The signed Eulerian numbers on involutions

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Abstract

We define an analog of signed Eulerian numbers $f_{n,k}$ for involutions of the symmetric group and derive some combinatorial properties of this sequence. In particular, we exhibit both an explicit formula and a recurrence for $f_{n,k}$ arising from the properties of its generating function.
Keywords: involution, ascent, signed Eulerian number.
AMS classification: 05A15, 05A19.

1 Introduction

Let σ be a permutation in S_n. We say that σ has a *descent* at position i whenever $\sigma(i) > \sigma(i+1)$. Analogously, we say that σ has an *ascent* at position i whenever $\sigma(i) < \sigma(i+1)$. The number of descents (ascents) of a permutation σ is denoted by $des(\sigma)$ ($asc(\sigma)$). The polynomial

$$A_n(t) = \sum_{\sigma \in S_n} t^{des(\sigma)} = \sum_{k=0}^{n-1} e_{n,k} t^k,$$

is known as the *Eulerian polynomial*, and the integers $e_{n,k}$, i.e., the number of permutations $\sigma \in S_n$ with $des(\sigma) = k$, are called the *Eulerian numbers*.
We recall that the Eulerian numbers satisfy the property $e_{n,k} = e_{n,n-1-k}$, that implies

$$\sum_{\sigma \in S_n} t^{des(\sigma)} = \sum_{\sigma \in S_n} t^{asc(\sigma)}. \tag{1.1}$$

The study of the distribution of the descent statistic has been carried out both in the case of the symmetric group and of some particular subsets of permutations. For example, Eulerian distribution on the set $\mathscr{I}_n$ of involutions has been deeply investigated by several authors ([1], [4], [5], and [6]).

Loday [8], in his study of the cyclic homology of commutative algebras, introduced the sequence $b_{n,k}$ of *signed Eulerian numbers*, namely, the coefficients of the polynomial

$$B_n(t) = \sum_{\sigma \in S_n} sgn(\sigma) t^{asc(\sigma)} = \sum_{k=0}^{n-1} b_{n,k} t^k.$$

In [3], Désarménien and Foata proved several combinatorial properties satisfied by the integers $b_{n,k}$ by exploiting their relations with the Eulerian numbers.

In this paper we study the signed Eulerian numbers on involutions, i.e., the coefficients of the *signed Eulerian polynomial on involutions*

$$F_n(t) = \sum_{\sigma \in \mathscr{I}_n} sgn(\sigma) t^{asc(\sigma)} = \sum_{k=0}^{n-1} f_{n,k} t^k.$$

Our basic tool is a map introduced in [2] that associates an involution with a family of generalized involutions, which share the shape of the corresponding semistandard Young tableau. This allows us to define the sign of a generalized involution as the sign of the corresponding involution, and give an explicit formula for the number of odd and even generalized involutions of length n over a fixed alphabet. This formula yields an explicit formula for the signed Eulerian numbers on involutions and the following expression for the generating function of the signed Eulerian polynomials:

$$\sum_{n\geq 0} F_n(t)\frac{u^n}{(1-t)^{n+1}} = \sum_{r\geq 0} t^r \frac{(1+u)^{r+1}}{(1+u^2)^{\binom{r+2}{2}}}.$$

Applying the Maple package $ZeilbergerRecurrence(T, n, k, s, 0..n)$ to this last identity, we find a recurrence satisfied by the integers $f_{n,k}$.
Moreover, we exhibit an explicit formula, a recurrence, and a generating function for the sequence $F_n(1)$, with $n \in \mathbb{N}$, namely, the difference between the number of even and odd involutions in $\mathscr{I}_n$.

The last section is devoted to various analogs of Worpitzky's Identity. Our starting point is a known interpretation of the classical Worpitzky's Identity (see, e.g., [12]), which is based on a map from the set of functions $f : [n] \to [m]$, and the symmetric group S_n. In this perspective, the identities relating the numbers of (signed) generalized involutions on a given alphabet and (signed) involutions with a fixed number of ascents can be seen as variations of Worpitzky's Identity.

