An instance of umbral methods in representation theory: the parking function module

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Abstract

We test the umbral methods introduced by Rota and Taylor within the theory of representation of the symmetric group. We prove that the volume polynomial of Pitman and Stanley represents the Frobenius characteristic of the Haiman parking function module, when the set of its variables consists of suitable umbrae. We also show that the volume polynomial in any set of similar and uncorrelated umbrae is umbrally equivalent, up to a constant term, to an Abel-like umbral polynomial. An analogous treatment of the parking function module of type B is given.
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1 Introduction

Parking functions were introduced by Konheim and Weiss [20] in the Sixties. Afterwards several authors gave a strong contribution to the development of this subject in the context of combinatorics and representation theory [15],[21], [24, 25], [39]. Recently Pitman and Stanley [26] have introduced the so-called volume polynomial, that arises naturally in several different settings, in particular in the study of plane partitions and parking functions. Haiman [17] has defined the parking function module considering the standard action of the symmetric group $\mathfrak{S}_n$ on the set of all parking functions of length n. In this context, parking functions give a combinatorial description of the ring R_n of diagonal coinvariants of $\mathbb{C}[\mathbf{x}, \mathbf{y}]$, seen as a representation of the symmetric group. See [18] for a survey.
The systematic study of noncrossing partitions was started in the Seventies by Kreweras [19] and Poupard [27]. This subject arises in the wide-ranging of connections between algebra and combinatorics [12, 13], [14], [34], [37], an overview can be found in [35]. In detail, the noncrossing partition lattice turns out to have strong symmetry properties which give rise to a symmetric function corresponding to a representation of the symmetric group. Indeed, Stanley [38] has recovered the Haiman action as a local action of the symmetric group on the maximal chains of the lattice of noncrossing partitions. This is done by defining an edge labelling of maximal chains with parking functions, each occuring once.
As shown by Biane [3] the lattice of noncrossing partitions can be embedded into the Cayley graph of the symmetric group. Moreover, Reiner [29] has introduced a class of noncrossing partitions for all classical reflection groups. In particular, type A reflection groups (i.e. symmetric groups) correspond to classical noncrossing partitions. The hyperoctahedral groups, that is type B reflection groups, correspond to the *noncrossing partitions of type B*, that is the noncrossing partitions of the set $[\pm n] = \{-n, \ldots, -1, 1, \ldots, n\}$ which are invariant under sign change. The results of Biane and Reiner have been generalized by Brady [5], Brady and Watt [6] and Bessis [2]. A poset $\mathcal{NC}(W)$ of noncrossing partitions can be defined for finite Coxeter systems (W, S) of each type. In particular, these posets have a nice description in terms of the length function

ℓ_T defined with respect to their sets T of reflections. The reader may refer [1] for an overview on the combinatorial aspect of this beautiful subject. The notion of parking function of type B, introduced by Stanley [38] and studied by Biane [4], parallels the classical one providing an edge labelling of maximal chains in the lattice of noncrossing partitions of type B.
The aim of this paper is to test the umbral methods introduced in Rota and Taylor [32] in this context. Applications of these methods are given by Zeilberger [42], where generating functions are computed for many difficult problems dealing with counting many combinatorial objects. Applications to bilinear generating functions for polynomial sequences are given by Gessel [16]. The ideas of Rota and Taylor have been developed by Di Nardo and Senato in [8, 9] and here we follow this last point of view. In this paper, we prove that the n-volume polynomial $V_n(x_1, \ldots, x_n)$ of Pitman and Stanley represents the Frobenius characteristic $\mathcal{PF}_n$ of the Haiman parking function module, when each variable x_i is replaced by a suitable umbra $\bar{\vartheta}_i$. We also use Abel polynomials $A_n(x, \alpha) = x(x - n.\alpha)^{n-1}$ of Rota, Shen and Taylor [31] to show that $n!V_n(\bar{\vartheta}_1, \ldots, \bar{\vartheta}_n)$ and $\bar{\vartheta}(\bar{\vartheta} + n.\bar{\vartheta})^{n-1}$ are umbrally equivalent. Analogous treatment is applied to the parking function module of type B. In this case, the polynomials $B_n(x, \alpha) = (x - n.\alpha)^n$ play a role of Abel polynomials of type B. Further applications of Abel polynomial can be found in [7], where an unifying framework for the cumulant theory, both classical, boolean and free, is given.

