Partially ordered patterns and their combinatorial interpretations

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Abstract

This paper is a continuation of the study of partially ordered patterns (POPs) introduced recently. We provide a general approach to code combinatorial objects using (POP-)restricted permutations. We give several examples of relations between permutations restricted by POPs and other combinatorial structures, such as labeled graphs, walks, binary vectors, and others. Also, we show how restricted permutations are related to Cartesian products of certain objects.
Keywords: pattern avoidance, segmented patterns, permutations, words, walks, (homeomorphically irreducible) labeled general graphs, binary vectors, coding.

1 Introduction

Below we define partially ordered generalized patterns (POPs) by first defining "classical patterns," and then defining generalized patterns (GPs) introduced by Babson and Steingrímsson (see [1]). We refer to [2, 7, 10] for motivations to study GPs, POPs, and classical patterns.

We write permutations as words $\pi = a_1 a_2 \cdots a_n$, whose letters are distinct and usually consist of the integers $1, 2, \ldots, n$.

An occurrence of a *pattern* τ in a permutation π is "classically" defined as a subsequence in π (of the same length as τ) whose letters are in the same relative order as those in τ. For example, the permutation 31425 has three occurrences of the pattern 123, namely the subsequences 345, 145, and 125.

Generalized permutation patterns (*GPs*) allow the requirement that two adjacent letters in a pattern must be adjacent in the permutation. In order to avoid confusion we write a "classical" pattern, say 231, as 2-3-1, and if we write, say 2-31, then we mean that if this pattern occurs in a permutation π, then the letters in π that correspond to 3 and 1 are adjacent. For example, the permutation $\pi = 516423$ has only one occurrence of the pattern 2-31, namely the subword 564, whereas the pattern 2-3-1 occurs, in addition, in the subwords 562 and 563.

In [5, 7], a further generalization of GPs was introduced, namely *partially ordered patterns* (*POPs*) also called *partially ordered generalized patterns* (*POGPs*). A POP is a GP some of whose letters are incomparable. For instance, if we write $p = 1\text{-}1'2'$ then we mean that in an occurrence of p in a permutation π the letter corresponding to the 1 in p can be either larger or smaller than the letters corresponding to $1'2'$. Thus, the permutation 31254 has three occurrences of p, namely 3-12, 3-25, and 1-25.

A POP with no dashes is called a *segmented POP (SPOP)*.

In this paper we continue the study of POPs considered in [5]–[9]. Sections 2 and 3 are devoted to POPs in permutations, which is the main concern of the paper. Section 4 deals with POPs in words.

Let $\mathcal{D}_n$ and $\mathcal{M}_n$ denote the set of *Dyck paths* of length $2n$ and *Motzkin paths* of length n respectively. What could be a natural combinatorial interpretation for the set $\mathcal{D}_i \times \mathcal{M}_{n-i}$ or $\mathcal{D}_{i_1} \times \mathcal{M}_{i_2} \times \mathcal{M}_{n-i_1-i_2}$? It turns out that such a combinatorial interpretation in these and many other cases is given by a set of (n-)permutations simultaneously avoiding certain sets of POPs. In section 3 we explain this phenomenon as well as suggest a general approach for looking for different connections between restricted permutations and other combinatorial objects. We provide several examples. This direction is related to coding combinatorial objects in terms of (POP-)restricted permutations. In section 2 we are mostly concerned with combinatorial interpretations for the sets of permutations avoiding certain sets of POPs of length at most 4.

Finally, in section 4 we briefly sketch a direction for further research related to combinatorial interpretations for restricted words.

Notation

We use x^k to denote $\underbrace{xx\cdots x}_{k \text{ times}}$.

Throughout the paper we assume that A_n (resp. $A(x)$, $G(x)$) denotes the number (resp. the exponential and ordinary generating functions for the number) of permutations that avoid a pattern or a set of patterns under consideration. Moreover, when we consider the patterns of length 4, it is clear that $A_0 = A_1 = 1$, $A_2 = 2$, and $A_3 = 6$, which we do not state explicitly in most cases.

2 Multi-avoidance of POPs in permutations

In this section we give some relations between multi-avoidance of POPs and other combinatorial objects such as certain walks and labeled general graphs.

