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On the special basis of a certain full symmetry class of tensors

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Abstract. The problem of existence of a special basis for the symmetry classes of tensors were studied by several authors. In [5], Holmes proved that if V is an n-dimensional inner product space over $\mathbb{C}$, $n \geq 3$ and λ is an irreducible character of S_m of the form

$$(m-1,\ 1), \quad m \geq 3,$$

then the full symmetry class of tensors associated with λ, i.e., $V_\lambda(S_m)$ is non-zero and does not have a special basis. The nonexistence of a special basis for the full symmetry class of tensors associated with irreducible character (l_1, l_2), $l_1 \geq 3$ of S_m is concluded by Dias da Silva and M. Torres in [3], by computation the orthogonal dimension of critical orbital sets. In this paper we prove this result by a new method. Indeed, we show that if V is an n-dimensional inner product space over $\mathbb{C}$, $n \geq 2$ and λ is an irreducible character of S_m of the form

$$(m-l,\ l), \quad m \geq 3l,$$

then $V_\lambda(S_m)$ is non-zero and has no special basis.

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1 Introduction and preliminaries

Let G be a subgroup of the full symmetric group S_m and let λ be a complex irreducible character of G. Let V be an n-dimensional inner product space over $\mathbb{C}$. The m-th tensor power of V is denoted by $\bigotimes^m V$ and for every $\sigma \in S_m$, $P(\sigma)$ denotes the linear operator of $\bigotimes^m V$ satisfying

$$P(\sigma)(v_1 \otimes \cdots \otimes v_m) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(m)}, \quad (v_1, \ldots, v_m \in V).$$

The linear operators on $\bigotimes^m V$ belonging to the linear closure of $\{P(\sigma) : \sigma \in S_m\}$ are called *symmetrizers*. The symmetrizer

$$T(G, \lambda) = \frac{\lambda(\mathrm{id})}{|G|} \sum_{\sigma \in G} \lambda(\sigma) P(\sigma),$$

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($|G|$ denotes the order of G and id the identity of G) is said to be associated with the *character* λ. The image of $T(G,\lambda)$

$$V_\lambda(G) = T(G,\lambda)\left(\bigotimes^m V\right)$$

is called the *symmetry class of tensors* associated with λ.

The elements of $V_\lambda(G)$ of the form $T(G,\lambda)(v_1 \otimes \cdots \otimes v_m)$ are called *decomposable symmetrized tensors* and denoted by $v_1 * \cdots * v_m$.

Let $\Gamma_{m,n}$ be the set of all sequences $\alpha = (\alpha_1, \dots, \alpha_m)$ with $1 \leq \alpha_i \leq n$. Then G acts on $\Gamma_{m,n}$ by

$$\alpha^\sigma = (\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(m)}),$$

where $\sigma \in G$ and $\alpha \in \Gamma_{m,n}$. We denote by G_α, the *stabilizer subgroup* of α, i.e.,

$$G_\alpha = \{\sigma \in G : \alpha^\sigma = \alpha\}.$$

Define

$$\Omega = \left\{\alpha \in \Gamma_{m,n} : \sum_{g \in G_\alpha} \lambda(g) \neq 0\right\}.$$

We consider the following equivalence relation in $\Gamma_{m,n}$: $\alpha \sim \beta (\mathrm{mod}\ G)$ if there exists $\sigma \in G$ such that $\alpha = \beta^\sigma$.

The equivalence classes for this relation are called *orbits*. Denote by Δ the system of representatives obtained by choosing in each orbit the smallest element in the lexicographic order and define $\bar{\Delta} = \Delta \bigcap \Omega$.

Let $\{e_1, \dots, e_n\}$ be a basis of V. The set

$$\{e_\alpha^\otimes = e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_m} : \alpha \in \Gamma_{m,n}\},$$

is a basis of $\bigotimes^m V$, hence

$$\{e_\alpha^* = e_{\alpha_1} * \cdots * e_{\alpha_m} : \alpha \in \Gamma_{m,n}\},$$

spans the symmetry class of tensors $V_\lambda(G)$.