2 The signed Eulerian numbers

A *generalized permutation* is defined to be a biword:

$$\alpha = \binom{x}{y} = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{pmatrix},$$

such that $x_1 \leq x_2 \leq \cdots \leq x_n$ and $x_i = x_{i+1} \Longrightarrow y_i \geq y_{i+1}$. If, in addition, we have that for every $1 \leq i \leq n$, there exists an index j with $x_i = y_j$ and $y_i = x_j$, the array α is called a *generalized involution*. The word $x = x_1 \cdots x_n$ is called the *content* of the generalized involution, and the integer n is called its *length*.
We say that an integer a is a *repetition* of multiplicity r for the generalized involution α if

$$x_i = y_i = x_{i+1} = y_{i+1} = \cdots = x_{i+r-1} = y_{i+r-1} = a.$$

For example, the integer 7 is a repetition of multiplicity 3 for the generalized involution

$$\alpha = \begin{pmatrix} 1 & 3 & 3 & 7 & 7 & 7 & 7 & 8 & 9 \\ 8 & 9 & 7 & 7 & 7 & 7 & 3 & 1 & 3 \end{pmatrix}.$$

In [2], the present authors introduced a map Π from the set of generalized involutions to the set of involutions $\mathscr{I}_n$ defined as follows: if

$$\alpha = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{pmatrix}$$

is a generalized involution, then $\Pi(\alpha)$ is the involution σ

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ y'_1 & y'_2 & \cdots & y'_n \end{pmatrix},$$

where $y'_i = 1$ if y_i is the least symbol occurring in the word y, $y'_j = 2$ if y_j is the second least symbol in y and so on. In the case $y_i = y_j$, with $i > j$, we consider y_i to be less then y_j. We call the involution $\sigma = \Pi(\alpha)$ the ***standardization*** of α. The standardization map can be defined analogously on generalized permutations.

For example, the standardization of the generalized involution

$$\alpha = \begin{pmatrix} 1 & 3 & 3 & 7 & 7 & 7 & 7 & 8 & 9 \\ 8 & 9 & 7 & 7 & 7 & 7 & 3 & 1 & 3 \end{pmatrix}$$

is the involution

$$\sigma = \Pi(\alpha) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 9 & 7 & 6 & 5 & 4 & 3 & 1 & 2 \end{pmatrix}.$$

We denote by $\mathrm{Gen}_m(\sigma)$ the set of generalized involutions, with symbols taken from $[m] = \{1, 2, \ldots, m\}$, whose standardization is σ. Then, we have the following result stated in [1]. We describe the proof in full details, since it will be useful in the following sections:

Proposition 1. Let $\sigma \in \mathscr{I}_n$ be an involution with t ascents. Then,

$$|\mathrm{Gen}_m(\sigma)| = \binom{n+m-t-1}{n}. \tag{2.1}$$

Proof: Choose an involution $\sigma \in \mathscr{I}_n$ with t ascents. It is easily seen that the set $\mathrm{Gen}_m(\sigma)$ corresponds bijectively to the set of contents $x = x_1 \ldots x_n$ with $1 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq m$, where the inequalities are strict in correspondence of the ascents of σ. Every such content is uniquely determined by the sequence $\delta := \delta_0\delta_1 \ldots \delta_n$, with

$$\delta_0 = x_1 - 1, \quad \delta_1 = x_2 - x_1, \quad \ldots, \quad \delta_n = m - x_n$$

which is a composition of the integer $m-1$ such that its i-th component δ_i is at least one whenever σ has an ascent at the i-th position. For this reason we can consider the word $\delta' = \delta'_0\delta'_1 \ldots \delta'_n$ defined as follows:

$$\delta'_i = \begin{cases} \delta_i - 1 & \text{if } \sigma \text{ has an ascent at the i-th position} \\ \delta_i & \text{otherwise} \end{cases},$$

which is a composition of the integer $m - t - 1$ in $n + 1$ parts. This gives the assertion.

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We define the ***sign*** of a generalized involution α ($sgn(\alpha)$) to be the sign of its standardization $\Pi(\alpha)$. The integer $sgn(\alpha)$ can be also described in terms of repetitions of α, as follows.
Recall that the sign of an involution $\sigma \in \mathscr{I}_n$ is determined by the number $fix(\sigma)$ of fixed points of σ. More precisely:

$$sgn(\sigma) = (-1)^{\frac{n-fix(\sigma)}{2}}.$$

It is easily seen that, given a generalized involution α and its standardization $\Pi(\alpha)$, every repetition of α of odd multiplicity yields a fixed point in $\Pi(\alpha)$, while repetitions of even multiplicity of α produce no fixed point of $\Pi(\alpha)$. For example, the standardization of the generalized involution