2 Parking functions, noncrossing partitions and volume polynomial

A *parking function* of length n is a sequence $\mathbf{p} = (p_1, \ldots, p_n)$ of n positive integers whose nondecreasing rearrangement $\mathbf{p}' = (p'_1, \ldots, p'_n)$ is such that $p'_j \leq j$. As in [26] we denote by $park(n)$ the set of all parking functions of length n. Its cardinality is $(n+1)^{n-1}$. The symmetric group $\mathfrak{S}_n$ acts on the set $park(n)$ by permuting the entries of parking functions (*standard action*). As introduced by Haiman [17], the *parking function module* is obtained by considering the standard action of $\mathfrak{S}_n$ on the $\mathbb{Q}$-vector space spanned by all parking functions of length n. The number of orbits of this action on the set $park(n)$ is equal to the n-th Catalan number C_n, that is

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

The Frobenius characteristic of the parking function module (i.e. the symmetric function associated to its character by the Frobenius map ch) is known as *parking function symmetric function* and is denoted by $\mathcal{PF}_n$. We have

$$\mathcal{PF}_n = \sum_{\mu \vdash n} \frac{(n)_{\ell(\mu)-1}}{m(\mu)!} h_\mu, \tag{2.1}$$

where the sum ranges over all integer partitions μ of n. More precisely, in (2.1) we have $m(\mu)! = m_1! \cdots m_n!$, m_i being the number of parts of μ equal to i, $\ell(\mu) = m_1 + \cdots + m_n$, and h_μ denotes the complete homogeneous symmetric function indexed by μ. There are at least three other ways to get the symmetric function $\mathcal{PF}_n$. Two of these arise within noncrossing partitions and are due to Stanley [38].
Let $[n]$ denote the set $\{1, \ldots, n\}$ of positive integers. A *noncrossing partition* of $[n]$ is a partition $\pi = \{B_1, \ldots, B_s\}$ of $[n]$, such that, if $1 \leq h < l < k < m \leq n$ with $h, k \in B_j$ and $l, m \in B_{j'}$, then $j = j'$. As usual, denote by $\mathcal{NC}_n$ the set of all noncrossing partitions of $[n]$. Its cardinality is $|\mathcal{NC}_n| = C_n$ too. A simple bijection between noncrossing partitions of $[n]$ and orbits of the set $park(n)$ with respect to the standard action of $\mathfrak{S}_n$ was given by Rattan [28].
As is well known, $\mathcal{NC}_n$ is a lattice of rank $n-1$ with respect to the refinement order. Denote by $\mathbf{0}_n$ and $\mathbf{1}_n$ its minimum and maximum element respectively. The number of maximal chains of $\mathcal{NC}_n$ is n^{n-2}. As shown by Stanley [38], maximal chains of $\mathcal{NC}_{n+1}$, whose number is $(n+1)^{n-1}$, can be labelled by parking functions, each occurring once. Moreover, if $V_{\mathcal{NC}_{n+1}}$ is the $\mathbb{Q}$-vector space spanned by all the maximal chains of $\mathcal{NC}_{n+1}$, a local action of the symmetric group $\mathfrak{S}_n$ can be defined on $V_{\mathcal{NC}_{n+1}}$ which turns out to have the same character of the parking function

module. In particular, this is obtained by transferring the action of $\mathfrak{S}_n$ on parking functions to their respective maximal chains. Stanley has also proved the following generalization. Let k be a positive integer. A k-parking function is a sequence $\mathbf{p} = (p_1, \ldots, p_n)$ of n positive integers whose nondecreasing rearrangement $\mathbf{p}' = (p'_1, \ldots, p'_n)$ is such that $p'_j \le kj$. Let $\mathcal{NC}_n^{(k)}$ denote the subset of all noncrossing partitions of $\mathcal{NC}_{kn}$ whose block's cardinalities are multiples of k. Maximal chains of $\mathcal{NC}_n^{(k)}$ are labelled by k-parking functions each occurring once. This yields a local action of $\mathfrak{S}_n$ on the $\mathbb{Q}$-vector space spanned by the maximal chains of $\mathcal{NC}_n^{(k)}$ which is isomorphic to the k-parking function module. The following results are proved in [38].