2.1 4-SPOPs and walks

In [6] a bijection was given between the set of $(n+1)$-permutations avoiding the SPOP $12'21'$ and the set of walks of n unit steps between lattice points, each in a direction N, S, E or W, starting from the origin and remaining in the positive quadrant. Propositions 2.2 and 2.3 below establish a connection between certain 1-dimensional walks and permutations avoiding the SPOPs $11'22'$ and $22'11'$ simultaneously.

Proposition 2.1. *For the set of patterns* $\{11'22', 22'11'\}$ *and* $n \geq 3$, $A_n = 2\binom{n}{\lfloor n/2 \rfloor}$.

Proof: Suppose a permutation π avoids the above patterns. One can see that whenever $\pi(i) < \pi(i+2)$ (resp. $\pi(i) > \pi(i+2)$) for some i, we must have that $\pi(i+1) > \pi(i+3)$ (resp. $\pi(i+1) < \pi(i+3)$). Thus, either the entries in even (resp. odd) positions of π are in increasing (resp. decreasing) order, or vice versa. We choose the letters of π in odd positions in $\binom{n}{\lfloor n/2 \rfloor}$ ways, then rearrange the odd and even positions in two ways.

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Using a result by Perrin [11] and proposition 2.1, we get the following proposition which we prove by finding an explicit bijection.

Proposition 2.2. *For $n \geq 3$, there is a bijection between the set of all n-permutations avoiding simultaneously the patterns $11'22'$ and $22'11'$, and the set of all $(n+1)$-step walks on the x-axis with the steps $a = (1,0)$ and $\bar{a} = (-1,0)$ starting from the origin but not returning to it.*

Proof: Using the symmetry of the class of walks with respect to the origin, and the structure of permutations avoiding the patterns (see the proof of proposition 2.1), it is sufficient to find a bijection between the set of permutations

$$\{\pi \mid \pi \in S_n(11'22', 22'11'), \pi(1) < \pi(3)\}$$

and the set of all the walks on the x-axis of length n starting at $X = (1, 0)$, with the first step $a = (1, 0)$ which never go to the left of X.

So, we know that the permutations under consideration have the letters in odd positions in increasing order, whereas the letters in even position in decreasing order. As the first step of our bijective proof, given a permutation π, we may reverse the even position letters to get π'. Thus π' is a permutation obtained by shuffling two increasing sequences. Moreover, clearly any walk under consideration can be coded by a word $w = w_1w_2 \cdots w_n$ over the alphabet $\{a, \bar{a}\}$ with the property that for any i, $1 \le i \le n$, the number of a's in $w_1w_2 \cdots w_i$ is not less than the number of $\bar{a}$'s there. Considering π''s and the walks, we find ourselves under conditions of the auxiliary problem in the proof of a bijective result [6, Prop. 9] related to walks in the positive quadrant and pattern avoidance. So we can use the *jumping procedure* to get the desired bijection. We follow [6, Prop. 9] in describing this procedure. However, we refer to the proof of [6, Prop. 9] for all required justifications, as well as for description of the reverse to that procedure.

Suppose we are given a walk $w = aa\bar{a}a\bar{a}a$. The i-th letter in w will correspond to the letter i in the n-permutation π' corresponding to w. We start with the barred permutation $12\bar{3}4\bar{5}6$. Now, the barred letters will move to the left jumping over the neighbor with no bar; if more than one of consecutive letters are barred, these barred consecutive letters form a group that will jump to the left over the closest neighbor with no bar. After a jump, we erase a bar from the largest letter in the group of letters that have jumped together to get the first approximation of π': $12\bar{3}546$. In general, only the rightmost group of barred letters jump in a time. Under this procedure a merging of two barred groups is possible. We are supposed to proceed until all the letters have no bar. In our example one more jump is required, and we get $\pi' = 132546$. The desired permutation is easily obtained from π' by reversing the sequence in even positions: $\pi = 162543$.

We provide an example here of implementing the jumping procedure in both directions:

$$\pi' = 31527486 \leftrightarrow 1\bar{3}527486 \leftrightarrow 12\bar{3}\bar{5}7486 \leftrightarrow 12\bar{3}4\bar{5}\bar{7}86 \leftrightarrow 12\bar{3}4\bar{5}6\bar{7}\bar{8} \leftrightarrow aa\bar{a}a\bar{a}a\bar{a}\bar{a}$$

Finally, we note that in order to apply our bijection to the walks starting in the negative direction, we may first switch a's and $\bar{a}$'s, and then, using the jumping procedure, obtain π' which consists of two shuffled increasing sequences, and finally reverse the letters of π' in odd positions to get the desired π. The converse of this operation is clear.