$$\alpha = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 3 & 3 & 3 & 3 & 4 & 5 & 5 & 5 & 5 \\ 4 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 5 & 3 & 3 & 3 & 1 & 5 & 5 & 5 & 3 \end{pmatrix}$$

with 3 repetitions of odd multiplicity is the involution

$$\Pi(\alpha) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 \\ 13 & 6 & 5 & 4 & 3 & 2 & 8 & 7 & 17 & 12 & 11 & 10 & 1 & 16 & 15 & 14 & 9 \end{pmatrix},$$

which has 3 fixed points. Note that the symbol 2 is a repetition of multiplicity 2 and does not produce any fixed point in $\pi(\alpha)$.

Hence, if we denote by $gfix(\alpha)$ the number of repetitions of odd multiplicity of α, we have $gfix(\alpha) = fix(\Pi(\alpha))$ and

$$sgn(\alpha) = (-1)^{\frac{n-gfix(\alpha)}{2}}.$$

The number of generalized involutions α of length n over $[m]$ with an assigned number of repetitions of odd multiplicity can be computed as follows:

Proposition 2. The number of generalized involutions α of length n over $[m]$ with $gfix(\alpha) = s$ is

$$\binom{m}{s}\binom{\binom{m+1}{2} + \frac{n-s}{2} - 1}{\frac{n-s}{2}}.$$

Proof: The s repetitions of odd multiplicity of α can be chosen in $\binom{m}{s}$ ways. In order to complete α, it suffices to choose $\frac{n-s}{2}$ pairs of the kind:

- $\binom{x_k}{y_k}$ with $x_k < y_k$. In this case, α must contain also the pair $\binom{y_k}{x_k}$;
- $\binom{x_k}{x_k}$. In this case, α must contain an additional occurrence of the same pair.

Since there are $\binom{m}{2}$ pairs of the former kind and m pairs of the latter kind, we get the assertion.

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Let by $A_{n,m}$ be the set of generalized involutions of length n over $[m]$ and $\hat{a}_{n,m}$ the integer

$$\hat{a}_{n,m} = \sum_{\alpha \in A_{n,m}} (-1)^{\frac{n-gfix(\alpha)}{2}}.$$

Then, Proposition 2 yields the following explicit formula for $\hat{a}_{n,m}$:

Proposition 3. We have:

$$\hat{a}_{n,m} = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \binom{\binom{m+1}{2} + j - 1}{j}\binom{m}{n-2j}. \tag{2.2}$$

Proof: The contribution in $\hat{a}_{n,m}$ of a generalized involution α with $gfix(\alpha) = s$ equals $(-1)^{\frac{n-s}{2}}$. Hence, by Proposition 2, we have:

$$\hat{a}_{n,m} = \sum_{\substack{s=0 \\ n-s \text{ even}}}^{n} (-1)^{\frac{n-s}{2}} \binom{\binom{m+1}{2} + \frac{n-s}{2} - 1}{\frac{n-s}{2}}\binom{m}{s},$$

that is equivalent to Identity (3) by setting $j = \frac{n-s}{2}$.

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Denote by $a^+_{n,m}$ ($a^-_{n,m}$) the number of even (odd) generalized involutions of length n over $[m]$. Proposition 1 implies that

$$a^+_{n,m} = \sum_{k=0}^{m-1} \binom{n+k}{k} f^+_{n,m-k-1} \tag{2.3}$$

$$a^-_{n,m} = \sum_{k=0}^{m-1} \binom{n+k}{k} f^-_{n,m-k-1}, \tag{2.4}$$

where $f^+_{n,k}$ and $f^-_{n,k}$ denote the number of positive and negative involutions in $\mathscr{I}_n$ with k ascents, respectively. These identities allow to state the following result:

Theorem 4. The signed Eulerian number $f_{n,k}$ can be computed as follows:

$$f_{n,k} = \sum_{m=0}^{k+1} (-1)^{k-m+1} \binom{n+1}{k-m+1} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \binom{\binom{m+1}{2}+j-1}{j} \binom{m}{n-2j}. \tag{2.5}$$

Proof: Obviously, $\hat{a}_{n,m} = a^+_{n,m} - a^-_{n,m}$. Then, Identities (2.3) and (2.4) imply

$$\hat{a}_{m,n} = \sum_{k=0}^{m-1} \binom{n+k}{k} f_{n,m-k-1}. \tag{2.6}$$

By inversion, we have:

$$f_{n,k} = \sum_{m=0}^{k+1} (-1)^{k-m+1} \binom{n+1}{k-m+1} \hat{a}_{n,m} =$$

$$= \sum_{m=0}^{k+1} (-1)^{k-m+1} \binom{n+1}{k-m+1} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \binom{\binom{m+1}{2}+j-1}{j} \binom{m}{n-2j},$$

as desired.

