, 14, 17]).

The following question is thus natural: is it true that for these infinite words, the cardinality of the set of return words over a factor w is always three?

In this direction, Justin and Vuillon [17] show that Arnoux–Rauzy infinite words [3], which are infinite words with complexity $2n + 1$ (see [3, 8, 9, 12]), has property R_3. In the sequel, we say that an infinite word have the property R_n if the cardinality of the set of return words over each factor w is exactly n. They give also the structure of return words in the context of Episturmian words [12, 17].

In the opposite, the work of Ferenczi [14] presents a nice substitutive infinite word with complexity $p(n) = 2n + 1$ given by the Chacon transformation. It is not difficult to show that the cardinality of the set of return words may be larger than three for this infinite word. This is the first example of infinite word with complexity $2n + 1$ without the property R_3.

Nevertheless, we find two other classes of infinite words with complexity $p(n) = 2n + 1$ having the property R_3. The first one is the coding of a rotation α in the unit circle with three intervals (with rationally independent lengths) and one of these intervals of length α. The second one is a generalization of this class namely the interval exchange transformation on three intervals.

The structure of the article is the following. First, we recall some basic definitions in combinatorics of words. Secondly, we present the infinite words given by codings of rotation. Then, we show that an infinite word given by a regular interval exchange with three intervals (resp. n intervals) have the property R_3 (resp. R_n). We compute also the length of each return word over a given finite word w. At last, we show that the coding of a rotation in the unit circle with three (resp. n intervals) intervals with rationally independent lengths has property R_3 (resp. R_n).

2 Basic definitions and examples

Let $\mathcal{A} = \{0, 1\}$ be a binary alphabet. We denote by $\mathcal{A}^*$ the set of finite words on $\mathcal{A}$ and by $\mathcal{A}^{\mathbb{N}}$ the one-sided infinite word U. A word w is a *factor* of a word $x \in \mathcal{A}^*$ if there exist some words $u, v \in \mathcal{A}^*$ such that $x = uwv$. An infinite word U is called *recurrent* if every factor of U appears infinitely many times in U. In the sequel, we assume that the word U is recurrent. For a finite word $w = w_1 w_2 \cdots w_n$, with $w_i \in \mathcal{A}$, the length of w is denoted by $|w|$ and is equal to n. The set of factors of U with length n is denoted by $L_n(U)$. The language $L(U) = \cup_n L_n(U)$ is the set of factors of U. For two finite words w and u, the number of occurrences of w on u is denoted by $|u|_w$ and $|u|_w = \text{Card}\,\{i \mid 0 \leq i \leq |u| - |w| \text{ s.t. } u_i u_{i+1} \cdots u_{i+|w|-1} = w_1 w_2 \cdots w_{|w|}\}$.

The *position set* of the word w is a set of integers $i(U, w) = \{i_1, i_2, \cdots, i_k, \cdots\}$ where i_k represents the position of the first letter of the k-th occurrence of the word w in the infinite word U. In a more formal way, $i_k \in i(U,w)$ if and only if $U_{i_k}U_{i_k+1}\cdots U_{i_k+|w|-1} = w$ and $|U_1 \cdots U_{i_k+|w|-1}|_w = k$. Since the infinite word U is recurrent the set $i(U,w)$ is infinite. For a recurrent word U, the set of *return words over* w is the set (denoted by $\mathcal{H}_{U,w}$) of all distinct words with the following form:

$$U_{i_k}U_{i_k+1}\cdots U_{i_{k+1}-1}$$

for all $k \in \mathbb{N}$, $k > 0$. This definition is best understood on an example. Let $U_1 = (0100100001)^\omega$ be an infinite word on the alphabet $\mathcal{A}$. By definition, the set of return words over 01 is $\mathcal{H}_{U_1,010} = \{010, 01000, 01\}$. Indeed, the infinite word U_1 can be written

$$(\underline{0}10\underline{0}1000\underline{0}1\underline{0}10\underline{0}1000\underline{0}1)^\omega,$$

where _ denotes the position of the first letter for each occurrence of the word 010. Remark that a return word on w could be of length lower than w. For instance, 01 is a return word on $w = 010$. Thus we could check that 010,01000 and 01 are elements of $\mathcal{H}_{U_1,010}$. We say that an infinite word have the property R_n if the cardinality of the set of return words over each factor w is exactly n (i.e. $\mathrm{Card}\,(\mathcal{H}_{U,w}) = k \ \ \forall w \in L(U)$).