For $\alpha \in \Delta$, $V_\alpha^* = \langle e_{\alpha^\sigma}^* : \sigma \in G\rangle$ is called the *orbital subspace* of $V_\lambda(G)$, and we can easily prove that

$$V_\lambda(G) = \bigoplus_{\alpha \in \bar{\Delta}} V_\alpha^*. \tag{1}$$

In [6], it is proved that

$$\dim V_\alpha^* = \frac{\lambda(\mathrm{id})}{|G_\alpha|} \sum_{g \in G_\alpha} \lambda(g) = \lambda(\mathrm{id})[\lambda, 1_{G_\alpha}], \tag{2}$$

where $[,]$ denotes the inner product of characters.

We assume that $\bigotimes^m V$ equipped with the inner product induced by the inner product of V. If $\mathcal{B}_\alpha = \{e^*_{\alpha^{\sigma_1}}, \ldots, e^*_{\alpha^{\sigma_{t_\alpha}}}\}$ is a basis of the orbital subspace $V^*_\alpha (\alpha \in \bar{\Delta})$ extracted from the *orbital set* $\{e^*_{\alpha^\sigma} : \sigma \in G\}$, then

$$\mathcal{B} = \bigcup_{\alpha \in \bar{\Delta}} \mathcal{B}_\alpha$$

is a basis for $V_\lambda(G)$ extracted from $\{e^*_\alpha : \alpha \in \Gamma_{m,n}\}$. It is well known that, if the basis $\{e_1, \ldots, e_n\}$ is orthogonal, then the direct sum (1) is orthogonal. If we can choose $\mathcal{B}_\alpha$ orthogonal, for each $\alpha \in \bar{\Delta}$, then the basis $\mathcal{B}$ is also orthogonal.

An orthogonal basis of $V_\lambda(G)$ extracted from $\{e^*_\alpha : \alpha \in \Gamma_{m,n}\}$ is called a *special basis* of $V_\lambda(G)$.

It is well known that if λ is a linear character, then for every $\alpha \in \bar{\Delta}$, $\dim V^*_\alpha = 1$. Therefore for symmetry classes of tensors associated with these characters, we can find an orthogonal basis $\mathcal{B}_\alpha$ for each $\alpha \in \bar{\Delta}$, thus a special basis of $V_\lambda(G)$.

Also if the basis $\{e_1, \ldots, e_n\}$ is orthonormal, then

$$\|e^*_\alpha\|^2 = \frac{\dim V^*_\alpha}{[G : G_\alpha]}, \tag{3}$$

for every $\alpha \in \bar{\Delta}$. (see [6])

In this note we use the Theorem (3.1) to investigate the nonexistence of a special basis of $V_\lambda(S_m)$ for irreducible characters of S_m of the form

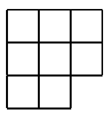
$$\lambda = (m - l,\ l), \quad m \geq 3l.$$

2 Ordinary representations of S_m

By *partition* of m we mean a non-increasing finite sequence of nonnegative integers with sum m. If λ is a partition of m we denote $\lambda \vdash m$. The nonzero terms of a partition are called *parts*. A subsequence of λ with k parts each part equal to i is denoted by i^k. If $\lambda = (\lambda_1, \ldots, \lambda_t)$ is a partition of m, from now on we fix $\lambda_l = 0$, for $l > t$. Let $\lambda = (\lambda_1, \ldots, \lambda_t)$ be a partition of m and

$$\acute{\lambda}_i = |\{j \in \{1, \ldots, t\} : \lambda_j \geq i\}|, \quad i = 1, \ldots, \lambda_1.$$

The sequence $\acute{\lambda} = (\acute{\lambda}_1, \ldots, \acute{\lambda}_{\lambda_1})$ is a partition of m, called *conjugate partition* of λ. A partition $\lambda = (\lambda_1, \ldots, \lambda_t)$ is usually represented by a collection of m boxes arranged in t rows such that the number of boxes of row i is equal to λ_i, $i = 1, \ldots, t$. This collection is called *Young diagram* associated with λ and denoted by $[\lambda]$, e.g.