Theorem 2.1. *If $H(t) = 1 + \sum_{n\ge1} h_n t^n$ and $\mathcal{PF}(t) = 1 + \sum_{n\ge1} \mathcal{PF}_n t^n$, then we have*

$$t\mathcal{PF}(t) = \left[\frac{t}{H(t)}\right]^{<-1>}, \tag{2.2}$$

where $^{<-1>}$ denotes the compositional inverse. More generally, if k is a positive integer, $\mathcal{PF}_n^{(k)}$ is the Frobenius characteristic of the k-parking function module and $\mathcal{PF}^{(k)}(t) = 1+\sum_{n\ge1} \mathcal{PF}_n^{(k)} t^n$, then

$$t\mathcal{PF}^{(k)}(t) = \left[\frac{t}{H(t)^k}\right]^{<-1>}. \tag{2.3}$$

A second way to obtain the parking function symmetric function is the following. For each $S \subseteq [n-1]$ the *Gessel's quasi-symmetric function* $\mathcal{Q}_S$ is defined by

$$\mathcal{Q}_S = \sum_{\substack{i_1 \le \cdots \le i_n \\ i_j < i_{j+1} \text{ if } j \in S}} x_{i_1} \cdots x_{i_n}.$$

Let r denote the rank function of the lattice $\mathcal{NC}_{n+1}$. If $S \subseteq [n-1]$ and $|S| = s-1$, then let $\alpha_{\mathcal{NC}_{n+1}}(S)$ be the number of chains $\mathbf{0}_{n+1} = \pi_0 < \pi_1 < \ldots < \pi_s = \mathbf{1}_{n+1}$ such that $S = \{r(\pi_1), \ldots, r(\pi_{s-1})\}$. Define $\beta_{\mathcal{NC}_{n+1}}(S)$ to be the following integer:

$$\beta_{\mathcal{NC}_{n+1}}(S) = \sum_{T\subseteq S} (-1)^{|S-T|} \alpha_{\mathcal{NC}_{n+1}}(S).$$

The functions $\alpha_{\mathcal{NC}_{n+1}}$ and $\beta_{\mathcal{NC}_{n+1}}$ are named *flag f-vector* and *flag h-vector* respectively, of $\mathcal{NC}_{n+1}$. The connection between $\beta_{\mathcal{NC}_{n+1}}$ and $\mathcal{PF}_n$ is shown by the following theorem.

Theorem 2.2 ([38]). *Let $\beta_{\mathcal{NC}_{n+1}}$ and $\mathcal{Q}_S$ be defined as above. Then the polynomial*

$$F_{\mathcal{NC}_{n+1}} = \sum_{S\subseteq[n-1]} \beta_{\mathcal{NC}_{n+1}}(S)\mathcal{Q}_S,$$

is symmetric and it is such that

$$\omega F_{\mathcal{NC}_{n+1}} = \mathcal{PF}_n, \tag{2.4}$$

ω being the involution of the ring of the symmetric functions mapping complete homogeneous symmetric functions h_n onto elementary symmetric functions e_n.

Third approach to $\mathcal{PF}_n$ is essentially based on an involution ψ on the ring of symmetric functions defined by Macdonald [23] (see also [17] and [22] on this subject). The map ψ is defined by $\psi(h_n) = h_n^*$, where h_n^* are symmetric functions whose generating function $H^*(z) = 1 + \sum_{n\ge1} h_n^* z^n$ has the following property:

$$zH^*(z) = [zH(z)]^{<-1>}.$$

Lagrange inversion gives

$$(-1)^n h_n^* = \sum_{\mu\vdash n} \frac{(n)_{\ell(\mu)-1}}{m(\mu)!} e_\mu,$$

thus, by virtue of (2.1) we have

$$(-1)^n \omega(h_n^*) = \mathcal{PF}_n. \tag{2.5}$$

Let us recall the notion of volume polynomial. If $\mathbf{x} = \{x_1, \ldots, x_n\}$ is a set of commuting variables, and $\mathbf{c} = (c_1, \ldots, c_l)$ is a sequence of positive integers with $l \leq n$ and $c_i \leq n$ for $1 \leq i \leq l$, then we set

$$\begin{aligned} \mathbf{x_c} &= x_{c_1} \cdots x_{c_l}, \\ \mathbf{x^c} &= x_1^{c_1} \cdots x_n^{c_n}. \end{aligned}$$

Following Pitman and Stanley [26], we define the *n-volume polynomial* in the set of variables $\mathbf{x}$ to be the polynomial $V_n(\mathbf{x}) = V_n(x_1, \ldots, x_n)$ such that

$$V_n(\mathbf{x}) = \frac{1}{n!} \sum_{\mathbf{p} \in park(n)} \mathbf{x_p}. \tag{2.6}$$

Straightforward computations provides the following expression of $V_n(\mathbf{x})$:

$$V_n(\mathbf{x}) = \sum_{\mu \vdash n} \frac{(n)_{\ell(\mu)-1}}{m(\mu)!\mu!} \mathbf{x}^\mu, \tag{2.7}$$

where $\mathbf{x}^\mu = x_1^{\mu_1} \cdots x_l^{\mu_l}$ and $\mu! = \mu_1! \cdots \mu_l!$ whenever $\mu = (\mu_1, \ldots, \mu_l)$.