Alternatively, we can use Catalan factorization to implement the same bijection as that given by the jumping procedure. We map a sequence of a's and $\bar{a}$'s as described above onto an increasing string of positions $12 \ldots n$ where the positions of $\bar{a}$'s are barred and positions of a are unbarred. At each position i, we count the number $m(i)$ of unbarred positions and the number $\bar{m} = i - m(i)$ of barred positions up to each position $i = m(i) + \bar{m}(i)$ and, for each $k \ge 1$, find positions $i_k = \max\{i \mid m(i) - \bar{m}(i) = k - 1\} + 1$. Then i_k is the leftmost position such that $m(j) - \bar{m}(j) \ge k$ for any $j \ge i_k$. Then the word on each segment on positions $i_k + 1$ through $i_{k+1} - 1$ corresponds to a Dyck word on $\{a, \bar{a}\}$ with equal number of barred and unbarred positions. Now we permute the entries of our original permutation so that each resulting Dyck segment on $2i$ letters is mapped onto the segment $(\bar{a}a)^i$ and both barred and unbarred positions increase left to right, while leaving the letters i_k fixed.

$$aa\bar{a}aaa\bar{a}\bar{a}a = 12\bar{3}456\bar{7}\bar{8}9 = \mathbf{1}2\bar{3}\mathbf{4}56\bar{7}\bar{8}\mathbf{9} \mapsto \mathbf{1}32\mathbf{4}75869 \mathbf{}\!\!\!\! = 132475869 = \pi'$$

Again, the converse map is not difficult to find. Values $\pi(i)$ of descents i of π' correspond exactly to barred positions.

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Remark 2.1. Note that, alternatively, we can use the steps $a = (1, 1)$ and $\bar{a} = (1, -1)$ in two dimensions to obtain prefixes of either positive or negative elevated Dyck paths.

The statement of proposition 2.2 can be slightly generalized.

Proposition 2.3. *Let $k \ge 0$ be an integer. For $n \ge 3$, there is a bijection between the set of all n-permutations avoiding simultaneously the patterns $11'22'$ and $22'11'$, and the set of all*

$(n+k)$-step walks on the x-axis with the steps $a=(1,0)$ and $\bar{a}=(-1,0)$ starting from the origin but not returning closer than k units to it. That is, for $k\geq 1$, the points $(0,0)$, $(\pm 1,0)$, $(\pm 2,0),\ldots,(\pm(k-1),0)$ may be visited only once by a legal walk, whereas for $k=0$, a legal walk is not allowed to cross the origin, it is only allowed to touch it.

2.2 4-SPOPs and semi-alternating permutations

In this section we consider the set of permutations avoiding the pair of patterns $(121'2',212'1')$. Since this set of patterns is closed under reversal and complementation, the set $S_n(121'2',212'1')$ is also closed under reversal and complementation.

The permutations $\pi\in S_n(121'2',212'1')$ are characterized as follows: $\pi\in S_n(121'2',212'1')$ if and only if for each $t\geq 1$, $\pi(t)>\pi(t+1)$ exactly when $\pi(t+2)<\pi(t+3)$ and $\pi(t)<\pi(t+1)$ exactly when $\pi(t+2)>\pi(t+3)$. We call such π *semi-alternating*.

Given a permutation in S_n we can map it onto its *updown word* of length $n-1$ which has letter u (resp. d) at position i if $\pi(i)<\pi(i+1)$ (resp. $\pi(i)>\pi(i+1)$). It is easy to see that exactly half the permutations in $S_n(121'2',212'1')$ begins with an ascent, so we consider the updown words of $\pi\in S_n(121'2',212'1')$ that start with u. Those fall into 8 different classes: $(uudd)^*$, $(uudd)^*u$, $(uudd)^*uu$, $(uudd)^*uud$, $(uddu)^*$, $(uddu)^*u$, $(uddu)^*ud$, $(uddu)^*udd$. It is also easy to see that the classes $(uudd)^*u$ and $(uddu)^*u$ are equinumerous, as are $(uudd)^*uud$ and $(uddu)^*udd$. Note that the empty string and the string u are contained in two classes. We will take that into account later.