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The next table contains the first values of the sequence $f_{n,k}$:

n/k	0	1	2	3	4	5	6	7	8	9
1	1									
2	-1	1								
3	-1	-2	1							
4	1	-2	-2	1						
5	1	6	0	-2	1					
6	-1	3	14	2	-3	1				
7	-1	-12	-15	12	-1	-4	1			
8	1	-4	-51	-76	4	-3	-4	1		
9	1	20	67	-10	-80	30	3	-4	1	
10	-1	5	137	517	414	66	75	7	-5	1

3 The signed Eulerian polynomial on involutions

We define the n-th *signed Eulerian polynomial for involutions* to be the polynomial:

$$F_n(t) = \sum_{\sigma \in \mathscr{I}_n} sgn(\sigma) t^{asc(\sigma)} = \sum_{k=0}^{n-1} f_{n,k} t^k.$$

As shown in the previous section, the relation between involutions and generalized involutions is crucial in our analysis. We denote by

$$R_n(t) = \sum_{m \geq 0} \hat{a}_{n,m} t^m$$

$$C_m(t) = \sum_{n \geq 0} \hat{a}_{n,m} t^n$$

the row and column generating functions of the array $\hat{a}_{n,m}$. As seen in the proof of Theorem 4, we have

$$\hat{a}_{n,m} = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \binom{\binom{m+1}{2}+j-1}{j} \binom{m}{n-2j}$$

and hence

$$C_m(t) = \frac{(1+u)^m}{(1+u^2)^{\binom{m+1}{2}}}.$$

Theorem 5. The polynomial $F_n(t)$ satisfies the identity

$$\sum_{n\geq 0} F_n(t) \frac{u^n}{(1-t)^{n+1}} = \sum_{m\geq 0} \frac{(1+u)^{m+1}}{(1+u^2)^{\binom{m+2}{2}}} t^m. \tag{3.1}$$

Proof: The binomial relation between the sequences $f_{n,k}$ and $\hat{a}_{n,m}$ (Identity (2.5)) yields the following identity:

$$\frac{tF_n(t)}{(1-t)^{n+1}} = R_n(t).$$

Hence:

$$\sum_{n\geq 0} F_n(t) \frac{u^n}{(1-t)^{n+1}} = \sum_{n\geq 0}\sum_{m\geq 0} \hat{a}_{n,m+1} t^m u^n =$$

$$= \sum_{m\geq 0} C_{m+1} t^m = \sum_{m\geq 0} \frac{(1+u)^{m+1}}{(1+u^2)^{\binom{m+2}{2}}} t^m.$$

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Identity (3.1) yields the following recurrence formula:

Theorem 6. The signed Eulerian numbers satisfy the polynomial recurrence:

$$\begin{aligned} nf_{n,k} &= (2+k-n)f_{n-1,k} + (2n-k-1)f_{n-1,k-1} - (n+3k+k^2)f_{n-2,k}+ \\ &+(-2+4k+2k^2-2kn)f_{n-2,k-1} + (2-k-k^2+2kn-n^2)f_{n-2,k-2}+ \\ &+(-n-k^2-2k+2)f_{n-3,k} + (-7+4k+3k^2+2n-2kn)f_{n-3,k-1}+ \\ &+(8-2k-3k^2-2n+4kn-n^2)f_{n-3,k-2} + (-3+k^2+n-2kn+n^2)f_{n-3,k-3}. \end{aligned}$$