Remark 2.1. *We stress a direct connection between $\mathcal{PF}_n$ and this last expression of $V_n(\mathbf{x})$: by making the symbolic substitution $\mathbf{x}^\mu \to \mathbf{x}_\mu$ in (2.7), and then setting $x_i = i!h_i$, we recover the parking function symmetric function $\mathcal{PF}_n$. This fact suggests to us the introduction of umbral notations.*

3 Umbrae and Abel polynomials

Classical umbral calculus is a strongly symbolic method for the manipulation of sequences $(1, a_1, a_2, \ldots)$, where a_i belongs to some ring R whose quotient field is of characteristic zero. This way of dealing with sequences of numbers has been applied to combinatorial and algebraic subjects [16, 31, 40, 42], wavelets theory [33, 36], and difference equations [41]. It has also led to a finely adapted language for random variables theory, as shown in [8, 9, 30]. The basic device is to represent an unital sequence of numbers by a symbol α, named umbra, that is to associate the sequence $1, a_1, a_2, \ldots$ to the sequence $1, \alpha, \alpha^2, \ldots$ of powers of α through an operator E that resembles the expectation operator of random variables. Hence, an umbra carries the structure of a random variable, while making no reference to a probability space. Classical umbral calculus essentially consists of the following data:

1. a set $A = \{\alpha, \gamma, \delta, \ldots\}$, called the *alphabet*, whose elements are named *umbrae*,
2. a linear functional E, called *evaluation*, defined on the polynomial ring $R[A]$ and taking value in R, such that
   - $E[1] = 1$,
   - $E[\alpha^i \gamma^j \cdots \delta^k] = E[\alpha^i] E[\gamma^j] \cdots E[\delta^k]$ for all pairwise distinct umbrae $\alpha, \gamma, \ldots, \delta$ (*uncorrelation property*),
3. two special umbrae ε (*augmentation*) and u (*unity*) such that

   $$E[\varepsilon^i] = \delta_{0,i}, \quad \text{for } i = 0, 1, 2, \ldots,$$

   and

   $$E[u^i] = 1, \quad \text{for } i = 0, 1, 2, \ldots.$$

A sequence $(1, a_1, a_2, \ldots)$ is said to be *represented* by an umbra α if $E[\alpha^i] = a_i$ for $i = 1, 2, \ldots$ (note that $E[\alpha^0] = 1$ for all α). In this case we say a_i is the i-th *moment* of α. Two umbrae α and γ are said to be *umbrally equivalent*, denoted by $\alpha \simeq \gamma$, if $E[\alpha] = E[\gamma]$. They are said to be *similar*, written $\alpha \equiv \gamma$, if α and γ represent the same sequence of moments, that is $\alpha^i \simeq \gamma^i$ for all $i = 1, 2, \ldots$. We can extend coeffincientwise the action of E to exponential formal power series

$$e^{\alpha t} = \sum_{i \geq 0} \alpha^i \frac{t^i}{i!}$$

obtaining in this way the *generating function* $f(\alpha, t)$ of α:

$$f(\alpha, t) = E[e^{\alpha t}] = \sum_{i \geq 0} E[\alpha^i] \frac{t^i}{i!} = 1 + \sum_{i \geq 1} a_i \frac{t^i}{i!}.$$

Note that $\alpha \equiv \gamma$ if and only if $f(\alpha, t) = f(\gamma, t)$. The generating functions of the augmentation ε and the unity u are respectively

$$f(\varepsilon, t) = E[e^{\varepsilon t}] = \sum_{i \geq 0} E[\varepsilon^i] \frac{t^i}{i!} = 1$$

and

$$f(u, t) = E[e^{ut}] = \sum_{i \geq 0} E[u^i] \frac{t^i}{i!} = 1 + \sum_{i \geq 1} \frac{t^i}{i!} = e^t.$$