Note that the letter n corresponds to the initial d or the final u or some segment ud, while the letter 1 corresponds to the initial u or the final d or some segment du.

We now define exponential generating functions for the numbers of permutations $\pi\in S_n(121'2',212'1')$ with a given updown word, in the independent variable t. We will also give the values of of each function and its first three derivatives at $t=0$.

updown word	e.g.f.	initial values at $t=0$ of e.g.f. and first three derivatives
$(uudd)^*uu$	$u(t)$	$0,0,0,1$
$(uudd)^*$	$v(t)$	$0,1,0,0$
$(uudd)^*uud$, $(uddu)^*udd$	$w(t)$	$1,0,0,0$
$(uudd)^*u$, $(uddu)^*u$	$x(t)$	$0,0,1,0$
$(uddu)^*$	$y(t)$	$0,1,0,0$
$(uddu)^*ud$	$z(t)$	$0,0,0,2$

We can now split each π into $\pi_1 n\pi_2$ or $\pi'1\pi''$, where each of π_1,π_2,π',π'' avoids $121'2'$ and $212'1'$. This results in the following system of quadratic differential equations:

$$\begin{cases} u' &= wx \\ v' &= w^2 = x^2+1 \\ w' &= xy \\ x' &= wy = v+xz \\ y' &= w+yz \\ z' &= y^2 = 2x+z^2 \end{cases}$$

Using the initial conditions, we can express each of the functions as a function of y, as follows. Let $Y(t)=\int_0^t y(s)ds$, then $u=\frac{1}{2}\int_0^t \sinh(2Y(s))ds$, $v=\frac{1}{2}t+\frac{1}{2}\int_0^t\cosh(2Y(s))ds$, $w=\cosh(Y)$, $x=\sinh(Y)$, $z=\int_0^t y(s)^2ds$. However, we were unable to solve the above system explicitly. The exponential generating function for $|S_n(121'2',212'1')|$ is given by

$$2(u+v+2w+2x+y+z)-3t-1=4e^Y+\int_0^t e^{2Y(s)}ds+2y+2\int_0^t y(s)^2ds-2t-1.$$

2.3 POPs and general graphs

A *general graph* (*pseudograph*) is a graph in which both graph loops and multiple edges are permitted. A *homeomorphically irreducible* general graph is a graph with multiple edges and loops and without nodes of degree two (see [12, A060580]).

It is easy to see that the number of homeomorphically irreducible general graphs on two labeled nodes (2-HIGG) with $n \geq 3$ edges is $\binom{n+2}{2} - 4 = \frac{n^2+3n-6}{2}$. Indeed, suppose L and R are the nodes of a 2-HIGG with n edges. Then the number of such graphs is the number of integer non-negative solutions to $x_L + x_R + x_{LR} = n$, where x_L and x_R are the number of loops of L and R respectively, and x_{LR} is the number of edges between L and R; we must also subtract the number of "bad" graphs. There are 4 such graphs: two graphs with $x_{LR} = 2$ and one of $x_L = 0$ or $x_R = 0$, and two graphs with $x_{LR} = 0$ and one of $x_L = 1$ or $x_R = 1$. We represent a 2-HIGG by the triple (x_L, x_{LR}, x_R).

Proposition 2.4. *For $n \geq 2$, and the SPOPs $p_1 = 121'2'$, $p_2 = 211'2'$, $p_3 = 121'3$, we have*

$$A_n = \frac{n^2 + 3n - 6}{2}.$$

Proof: The result is clearly true for $n = 2, 3$. Suppose $n \geq 4$ and a permutation $\pi = \pi_1\pi_2 \cdots \pi_n$ avoids the patterns.

If $\pi_i = n$ then $i \leq 3$, since otherwise p_1 or p_2 occur in π. We consider three cases:

1) If $\pi_1 = n$ then either $\pi_2 < \pi_3 > \pi_4 > \cdots > \pi_n$ or $\pi_2 > \pi_3 > \cdots > \pi_n$, otherwise p_1 or p_2 occurs in π. So, we have $(n-1)$ permutations in this case.