$[(3, 3, 2)] =$ 

The *Young subgroup* corresponding to $\lambda \vdash m$ is the internal direct product

$$S_\lambda = S_{\lambda_1} \times \cdots \times S_{\lambda_t}.$$

Denote by $1_{S_\lambda} = 1_\lambda$ the principal character of S_λ. Since $1_\lambda^{S_m}$ is a character of S_m, there must exist integers $K_{\mu,\lambda}$ such that

$$1_\lambda^{S_m} = \sum_{\mu \vdash m} K_{\mu,\lambda}\ \mu.$$

The numbers $K_{\mu,\lambda} = [1_\lambda^{S_m}, \mu]$ are called *Kostka coefficients.*

In [6], it is proved that the *Kostka coefficient* $K_{\mu,\mu} = 1$ for all $\lambda \vdash m$.

In the set of partitions of m we consider the dominance partial ordering. Let λ and μ be partitions of m. We say that μ *dominates* λ and denote $\lambda \preceq \mu$ if

$$\lambda_1 + \lambda_2 + \cdots + \lambda_i \le \mu_1 + \mu_2 + \cdots + \mu_i$$

for all i. Since there is a standard one-to-one correspondence between the complex irreducible characters of S_m and the partitions of m, we use the same symbol to denote an irreducible character of S_m and the partition of m corresponding to it.

Let $\lambda = (\lambda_1, \ldots, \lambda_t)$ be a partition of m. For each ordered pair (i, j), $1 \le i \le t$, $1 \le j \le \lambda_i$, there is corresponding a box, B_{ij}, in Young diagram $[\lambda]$. B_{ij} determines a unique *hook* in $[\lambda]$ consisting of B_{ij} itself, all the boxes in row i of $[\lambda]$ to the right of B_{ij}, and all boxes in column j of $[\lambda]$ below B_{ij}. The number of boxes in the hook determined by B_{ij} is its *length*, $h_{ij} = (\lambda_i - i) + (\acute{\lambda}_j - j) + 1$. For example, the hook lengths of $[(3,\ 3,\ 2)]$ are

5	4	2
4	3	1
2	1	

It is well known (see [6]) that if $\lambda = (\lambda_1, \ldots, \lambda_t)$ is an irreducible character of S_m, then the degree of irreducible character λ of S_m is

$$\lambda(\mathrm{id}) = \frac{m!}{\prod_{i=1}^{t} \prod_{j=1}^{\lambda_i} h_{ij}}\ . \tag{4}$$

If $\alpha \in \Gamma_{m,n}$, denote by $m_t(\alpha) = |\alpha^{-1}(t)|$ the multiplicity of t in the sequence α. The *multiplicity partition* of α, usually denoted by $M(\alpha)$, is the partition of m obtained by reordering in a decreasing way the n-tuple

$$(|\alpha^{-1}(1)|, \ldots, |\alpha^{-1}(n)|).$$

It is easy to see that $\alpha \sim \beta (\mathrm{mod}\, S_m)$ if and only if $|\alpha^{-1}(i)| = |\beta^{-1}(i)|$ for every $i = 1, 2, \ldots, n$.

Therefore $M(\alpha) = M(\beta)$ if $\alpha \sim \beta (\mathrm{mod}\, S_m)$.

Merris [7] proved that in $V_\lambda(S_m)$, $e_\alpha^* \ne 0$ if and only if $\lambda \succeq M(\alpha)$.

3 Main result

The study of symmetry classes of tensors is motivated by many branches of pure and applied mathematics: combinatorial theory, matrix theory, operator theory, group representation theory, differential geometry, partial differential equations, quantum mechanics and other areas. See [6] for some general background. Several papers are devoted to the investigation of the non-vanishing nature and existence of a special of $V_\lambda(G)$, see for example [1, 8, 9, 10]. Recently the existence of special basis of $V_\lambda(G)$, in the case $G = S_m$, have been studied by several authors, see example [2, 5].

In [5], Holmes proved that if V is an n-dimensional inner product space over $\mathbb{C}$, $n \geq 3$ and λ is an irreducible character of S_m of the form

$$(m-1,\ 1), \quad m \geq 3,$$

then the full symmetry class of tensors associated with λ, i.e., $V_\lambda(S_m)$ is non-zero and does not have a special basis. In this note we consider the case $\lambda = (m-l,\ l)$, we prove an analogue theorem for the space $V_\lambda(S_m)$.