If α and γ are two umbrae, then the generating function $f(\alpha + \gamma, t)$ is given by $f(\alpha, t) f(\gamma, t)$. In fact

$$f(\alpha + \gamma, z) = E[e^{(\alpha+\gamma)t}] = E[e^{\alpha t} e^{\gamma t}] = E[e^{\alpha t}] E[e^{\gamma t}] = f(\alpha, z) f(\gamma, z).$$

The *Bell umbra* β is defined to be an umbra representing the sequence of Bell numbers $\mathcal{B}_i$, that is $\beta^i \simeq \mathcal{B}_i$. In this way

$$f(\beta, t) = e^{e^t - 1}.$$

The singleton umbra χ has moments $\chi^i \simeq 1$ if $i = 0, 1$, and $\chi^i \simeq 0$ otherwise. Its generating function is

$$f(\chi, t) = 1 + t.$$

We work with a *saturated umbral calculus*, see [32], if we extend the action of the evaluation E to the ring $R[A \cup B]$, where B is the *auxiliary alphabet* whose elements, named *auxiliary umbrae*, are defined starting from the umbrae in A. Umbral equivalence and similarity are extended via E to polynomials p and q in $R[A \cup B]$. Given $\alpha \in A$, first auxiliary umbra we introduce is denoted by $-1 . \alpha$. It is uniquely determined (up to similarity) by the condition

$$\alpha + (-1 . \alpha) \equiv \varepsilon. \tag{3.1}$$

Its generating function is $f(\alpha, t)^{-1}$. Indeed, from (3.1) we have

$$1 = f(\varepsilon, t) = f[\alpha + (-1 . \alpha), t] = f(\alpha, t) f(-1 . \alpha, t).$$

More generally, if n is an integer, the umbra denoted by $n . \alpha$ is such that

$$n . \alpha \equiv \alpha_1 + \cdots + \alpha_n,$$

$\alpha_1, \ldots, \alpha_n$ being uncorrelated umbrae similar to α. We have

$$f(n . \alpha, t) = f(\alpha, t)^n.$$

As introduced by Rota, Shen and Taylor [31], an Abel polynomial in the variable x is a polynomial in $R[A \cup B][x]$ of type

$$A_n(x, \alpha) = x(x - n . \alpha)^{n-1}.$$

The following theorem states that $n! V_n(\alpha_1, \ldots, \alpha_n)$ and $A_n(\alpha, -1 . \alpha)$ are in the same class of umbral equivalence for all $n \geq 1$.

Theorem 3.1. *Let α be an umbra, $\alpha_1,\dots,\alpha_n$ be n uncorrelated umbrae similar to α, and $V_n(\alpha_1,\dots,\alpha_n)$ be the n-volume polynomial (2.6) in $x_i=\alpha_i$. Then we have*

$$n! V_n(\alpha_1,\dots,\alpha_n) \simeq A_n(\alpha,-1.\alpha) \simeq \alpha(\alpha+n.\alpha)^{n-1}. \tag{3.2}$$

Proof: We denote by a_i the i-th moment of α. Then, the k-th moment of $n.\alpha$ is given by

$$(n.\alpha)^k \simeq \sum_{\lambda\vdash k} \mathrm{d}_\lambda\,(n)_{\ell(\lambda)}\,a_\lambda,$$

where $\mathrm{d}_\lambda = k!/(\lambda! m(\lambda)!)$ and $a_\lambda = a_{\lambda_1}\cdots a_{\lambda_l}$, whenever $\lambda=(\lambda_1,\dots,\lambda_l)$. Thus, we have

$$\alpha(\alpha+n.\alpha)^{n-1} \simeq \sum_{1\le k\le n}\ \sum_{\lambda\vdash n-k} \binom{n-1}{k-1} \mathrm{d}_\lambda\,(n)_{\ell(\lambda)}\,a_k a_\lambda.$$

Let $Par(n)$ the set of all the integer partitions of n. If $S_n=\{(k,\lambda)\,|\,1\le k\le n,\ \lambda\vdash n-k\}$, then the previous equivalence can be rewritten as

$$\alpha(\alpha+n.\alpha)^{n-1} \simeq \sum_{(k,\lambda)\in S_n} \binom{n-1}{k-1} \mathrm{d}_\lambda\,(n)_{\ell(\lambda)}\,a_k a_\lambda.$$