2) If $\pi_2 = n$ then to avoid p_1 and p_2 we must have $\pi_3 > \pi_4 > \cdots > \pi_n$. So, this case gives us $(n-1)$ permutations too.

3) If $\pi_3 = n$ then, as above, we must have $\pi_4 > \pi_5 > \cdots > \pi_n$. If $\pi_1 > \pi_2$, there are no additional restrictions and we get $\binom{n-1}{2}$ good permutations in this case. If $\pi_1 < \pi_2$ then in order to avoid p_3 we must have $\pi_2 = n - 1$. So, we get $(n-2)$ permutations in this case.

Summing over the cases, we get the desired result.

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Using proposition 2.4 and the considerations preceding it, we get the truth of the following proposition which we prove combinatorially.

Proposition 2.5. *For $n \geq 3$, there is a bijection between the set $S_n(p_1, p_2, p_3)$ of n-permutations avoiding the SPOPs $p_1 = 121'2'$, $p_2 = 211'2'$, and $p_3 = 121'3$ and the set of homeomorphically irreducible general graphs on two labeled nodes with n edges.*

Proof: Suppose $n \geq 4$ (if $n = 3$, the bijection described below does not work; however, in this case there are only six objects in each set, and this is straightforward to find a bijection). We describe a map Ψ from $S_n(p_1, p_2, p_3)$ to the set of graphs which will be easy to see to be a bijection. See considerations before proposition 2.4 to recall that we denote our graphs by the triples (x_L, x_{LR}, x_R).

According to the proof of proposition 2.4, we can subdivide $S_n(p_1, p_2, p_3)$ into three subsets W_1, W_2, and W_3 consisting of permutations having $\pi_1 = n$, or $\pi_2 = n$, or $\pi_3 = n$ respectively.

If $\pi \in W_1$, and $\pi = ni\pi_3 \cdots \pi_n$, we consider two subcases. If $i = 1$ then corresponding $x_L = 0$, $x_{LR} = 1$, and $x_R = n - 1$; if $i > 1$ then $x_L = 0$, $x_{LR} = i + 1$, and $x_R = n - i - 1$. So, the permutations from W_1 are mapped to good graphs in which node L has no loops.

If $\pi \in W_2$, and $\pi = in\pi_3 \cdots \pi_n$, we consider three subcases. If $i = 1$ then $x_L = n - 1$, $x_{LR} = 1$, and $x_R = 0$; if $n - 1 > i > 1$ then $x_L = 0$, $x_{LR} = i + 1$, and $x_R = n - i - 1$ (note, that we do not count second time the triple $(0, n, 0)$); if $i = n - 1$, we map the permutation to the triple $(n, 0, 0)$. So, the permutations from W_2 are mapped to the good graphs in which node R has no loops.

If $\pi \in W_3$ and $\pi = \pi(i, j) = ijn\pi_4 \cdots \pi_n$ then $\Psi(\pi(1, n-1)) = (0, 0, n)$; $\Psi(\pi(i, n-1)) = (i, 0, n-i)$ (nodes L and R get at least two loops). The remaining case is $i > j$. In this case $\Psi(\pi(i, j)) = (n - i, i - j, j)$ (note that each of the components of the last vector is greater than

0, which distinguishes this case from the preceding two; also, the degrees of L and R are greater than 2, and thus we get a legal graph).

Finding the inverse of Ψ is straightforward.

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Using similar bijections we can get a bunch of combinatorial results connecting graphs and restricted permutations. For instance, if $p_3 = 123$ in proposition 2.5, then the structure of permutations avoiding $p_1 = 121'2'$, $p_2 = 211'2'$, and $p_3 = 123$ is almost the same as that in proposition 2.5, except for if $\pi_3 = n$ then it must be $\pi_1 > \pi_2$. Thus we get the following result right away.

Proposition 2.6. *For $n \geq 3$, there is a bijection between the set of n-permutations simultaneously avoiding the SPOPs $p_1 = 121'2'$, $p_2 = 211'2'$, and $p_3 = 123$ and the set of 2-HIGG with n edges such that either $x_{LR} > 0$ or $x_{LR} = 0$ and $x_L = n$.*

Imposing two extra restrictions, namely, $p_4 = [21'1$ and $p_5 = [1'21$ (here we use Babson-Steingrímsson notation, where "[" in $p = [xy\cdots$ means that an occurrence of p in a permutation must begin from the leftmost letter of the permutation) we see that the sets W_1 and W_2 in the proof of proposition 2.5 become prohibited; also, as above, if $\pi_3 = n$ then $\pi_1 > \pi_2$. Thus we obtain the following proposition.