We need the following theorem, which is proved in [10].

THEOREM 3.1 *Let V be an n-dimensional inner product space over $\mathbb{C}$. If λ is a non-linear irreducible character of G and there is $\alpha \in \Gamma_{m,n}$ such that*

$$\frac{\sqrt{2}}{2} < \|e^*_\alpha\| < 1,$$

then $V_\lambda(G)$ has no special basis.

We are now ready to state the main result.

THEOREM 3.2 *Let V be an n-dimensional inner product space over $\mathbb{C}$, $n \geq 2$. Let λ be an irreducible character of S_m of the form*

$$\lambda = (m-l,\ l), \quad m \geq 3l.$$

Then the full symmetry class of tensors associated with λ, i.e., $V_\lambda(S_m)$ has no special basis.

Proof. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of V. Put $\alpha = (\underbrace{1,\ 1,\ \ldots,\ 1}_{m-l \text{ times}}, \underbrace{2,\ 2,\ \ldots,\ 2}_{l \text{ times}})$, then $n \geq 2$ implies that $\alpha \in \Gamma_{m,n}$. Consider the action of S_m on $\Gamma_{m,n}$ and choose a system Δ of representatives such that $\alpha \in \Delta$. It is easy to see that

$$(S_m)_\alpha \cong S_{m-\ l} \times S_l = S_\lambda,$$

where $(S_m)_\alpha$ is the stabilizer subgroup of α and S_λ is the Young subgroup corresponding to $\lambda \vdash m$.

Therefore,

$$\begin{aligned}\frac{1}{|(S_m)_\alpha|}\sum_{\sigma\in(S_m)_\alpha}\lambda(\sigma) &= [\lambda,\ 1_{(S_m)_\alpha}]\\ &= [\lambda,\ 1_\lambda]_{S_\lambda}\\ &= [\lambda,\ 1_\lambda^{S_m}]_{S_m}\\ &= \left[\lambda,\ \sum_{\mu\vdash m} K_{\mu,\lambda}\ \mu\right]_{S_m}\\ &= \sum_{\mu\vdash m} K_{\mu,\lambda}[\lambda,\mu]\\ &= K_{\mu,\mu}\\ &= 1\neq 0,\end{aligned}$$

by the Frobenius Reciprocity Theorem.

This implies that $e_\alpha^*\neq 0$ and we obtain $\alpha\in\bar{\Delta}$. Moreover, by using (2) we get $\dim V_\alpha^*=\lambda(\mathrm{id})$.

It can easily verify that the product of the hook lengths of $[\lambda]$ are

$$\begin{aligned}\Pi_{i=1}^{2}\Pi_{j=1}^{\lambda_i}h_{ij} &= (m-l+1)(m-l)\dots(m-2l+2)(m-2l)\dots 2\times l(l-1)\dots 2\\ &= \frac{l!(m-l+1)!}{(m-2l+1)}.\end{aligned}$$

Hence, by (4),

$$\dim V_\alpha^* = \frac{m!\ (m-2l+1)}{l!\ (m-l+1)!}.$$

Now, since

$$[S_m:(S_m)_\alpha] = \frac{m!}{(m-l)!\ l!},$$

using (3), we obtain

$$\|e_\alpha^*\|^2 = \frac{m-2l+1}{m-l+1}.$$

Observe that $\frac{1}{2}<\|e_\alpha^*\|^2<1$ if and only if $m\geq 3l$. Therefore the result follows from Theorem (3.1). $\square$

Remark. Let λ be an irreducible character of S_m of the form $\lambda=(m-l,\ l)$. Consider $\alpha=(\underbrace{1,\ 1,\ \dots,\ 1}_{m-l\ \text{times}},\ \underbrace{2,\ 2,\ \dots,\ 2}_{l\ \text{times}})\in\Gamma_{m,n}$. Clearly, the multiplicity partition of α is $M(\alpha)=\lambda$. Since $\lambda\succeq M(\alpha)$, hence $e_\alpha^*\neq 0$. Therefore we conclude that if $n\geq 2$ and $m\geq 2l$, then $V_\lambda(S_m)\neq 0$. $\square$

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