Let $\Delta : S_n \hookrightarrow Par(n)$ be defined by $\Delta(k,\lambda) = k\cup\lambda$, where $k\cup\lambda$ is the integer partition obtained adding the part k to λ. If $\Delta(k,\lambda)=\mu$ and $m(\mu)=(m_1,m_2,\dots,m_n)$, then $a_k a_\lambda = a_\mu$, $\ell(\lambda)=\ell(\mu)-1$, $k!\lambda! = \mu!$ and $m(\lambda)! = m(\mu)!/m_k$, so that

$$\binom{n-1}{k-1}\mathrm{d}_\lambda\,(n)_{\ell(\lambda)} a_k a_\lambda = \frac{n!}{\mu!}(n)_{\ell(\mu)-1}\frac{k}{n}\frac{m_k}{m(\mu)!}a_\mu.$$

Let $\{\mu\}$ denote the set of all distinct parts of μ, that is the set whose elements are the parts of μ each occurring once. Since $\sum_{k\in\{\mu\}} k m_k = n$, finally we gain

$$\alpha(\alpha+n.\alpha)^{n-1} \simeq n! \sum_{\mu\vdash n} \frac{(n)_{\ell(\mu)-1}}{m(\mu)!\mu!} a_\mu.$$

Equivalence (3.2) follows from (2.7) and from the fact that if $x_i=\alpha_i$ then $\mathbf{x}^\mu = \alpha_1^{\mu_1}\cdots\alpha_l^{\mu_l} \simeq a_\mu$.

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Remark 3.1. *Theorem 3.2 parallels a known result involving $V_n(\mathbf{x})$ proved by Pitman and Stanley in [26]. More precisely, for all $a\in\mathbb{C}$ we have*

$$V_n(a,\dots,a) = a(a+na)^{n-1},$$

so that $V_n(a,\dots,a)$ is obtained by evaluating $x=a$ in the Abel polynomial $A_n(x,-a)=x(x+na)^{n-1}$.

We assume $R=\mathbb{Q}[\mathbf{x}]$ in the umbral setting, that is R is the ring of polynomials with rational coefficient in the set of variables $\mathbf{x}$. Evaluation E maps the umbrae of the base alphabet A to polynomials of $\mathbb{Q}[\mathbf{x}]$. For this reason we call them *polynomial umbrae.* Let $\bar{\epsilon}$ be a polynomial umbra such that

$$\bar{\epsilon} \equiv \chi_1 x_1 + \cdots + \chi_n x_n,$$

where $\chi_1,\dots,\chi_n$ are n uncorrelated umbrae similar to χ. Its generating function is

$$f(\bar{\epsilon},t) = \prod_{i=1}^n f(\chi_i x_i,t) = \prod_{i=1}^n (1+x_i t) = 1+\sum_{n\ge1} e_i t^n = E(t),$$

where $e_i=e_i(\mathbf{x})$ is the i-th elementary symmetric function in the variables $\mathbf{x}$. Thus, i-th moment of $\bar{\epsilon}$ is given by

$$\bar{\epsilon}^i \equiv i! e_i.$$

In [11] a polynomial umbra ϵ representing elementary symmetric functions (that is $\epsilon^i \simeq e_i$) was defined, from which the choice of the symbol $\bar{\epsilon}$. We define a new polynomial umbra $\bar{\vartheta}$ as follows:

$$\bar{\vartheta} \equiv -1. - \bar{\epsilon}. \tag{3.3}$$

Since for every umbra α we have

$$f(-\alpha, t) = f(\alpha, -t),$$

it is clear that $f(\bar{\vartheta}, t) = f(\bar{\epsilon}, -t)^{-1} = E(-t)^{-1}$. Moreover, since $H(t)E(-t) = 1$ we have $f(\bar{\vartheta}, t) = H(t)$, and

$$\bar{\vartheta}^i \simeq i!h_i.$$

Theorem 3.2. *If $\bar{\vartheta}_1, \ldots, \bar{\vartheta}_n$ are n uncorrelated umbrae similar to $\bar{\vartheta}$ and $V_n(\mathbf{x})$ is the n-volume polynomial* (2.6)*, then*

$$V_n(\bar{\vartheta}_1, \ldots, \bar{\vartheta}_n) \simeq \mathcal{PF}_n. \tag{3.4}$$

Proof: Since $\bar{\vartheta}_1^{\mu_1} \cdots \bar{\vartheta}_l^{\mu_l} \simeq \mu!h_\mu$ if μ_i are the parts of μ, by virtue of identities (2.1) and (2.7) we have proved the theorem.