The *complexity function* $p : \mathbb{N} \to \mathbb{N}$ of an infinite word U counts the number of distinct factors of U of given length:

$$p(n) = \mathrm{Card}\,\{w \mid w \in L_n(U)\}.$$

3 A negative result

Let consider the following substitution extensively studied by Ferenczi (see [14]) which is a recoding of the Chacon substitution:

$$\sigma(1) = 12,$$
$$\sigma(2) = 312,$$
$$\sigma(3) = 3312.$$

The fix point $\sigma(x) = x$ of the substitution begins with

$$x = 123123312123123312331212312123123312123123 1233\cdots$$

It is easy to check with few terms of the fix point x that the number of return words over 23 is upper than 3, indeed:

$$\mathcal{H}_{x,23} = \{231, 2331, 23121, 233121\}.$$

One open question is to compute the number of return words for each word of the Chacon substitution.

4 Codings of rotation

The aim of this section is to introduce codings of irrational rotation on the unit circle. For $p \geq 2$, let $F = \{\beta_0 < \beta_1 < \ldots < \beta_{p-1}\}$ be a set of p consecutive points of the unit circle (identified in all that follows with $[0, 1[$ or with the unidimensional torus $\mathbb{R}/\mathbb{Z}$) and let $\beta_p = \beta_0$. Let α be an irrational number in $]0, 1[$ and let us consider the *positive orbit* of a point x of the unit circle under the rotation by angle α, i.e., the set of points $\{\{\alpha n + x\},\ n \in \mathbb{N}\}$ where $\{.\}$ denotes the fractional part.

The *coding* of the orbit of x under the rotation by angle α with respect to the partition $\{[\beta_0, \beta_1[, [\beta_1, \beta_2[, \ldots, [\beta_{p-1}, \beta_p[\}$ is the infinite word u defined on the finite alphabet $\Sigma = \{0, \ldots, p-1\}$ as follows:

$$u_n = k \Leftrightarrow \{x + n\alpha\} \in [\beta_k, \beta_{k+1}[, \quad \text{for} \quad 0 \leq k \leq p-1.$$

A coding of the rotation Rot means the coding of the orbit of a point x of the unit circle under the rotation *Rot* with respect to a finite partition of the unit circle consisting of left-closed and right-open intervals.

4.1 Factors

With the above notation, consider a coding u of the orbit of a point x under the rotation by angle α with respect to the partition $\{[\beta_0, \beta_1[, [\beta_1, \beta_2[, \ldots, [\beta_{p-1}, \beta_p[\}$. Let $I_k = [\beta_k, \beta_{k+1}[$, Rot_α denotes the rotation by angle α, and $Rot_\alpha^n(x)$ the element $\{\alpha n + x\}$ (see Figure 1).

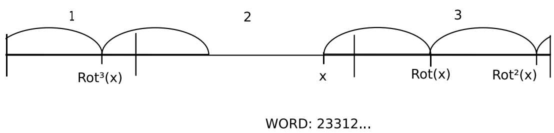

Figure 1: Coding of rotation with 3 intervals

LEMMA 1 *The word* $w_1 \cdots w_n$ *defined on the alphabet* $\Sigma = \{0, 1, \ldots, p-1\}$ *is a factor of the infinite word* u *if and only if* $I(w_1, \ldots, w_n) \neq \emptyset$ *where* $I(w_1, \ldots, w_n) = \bigcap_{j=0}^{n-1} Rot_\alpha^{-j}(I_{w_{j+1}})$.

Proof. A finite word $w_1 \cdots w_n$ defined on the alphabet $\Sigma = \{0, 1, \ldots, p-1\}$ is a factor of the infinite word u if and only if there exists an integer k such that

$$\{x + k\alpha\} \in I(w_1, \ldots, w_n) = \bigcap_{j=0}^{n-1} Rot_\alpha^{-j}(I_{w_{j+1}}).$$

As α is irrational, the sequence $(\{x + n\alpha\})_{n\in\mathbb{N}}$ is dense in the unit circle, which implies that $w_1 w_2 \cdots w_n$ is a factor of u if and only if $I(w_1, \ldots, w_n) \neq \emptyset$. In particular, the set of factors of a coding does not depend on the initial point x of this coding. Furthermore, the connected components of these sets are bounded by the points

$$\{k(1-\alpha)+\beta_i\}, \text{ for } 0 \le k \le n-1,\ 0 \le i \le p-1.$$

These sets consist of finite unions of intervals. More precisely, if for every k, $\beta_{k+1}-\beta_k \le \sup(\alpha, 1-\alpha)$, then these sets are connected; if there exists K such that $\beta_{K+1}-\beta_K > \sup(\alpha,1-\alpha)$, then the sets are connected except for $w_1 \dots w_n$ of the form a_K^n (the notation a_K^n denotes the word of length n obtained by successive concatenations of the letter a_K). Roughly speaking, this case appears when an interval I and the interval $Rot_{-1}^{\alpha}(I)$ have in common their beginning and their end (see [1]). Let us note for a question of lengths of intervals, that there exists at most one integer K satisfying $\beta_{K+1}-\beta_K > \sup(\alpha, 1-\alpha)$. □