Proof: Following along the lines of the proof of Theorem 2.2 in [6], we apply the Maple package ZeilbergerRecurrence$(T, n, k, s, 0..n)$ to the sequence $f_{n,k}$, where we set

$$T(n,k) = (-1)^k \binom{x+k-1}{k}\binom{y}{n-2k}$$

and

$$s(n) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} T(n,k).$$

The Maple package yields the following polynomial recurrence for $s(n)$:

$$n\,s(n) = -(2x-y+n-3)s(n-3) - (2x+n-2)s(n-2) + (y-n+1)s(n-1).$$

Setting $x = \binom{m+1}{2}$ and $y = m$, this identity is equivalent to

$$n\,s(n) = -(m(m-1)+m+n-3)s(n-3) - (m(m-1)+2m+n-2)s(n-2) + (m-n+1)s(n-1),$$

that can be rephrased as follows:

$$\begin{aligned} \frac{nI_n(t)}{(1-t)^{n+1}} &= -t^2\left(\frac{I_{n-3}(t)}{(1-t)^{n-2}}\right)'' - t\left(\frac{I_{n-3}(t)}{(1-t)^{n-2}}\right)' - (n-3)\frac{I_{n-3}(t)}{(1-t)^{n-2}}+ \\ &-t^2\left(\frac{I_{n-2}(t)}{(1-t)^{n-1}}\right)'' - 2t\left(\frac{I_{n-2}(t)}{(1-t)^{n-1}}\right)' - (n-2)\frac{I_{n-2}(t)}{(1-t)^{n-1}}+ \\ &+t\left(\frac{I_{n-1}(t)}{(1-t)^{n}}\right)' + (1-n)\frac{I_{n-1}(t)}{(1-t)^{n}}. \end{aligned}$$

Comparing the coefficients of t^k in both sides of this identity we get the assertion.

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In conclusion, we study the combinatorial properties of the sequence $F_n(1)$, with $n \in \mathbb{N}$. Obviously, $F_n(1)$ is the difference between the number i_n^+ of even involutions on n objects and the number i_n^- of odd involutions. First of all, we have:

Proposition 7. The evaluation of the polynomial $F_n(t)$ at 1 is

$$F_n(1) = 2\sum_{h=0}^{\lfloor \frac{n}{4} \rfloor} \frac{n!}{(2h)!(n-4h)!2^{2h}} - \sum_{h=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{(n-2h)!2^h}.$$

Proof: Fix an integer $h \leq \lfloor \frac{n}{4} \rfloor$. We count the number of involutions whose cycle decomposition consists of $2h$ transpositions. Choose a word $w = w_1 \cdots w_n$ consisting of distinct letters taken from $[n]$. We have $n!$ choices for w. This word corresponds to a unique even involution τ with $n-4h$ fixed points defined by the following conditions:

- $\tau(w_{2i-1}) = w_{2i}$ with $1 \leq i \leq 2h$,
- $\tau(w_{4h+j}) = w_{4h+j}$ with $1 \leq j \leq n-4h$.

It is easily checked that the involution τ arises from $(n-4h)!(2h)!2^{2h}$ different words w. These considerations imply that

$$i_n^+ = \sum_{h=0}^{\lfloor \frac{n}{4} \rfloor} \frac{n!}{(2h)!(n-4h)!2^{2h}}.$$

On the other hand, it is well known that

$$|\mathscr{I}_n| = \sum_{h=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{(n-2h)!2^h}.$$

Hence, since $F_n(1) = i_n^+ - i_n^- = 2i_n^+ - |\mathscr{I}_n|$, we get the assertion.

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Proposition 8. The sequence $F_n(1)$ satisfies the recurrence

$$F_n(1) = F_{n-1}(1) - (n-1)F_{n-2}(1).$$

Proof: Consider an even involution $\tau \in \mathscr{I}_n$. If $\tau(1) = 1$, the restriction of τ on the set $\{2, \ldots, n\}$ is an even involution on $n-1$ objects. If $\tau(1) = j \neq 1$, the restriction of τ on $\{2, \ldots, n\} \setminus \{j\}$ is an odd involution on $n-2$ objects. This implies that:

$$i_n^+ = i_{n-1}^+ + (n-1)i_{n-2}^-$$

and, analogously,

$$i_n^- = i_{n-1}^- + (n-1)i_{n-2}^+.$$

These identities give immediately the assertion.