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The relation between $V_n(\mathbf{x})$ and the symmetric functions $F_{\mathcal{NC}_{n+1}}$ and h_n^* introduced in the previous section is stated in the following theorem.

Theorem 3.3. *If $\bar{\epsilon}_1, \ldots, \bar{\epsilon}_n$ are n uncorrelated umbrae similar to $\bar{\epsilon}$ and $V_n(\mathbf{x})$ is the n-volume polynomial* (2.6)*, then*

$$\begin{aligned} V_n(\bar{\epsilon}_1, \ldots, \bar{\epsilon}_n) &\simeq F_{\mathcal{NC}_{n+1}}, \\ V_n(-\bar{\epsilon}_1, \ldots, -\bar{\epsilon}_n) &\simeq h_n^*. \end{aligned}$$

Proof: Observe that $\bar{\epsilon}_1^{\mu_1} \cdots \bar{\epsilon}_l^{\mu_l} \simeq \mu!e_\mu$ and $(-\bar{\epsilon}_1)^{\mu_1} \cdots (-\bar{\epsilon}_l)^{\mu_l} \simeq (-1)^n \mu!e_\mu$. The theorem is proved by comparing (2.7) with (2.4) and (2.5).

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In order to show the connection between $V_n(\mathbf{x})$ and $\mathcal{PF}_n^{(k)}$, the following auxiliary umbrae may be useful. For each umbra α, let $\alpha^{<-1>}$ denote an auxiliary umbra such that

$$f(\alpha^{<-1>}, t) - 1 = [f(\alpha, t) - 1]^{<-1>}.$$

The umbra $\alpha^{<-1>}$ is named the *compositional inverse* of α (see [8]). The α-*derivative* umbra, deeply studied in [10], is an auxiliary umbra α_D whose moments satisfies the identity

$$(\alpha_D)^i \simeq \partial_\alpha \alpha^i \simeq i\alpha^{i-1}, \quad i = 1, 2, \ldots.$$

We obtain

$$f(\alpha_D, t) \simeq \sum_{i \geq 0} (\alpha_D)^i \frac{t^i}{i!} \simeq \sum_{i \geq 0} \alpha^{i-1} \frac{t^i}{(i-1)!} \simeq 1 + t\, f(\alpha, t).$$

Let $\bar{\rho}$ be a polynomial umbra with moments $\bar{\rho}^i \simeq i!\mathcal{PF}_i$, then

$$f(\bar{\rho}, t) = \mathcal{PF}(t).$$

Identity (2.2) provides

$$\bar{\rho}_D \equiv (-1.\bar{\vartheta})_D{}^{<-1>}. \tag{3.5}$$

By means of (3.4) and (3.5) we have

$$n!V_n(\bar{\vartheta}_1, \ldots, \bar{\vartheta}_n) \simeq \frac{1}{n+1}[(-1.\bar{\vartheta})_D{}^{<-1>}]^{n+1}.$$

It is not too difficult to show that such an equivalence will be true even if we replace the umbra $\bar{\vartheta}$ with another umbra α. That is

$$n!V_n(\alpha_1, \ldots, \alpha_n) \simeq \frac{1}{n+1}[(-1.\alpha)_D{}^{<-1>}]^{n+1}, \tag{3.6}$$

for all $\alpha \in A$.

Theorem 3.4. *If $\bar{\vartheta}_1, \ldots, \bar{\vartheta}_n$ are n uncorrelated umbrae similar to $\bar{\vartheta}$, k is a positive integer and $V_n(\mathbf{x})$ is the n-volume polynomial* (2.6)*, then*

$$V_n(k.\bar{\vartheta}_1, \ldots, k.\bar{\vartheta}_n) \simeq \mathcal{PF}_n^{(k)}.$$

Proof: Let $\bar{\rho}^{(k)}$ denote an umbra such that $(\bar{\rho}^{(k)})^n \simeq n!\mathcal{PF}_n^{(k)}$. From (2.3) we have

$$n!\mathcal{PF}_n^{(k)} \simeq (\bar{\rho}^{(k)})^n \simeq \frac{1}{n+1}\{[(-k.\bar{\vartheta})_D]^{<-1>}\}^{n+1}.$$

Finally, since $-1.k.\bar{\vartheta} \equiv -k.\bar{\vartheta}$, from (3.6) we gain

$$n!V_n(k.\bar{\vartheta}_1, \ldots, k.\bar{\vartheta}_n) \simeq \frac{1}{n+1}[(-k.\bar{\vartheta})_D{}^{<-1>}]^{n+1} \simeq n!\mathcal{PF}_n^{(k)},$$

from which the theorem is proved.