5 Interval exchange transformations

This section deals with interval exchange transformations. This object is a natural generalization of the codings of rotation α where α is the length of at least one of the intervals. An interval exchange transformation is a piecewise affine transformation which maps a partition into intervals of the space to another partition according to a permutation. This transformation could be more complicated than a rotation. Indeed, irrational rotations are uniquely ergodic, unless a class of interval exchange transformations is non uniquely ergodic. In the sequel, we use the notations of Keane and Rauzy (see [18, 19]).

An interval exchange of k intervals is defined by a vector $(\lambda_1, \lambda_2, \cdots, \lambda_k)$ in $\mathbb{R}^k$ with strictly positive coordinates and $\sum_{i=1}^{k} \lambda_i = 1$, and a permutation σ of the set $\{1, 2, \cdots, k\}$. We set the partition of $[0, 1[$ in k intervals

$$X_i = \left[\sum_{j<i} \lambda_j, \sum_{j\le i} \lambda_j\right[\quad \text{for } 1 \le i \le k$$

(X_i has the length λ_i).

The interval exchange transformation associated to the ordered pair (σ, λ) is the transformation from $[0,1[$ to itself, defined as a piecewise affine transformation which maps the partition $(X_1, X_2, \cdots, X_k)$ to the partition $Y_i = \left[\sum_{k<\sigma^{-1}(i)} \lambda_{\sigma(k)}, \sum_{k\le\sigma^{-1}(i)} \lambda_{\sigma(k)}\right[$. The resulting partition is $Y_1, \dots, Y_k$, which can be ordered as $Y_{\sigma(1)}, \dots, Y_{\sigma(k)}$ with $Y_{\sigma(i)} = \left[\sum_{k<i} \lambda_{\sigma(k)}, \sum_{k\le i} \lambda_{\sigma(k)}\right[$.

The transformation T maps the point $x \in X_i$ to the point

$$T(x) = x + a_i,$$

where

$$a_i = \sum_{k<\sigma^{-1}(i)} \lambda_{\sigma(k)} - \sum_{k<i} \lambda_k.$$

Now, we construct an infinite word with values in the alphabet $\{1, 2, \cdots, k\}$ associated with a couple (σ, λ) by coding the positive orbit of a point x by the transformation T according to the partition $(X_1, X_2, \cdots, X_k)$. The infinite word $U(x)$ is given by

$$U(x)_n = \mathcal{I}(T^n(x)),\ \forall n \in \mathbb{N},$$

where $T^n(x) = T(T^{n-1}(x))$ and $\mathcal{I}(y) = i$ if $y \in X_i$ (see Figure 2).

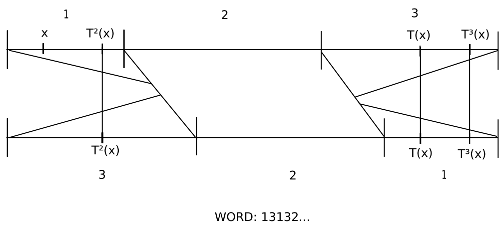

Figure 2: Interval exchange transformation with the $(3, 2, 1)$ permutation

In order to define a transformation such that each orbit is dense in $[0, 1[$, we add the following property: an interval exchange transformation is called regular if for all $0 = a_1 < a_2 < \cdots < a_{k+1} = 1$ of the intervals $X_i = [a_i, a_{i+1}[$ with $i \in \{1, \cdots, k\}$, we have

$$T^n a_i = a_j,\ i \text{ and } j \in \{2, 3, \cdots, k\}, n \in \mathbb{Z}$$

implies $n = 0, i = j$.

M. Keane shows the following result [18]:

Theorem 1 *An interval exchange transformation T is regular implies that for all point x in $[0, 1[$ the orbit $\cup_{n\in\mathbb{Z}}\{T^n x\}$ of x by T is dense in $[0, 1[$.*

A necessary condition to have a regular interval exchange transformation is to take an irreducible permutation. A permutation is called irreducible if no subset of $\{1, 2, \cdots, k\}$ is invariant by the permutation, i.e., $\sigma(\{1, 2, \cdots, \ell\}) \neq \{1, 2, \cdots, \ell\}$ for all $\ell < k$. From now on we use only irreducible permutations in this article.