Proposition 2.7. *For $n \geq 0$, there is a bijection between the set of $(n+3)$-permutations simultaneously avoiding the SPOPs $p_1 = 121'2'$, $p_2 = 211'2'$, $p_3 = 123$, $p_4 = [21'1$ and $p_5 = [1'21$ and the set of arbitrary general graphs on two labeled nodes and n edges.*

Let $A_{n,k}$ be the $k \times k$ adjacency matrix of a graph on k labeled nodes and n edges (multiple edges and loops are allowed). We assign labels $1, 2, \ldots, k^2$ to the entries of $A_{n,k}$ by reading the matrix from left to right and from top to bottom.

We define a class $\mathcal{C}_{n,k}(a_1, a_2, \ldots, a_\ell)$ of graphs by indicating $k^2 - \ell$ entries of $A_{n,k}$ that must be 0 (the other entries, having labels a_1, $a_2, \ldots, a_\ell$, may or may not be 0).

Definition 2.1. *Suppose $p = a_1 a_2 \cdots a_k$ is a permutation and, for fixed non-negative integers $\ell_1, \ell_2, \ldots, \ell_{k-1}$, the letters $b^{(i,j)}$, $1 \leq i \leq k-1$, $1 \leq j = j(i) \leq \ell_i$, are incomparable with each other and with the a_i's, $1 \leq i \leq k$. We call the SPOP*

$$a_1 b^{(1,1)} b^{(1,2)} \cdots b^{(1,\ell_1)} a_2 b^{(2,1)} b^{(2,2)} \cdots b^{(2,\ell_2)} a_3 \cdots a_{k-1} b^{(k-1,1)} b^{(k-1,2)} \cdots b^{(k-1,\ell_{k-1})} a_k$$

separated segmented POP (SSPOP). *For the SSPOP above we use the notation*

$$\tau_k(\ell_1, \ell_2, \ldots, \ell_{k-1}) = a_1|_{\ell_1} a_2|_{\ell_2} a_3 \cdots a_{k-1}|_{\ell_{k-1}} a_k.$$

We use "$|$" instead of "$|_1$".

SSPOPs were introduced in [6]. These patterns allow us to control the distance between certain letters in permutations and we use this property in Theorem 2.2. If we write, say, $p = [|_t xy$, then we mean that an occurrence of the pattern p in a permutation must start with the leftmost letter of the permutation, and the first t letters of the permutation can be arbitrary, while the relative order of x and y must be preserved.

In what follows we allow a SSPOP to contain dashes. We call such patterns *separated partially ordered patterns* and we abbreviate them S-POPs (to distinguish from SPOPs).

Let P' be a set of SPOPs (or, rather, SSPOPs) $\{[|_i 2|_{\ell-i-2} 1\}_{0 \leq i \leq \ell-2}$. We define P to be $P' \cup \{p_1 = |_{\ell-1} 12,\ p_2 = 12\text{-}3\}$.

We now state the main result in this subsection.

Theorem 2.2. *Let $\mathcal{C}_{n,k}(a_1, a_2, \ldots, a_\ell)$ and P be as defined above. There is a bijection between the set of graphs $\mathcal{C}_{n,k}(a_1, a_2, \ldots, a_\ell)$ and the set of $(n+\ell)$-permutations avoiding simultaneously the patterns from P.*

Proof: Suppose that $(n+\ell)$-permutation π avoids all the patterns from P. To avoid all the patterns from $P' \cup \{p_1\}$, we must have $\pi_\ell = n+\ell$. Then avoidance of p_1 and p_2 forces $\pi_1 > \pi_2 > \cdots > \pi_{\ell-1}$ and $\pi_{\ell+1} > \pi_{\ell+2} > \cdots > \pi_{n+\ell}$. Thus the number of good permutations is $\binom{n+\ell-1}{\ell-1}$, which is exactly the number of graphs in $\mathcal{C}_{n,k}(a_1, a_2, \ldots, a_\ell)$ (the number of non-negative integer solutions to $a_1 + a_2 + \cdots + a_\ell = n$).