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4 Parking functions of type B

As shown by Biane [3] the lattice of noncrossing partitions can be embedded into the Cayley graph of the symmetric group. Reiner [29] has introduced a class of noncrossing partitions for all classical reflection groups, that for type A reflection groups (i.e. symmetric groups) corresponds to $\mathcal{NC}_n$. In the case of hyperoctahedral groups, that is type B reflection groups, the *noncrossing partitions of type B* are defined to be the noncrossing partitions of the set $[\pm n] = \{-n, \ldots, -1, 1, \ldots, n\}$ which are invariant under sign change. Denote by $\mathcal{NC}_n^B$ the set of such partitions. These methods have been generalized by Brady [5], Brady and Watt [6] and Bessis [2]. A poset $\mathcal{NC}(W)$ of noncrossing partitions can be defined for finite Coxeter systems (W, S) of each type. In particular, these posets have a nice description in terms of the length function ℓ_T defined with respect to their respective sets T of reflections. The reader may refer [1] for an overview on the combinatorial aspect of this beautiful subject. Stanley [38] has noticed that maximal chains of $\mathcal{NC}_n^B$ are labeled by all sequences in $[n]^n$ each occurring once. From this analogy with parking functions, he has named them *parking functions of type B*. Biane [4] has completed the picture by showing that parking functions of length n of type A (i.e. classical ones) and B correspond to factorizations of the cycles $(1 \ldots n{+}1)$ and $(-n \ldots -1\, 1 \ldots n)$ respectively into products of reflections. The parking function module of type B can be defined by considering the standard action of $\mathfrak{S}_n$ on $[n]^n$. Following Stanley, we denote by $\mathcal{PF}_n^B$ its Frobenius characteristic.

Theorem 4.1 ([38]). *Let $\mathcal{PF}^B(t) = 1 + \sum_{n\geq 1} \mathcal{PF}_n^B t^n$, then we have*

$$\mathcal{PF}_n^B = [t^n]H(t)^n, \tag{4.1}$$

where $[t^n]$ means taking the coefficient of t^n in the power series.

We will define a type B Abel polynomial, denoted by $B_n(x, \alpha)$, which plays a role analogous to $A_n(x, \alpha)$ for the parking function module of type B. It is simply obtained by dividing $A_{n+1}(x, \alpha)$ by x, that is

$$B_{n-1}(x, \alpha) = (x - n.\alpha)^{n-1}.$$

Theorem 4.2. *Let $\mathcal{PF}^B$ be the Frobenius characteristic of the parking function module of type B and $\bar{\vartheta}$ be the polynomial umbra defined in (3.3). Then we have*

$$n!\mathcal{PF}_n^B \simeq B_n(-1.\bar{\vartheta}, -1.\bar{\vartheta}) \simeq (n.\bar{\vartheta})^n.$$

Proof: If $\bar{\rho}^B$ is a polynomial umbra such that $(\bar{\rho}^B)^i \simeq i!\mathcal{PF}^B$, then from (4.1) we have

$$(\bar{\rho}^B)^n \simeq (n.\bar{\vartheta})^n.$$

Finally, since $-1.\bar{\vartheta} + (n+1).\bar{\vartheta} \equiv n.\bar{\vartheta}$, then we have $B_n(-1.\bar{\vartheta}, -1.\bar{\vartheta}) \simeq (n.\bar{\vartheta})^n$ and the proof is completed.

Of course, if we introduce the polynomial $V_n^B(\mathbf{x})$ such that

$$V_n^B(\mathbf{x}) = \frac{1}{n!} \sum_{\mathbf{p} \in [n]^n} \mathbf{x_p} = \sum_{\mu \vdash n} \frac{(n)_{\ell(\mu)}}{m(\mu)! \mu!} \mathbf{x}^{\mu},$$

then we can complete the analogy with the results in the previous section.

Theorem 4.3. *The following umbral equivalence holds:*

$$n! V_n^B(\bar{\vartheta}_1, \ldots, \bar{\vartheta}_n) \simeq B_n(-1.\bar{\vartheta}, -1.\bar{\vartheta}).$$

Proof: It follows by simple computations.

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