Now we can state the main theorem of this section:

Theorem 2 *Let T be a regular interval exchange transformation with k intervals. The infinite word $U(x)$ associated with T has property R_k.*

In order to show this theorem, we first present a method to construct all the words of length n which are factors of the infinite word $U(x)$ and secondly a way to construct return words by using self-induction.

5.1 Factors

The construction of all the factors is not so far from the one used for rotations on the unit circle. The main tool is to consider the negative orbit of all the endpoints $0 = a_1 < a_2 < \cdots < a_{k+1} = 1$ of the intervals $X_i = [a_i, a_{i+1}[$ with $i \in \{1, \cdots, k\}$. The set of endpoints $\{a_i \mid 1 \le i \le k+1\}$ is called $X^{(1)}$.

A word w of length n is a factor of the infinite word $U(x)$ if and only if there exists an interval $I_w \subset [0, 1[$ and a point y in I_w such that the word w is the following concatenation of letters:

$$\mathcal{I}(y)\mathcal{I}(T(y)) \cdots \mathcal{I}(T^{n-1}) = w.$$

As the permutation is irreducible, the number of factors of length one is exactly the number of intervals in the exchange. That is the intervals with names $I_1 = X_1, I_2 = X_2, \cdots, I_k = X_k$ associated with the k letters of the alphabet $\{1, 2, \cdots, k\}$.

Proposition 1 (Arnoux, Rauzy) *Let a regular interval exchange transformation T with k intervals. The infinite word $U(x)$ associated with T has complexity $p(n) = n(k-1) + 1$, $\forall n \in \mathbb{N}$.*

Proof. We remark that the intersection of two non-empty intervals is always an interval. Thus the factors $w = w_1 w_2$ of length 2 are given by the set $\{\mathcal{I}(x)\mathcal{I}(T(x)), x \in [0, 1[\}$ or equivalently by the intervals $I_{w_1 w_2} = I_{w_1} \cap T^{-1} I_{w_2}$.

As the transformation T is a piecewise affine transformation, it is sufficient to find all the endpoints of the intervals I_w with $|w| = 2$. That is the positions of the endpoints of all intervals with form $I_{w_1} \cap T^{-1} I_{w_2}$. These endpoints are given by the ordered set

$$X^{(2)} = \{0 = b_1 < b_2 < \cdots < b_{2k} = 1\} = X^{(1)} \cup T^{-1} X^{(1)}.$$

Remark that the number of points in the union is $2k$ because the points 0 and 1 are elements of the intersection of the two sets $X^{(1)}$ and $T^{-1}X^{(1)}$.

The next step is to prove how associate an interval I_w to the word w.

By induction, let $w = w_1 w_2 \cdots w_n$ a factor of length n, then there exists a point x such that

$$w = \mathcal{I}(x)\mathcal{I}(T(x)) \cdots \mathcal{I}(T^{n-1}(x))$$

or equivalently

$$I_{w_1 w_2 \cdots w_n} = \cap_{i=0 \cdots n-1} T^{-i} I_{w_{i+1}}.$$

That is each endpoints of the interval associated with a word of length n is an element of the ordered set

$$X^{(n)} = \{0 = b_1 < b_2 < \cdots < b_{n(k-1)+2} = 1\} = X^{(1)} \cup T^{-1} X^{(1)} \cup \cdots \cup T^{-n+1} X^{(1)}.$$

Remark that the number of points in the union is $n(k-1) + 2$ because at each step, we add $k-1$ new points and for $n = 1$ we have $k+1$ points in the partition.

Furthermore, the intervals associated with factors of length n are connected. Indeed, the intervals are the intersection of connected intervals in $[0, 1[$.

In other words, for each word w with $|w| = n$ factor of the infinite word $U(x)$, there exists an interval $I_w = [b_\ell, b_{\ell+1}[$ where b_ℓ and $b_{\ell+1}$ are two consecutive points in the ordered set $X^{(n)}$.