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The preceding results allow to deduce the following expression for the exponential generating function of the sequence $F_n(1)$:

$$\sum_{n \geq 0} \frac{F_n(1)}{n!} t^n = e^{t - \frac{t^2}{2}}.$$

4 Worpitzky's identities

Denote by $e_{n,k}$ the (n,k)-th Eulerian number, namely, the number of permutations in S_n with k descents. It is well known that $e_{n,k}$ counts also permutations in S_n with k ascents. The famous Worpitzky's Identity

$$\sum_{k=0}^{n-1} e_{n,k}\binom{m+k}{n} = m^n$$

connects the polynomials m^n with the polynomials $\binom{m+k}{n}$. This identity can be also seen (see, e.g., [12]) as a relation between the Eulerian numbers $e_{n,k}$ and the number of functions $f:[n]\to[m]$. In fact, each function $f:[n]\to[m]$ can be regarded as the generalized permutation:

$$f = \left(\begin{array}{cccc} 1 & 2 & \cdots & n \\ f(1) & f(2) & \cdots & f(n) \end{array}\right).$$

Given a permutation $\sigma \in S_n$ with k ascents, the number f_σ of functions $f:[n]\to[m]$ whose standardization is equal to σ depends only on k. In fact, similar argumentations to those used in the proof of Theorem 1 lead to the following explicit formula for f_σ:

$$f_\sigma = \binom{m+n-k-1}{n}.$$

For example, set $m=3$ and $n=3$. The $\binom{3+1}{3}=4$ functions $f:[3]\to[3]$ that polarize to

$$\sigma = \left(\begin{array}{ccc} 1 & 2 & 3 \\ 2 & 1 & 3 \end{array}\right)$$

are

$$f_1 = \left(\begin{array}{ccc} 1 & 2 & 3 \\ 1 & 1 & 2 \end{array}\right) \quad f_2 = \left(\begin{array}{ccc} 1 & 2 & 3 \\ 2 & 1 & 3 \end{array}\right) \quad f_3 = \left(\begin{array}{ccc} 1 & 2 & 3 \\ 1 & 1 & 3 \end{array}\right) \quad f_4 = \left(\begin{array}{ccc} 1 & 2 & 3 \\ 2 & 2 & 3 \end{array}\right).$$

Hence:

$$m^n = \sum_{\sigma\in S_n} f_\sigma = \sum_{k=0}^{n-1}\binom{m+n-k-1}{n} e_{n,k}$$

that is equivalent to Worpitzky's Identity, recalling that $e_{n,k} = e_{n,n-k-1}$.

In the case of involutions, an analog of Worpitzky's Identity can be derived in a closely similar way. Denote by $i_{n,k}$ the number of involutions in S_n with k ascents (and also with k descents, as proved, for instance, in [11]). Recall that the number $|\mathrm{Gen}_m(\sigma)|$ of generalized involutions of length n over $[m]$ of fixed standardization σ depends only on the number k of ascents of σ, as stated in Theorem 1:

$$|\mathrm{Gen}_m(\sigma)| = \binom{n+m-t-1}{n}.$$

Hence, the total number $a_{n,m}$ of generalized involutions of length n over $[m]$ can be expressed in terms of the integers $i_{n,k}$:

$$a_{n,m} = \sum_{\sigma\in I_n} |\mathrm{Gen}_m(\sigma)| = \sum_{k=0}^{m-1}\binom{n+k}{k} i_{n,m-k-1}.$$

Since

$$a_{n,m} = \sum_{j=0}^{\lfloor \frac{n}{2}\rfloor}\binom{\frac{m(m-1)}{2}+j-1}{j}\binom{m+n-2j-1}{m-1},$$

we deduce the following analog of the Worpitzky's Identity:

$$\sum_{k=0}^{m-1} \binom{n+k}{k} i_{n,m-k-1} = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{\frac{m(m-1)}{2}+j-1}{j} \binom{m+n-2j-1}{m-1}.$$

This identity was proved in full details in [1].

From this perspective, Identity (2.6) yields a further analog of the Worpitzky's Identity for the case of signed involutions:

$$\sum_{j=0}^{s-1} \binom{n+j}{j} f_{n,s-j-1} = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \binom{\binom{s+1}{2}+j-1}{j} \binom{s}{n-2j}.$$

In conclusion, we recall that the following analogs of Worpitzky's Identity for the more general case of signed permutations appear in [3]:

$$\sum_{i=0}^{k} \binom{2n+i}{i} B_{2n,k-i} = k^n,$$

$$\sum_{i=0}^{k} \binom{2n-1+i}{i} B_{2n-1,k-i} = k^n.$$

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