With this construction, we find $n(k-1)+2$ points for the endpoints in the partition $X^{(n)}$. We have $n(k-1)+1$ intervals in the partition and then the complexity function for a regular interval exchange in k intervals is equal to $p(n, k) = n(k-1)+1$, $\forall (n, k) \in \mathbb{N}^2$. □

Thus, by the previous theorem applied to the cases $k = 2$, an infinite word associated with a regular interval exchange with two intervals is nothing but a Sturmian infinite word (i.e. an infinite word with complexity $p(n, 2) = n+1$ for all n). And for the case $k = 3$ an infinite word associated with a regular interval exchange with three intervals is an infinite word with complexity $p(n, 3) = 2n+1$ for all n.

5.2 Self-induction

Now, we focus on the construction of return words associated with a word w factor of the infinite word.

Theorem 3 *Let a regular interval exchange transformation T with k intervals. The infinite word $U(x)$ associated with T has property R_k.*

By the previous construction, we find a unique connected interval I_w associated with the word w. The main tool is to study the first return map in the adherence of the interval I_w. This method is used by Rauzy to give a generalized continued fraction algorithm [19, 20].

Proof. Let $r(y) = \inf_{t>0}\{T^{-t}y \in \overline{I_w}\}$ be the negative first return time in the interval $\overline{I_w}$. Keane shows that the points $\{T^{-r(y)}y \mid y \in \overline{I_w},\, k(y) < \infty\}$ give the endpoints of a partition of the interval $\overline{I_w}$ in exactly k intervals $I_{p_1}, I_{p_2}, \cdots, I_{p_k}$ (see [18]). Indeed, the endpoints of these intervals are given when the negative orbit of the points in $X^{(1)} = \{0 = a_1 < a_2 < \cdots < a_{k+1} = 1\}$ falls in $\overline{I_w}$ for the first time.

There are $k+1$ points in $X^{(1)}$. This shows that for a general interval $[\alpha, \beta]$, the number of induced points in $]\alpha, \beta[$ is $k+1$ and that the number of induced intervals is $k+2$. As the endpoints of $\overline{I_w}$ are both in the negative orbit of two different points in $X^{(1)}$. More precisely, as the interval exchange transformation is regular, we have

$$T^n a_i = a_j,\ i \text{ and } j \in \{2, 3, \cdots, k\},\ n \in \mathbb{Z}$$

implies $n = 0$, $i = j$. Thus

$$\alpha = T^{n_{a_0}} a_0,\ \beta = T^{n_{a_{k+1}}} a_{k+1}.$$

That is the number of induced points in the interior of $\overline{I_w}$ is $k-1$. Thus the number of induced intervals is k. Such intervals with endpoints on the negative orbit of a_0 and a_{k+1} are called acceptable intervals by Rauzy (see [19]). Indeed, for interval exchange transformation with k intervals, the induced transformation on an acceptable interval is also an interval exchange transformation with k intervals.

By construction, for each p_i there exists $y \in I_w$ and t such that the word

$$p_i = \mathcal{I}(y)\mathcal{I}(T(y))\cdots\mathcal{I}(T^t(y))\cdots\mathcal{I}(T^{t+|w|-1}(y)).$$

Furthermore, the prefix of length $|w|$ of p_i is exactly w (indeed $y \in I_w$) and the suffix of length $|w|$ of p_i is exactly w (indeed t is the positive first return time of y in the interval I_w defined by $r^+(y) = \inf_{t\geq 0}\{T^t y \in \overline{I_w}\}$). In other words, the number of return words over w is exactly the number of induced intervals. By construction I_w is an acceptable interval. Thus the number of return words over w is exactly k. That is, the infinite word associated with regular interval exchange transformation on k intervals has property R_k. □

Remark that the different lengths of the return words on w are exactly given by the set $\{r(x) \mid x \in I_w\}$.

6 Returns words for codings of rotation

As the codings of rotation by a rotation of length equal to the length of one of the intervals of the partition could be recoded in interval exchange transformations, we have the following corollary:

Corollary 1 *Let T be a coding of rotation by a rotation of length α equal to the length of one of the intervals on k intervals with rationally independent lengths and by a rotation of length α equal to the length of one of the intervals. The infinite word $U(x)$ associated with T has property R_k.*

Proof. As the rotation is defined on the circle, we can translate all the intervals in order to have the first interval with length equal to α. The coding of rotation of angle α on k intervals with the first interval of length α is equivalent to an interval exchange transformation with either the permutation $(2, 3, \cdots, k, 1)$ if $\alpha > 0$ or the permutation $(k, 1, 2, 3, \cdots, k-1)$ $\alpha < 0$. As the lengths of the intervals are rationally independent in the coding of rotation then the associated interval exchange transformation is regular. Consequently, the infinite word $U(x)$ associated with T has property R_k. □

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