Suppose $\pi = \pi(i_1, i_2, \ldots, i_{\ell-1}) = i_1 i_2 \cdots i_{\ell-1}(n+\ell)\pi_{\ell+1} \cdots \pi_{n+\ell}$, where $i_1 > i_2 > \cdots > i_{\ell-1}$. Clearly, we can represent the graphs from $\mathcal{C}_{n,k}(a_1, a_2, \ldots, a_\ell)$ by ℓ-tuples where the i-th entry will correspond to the number of edges associated with a_i. The desired bijection now is the map Ψ such that

$$\Psi(\pi(i_1, i_2, \ldots, i_{\ell-1})) = (i_{\ell-1} - 1, i_{\ell-2} - i_{\ell-1} - 1, \ldots, i_1 - i_2 - 1, n + \ell - i_1 - 1).$$

The reverse to Ψ is easy to see.

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As a corollary to Theorem 2.2, we have the following statement.

Corollary 2.1. *There is a bijection between the set of all graphs on k nodes with n edges and the set of $(n+k^2)$-permutations that avoid simultaneously all the patterns from the set* $\{[|_i 2|_{k^2-i-2} 1\}_{0 \le i \le k^2-2} \cup \{|_{k^2-1} 12,\ 12\text{-}3\}$.

3 A general approach for looking for connections between restricted permutations and other combinatorial objects

3.1 A general approach

A standard approach to find relations between restricted permutations and other combinatorial objects seems to be as follows. First one considers a particular set of patterns P; then one either finds a formula for, say, a number of permutations avoiding P or makes a (computer) experiment to find initial values of the number of permutations avoiding P; thereafter one may check if the numbers appear in [12], which might establish relations to other combinatorial objects. Of course, we may start from other combinatorial objects, and then find out using [12] if there is a relation to restricted permutations. The probability of getting a positive result with the last approach is very low, though.

We suggest to start from consideration of a structure of permutations that are supposed to avoid some set of patterns. The idea is to consider those structures that can be "controlled," that is, for which we can find a set of patterns that force our permutations to have the prescribed structure. Moreover, in order to increase the probability of obtaining relations to other objects, we may try to make the number of permutations having our structure be of a "nice" form, like expressions containing binomial coefficients, or, say, powers of 2, etc. As the next step we can either go to [12], or we may recognize the formula involved to count objects known to us. Then we try to find a bijection between our object and the other one.

Creating the structures described by prohibited patterns is the place when POPs are of great importance, since if we use just GPs, we would be forced to deal with huge sets of patterns, which are difficult to control. Of course, although sets of POPs are convenient abbreviation of sets of GPs, they still can be large in many cases.

Below, we list some of the tricks that may be helpful when creating structures (one also meets other tricks in our considerations below). We assume that we consider n-permutations. We also give the notation for particular sets of patterns, some of which we will be using below. Those sets are not unique, but our intention is to find minimal sets of forbidden patterns that give rise to a given structure.

(1) To make n occur in position i and not to create extra restrictions, consider the set of prohibited patterns $\mathcal{P}_1 = \{[|_k 2|_{i-k-2} 1\}_{0 \le k \le i-2} \cup \{[|_{i-2} 1\text{-}2\}$;

(2) To make n occur in position i and to have all the letters to the right of n in decreasing order (this gives the binomial coefficient for the number of ways to choose the first $i-1$ letters, namely $\binom{n-1}{i-1}$), consider the set $\mathcal{P}_2 = \{[|_k 2|_{i-k-2}1\}_{0\leq k\leq i-2} \cup \{|_{i-2}12\}$;

(3) To make the letter in position $j+1$ be the largest one among the first i letters of a permutation ($j < i$), use $\mathcal{P}_3 = \{[|_k 2|_{j-k-1}1\}_{0\leq k\leq i-1} \cup \{[|_j 1|_k 2\}_{0\leq k\leq i-j-2}$;

(4) To make ℓ consecutive letters of a permutation be decreasing starting from position t, consider the set $\mathcal{P}_4 = \{[|_{k+t-1}1|_{\ell-k-2}2\}_{0\leq k\leq \ell-2}$;

(5) To make any of the first i letters of an n-permutation be greater than any letter in positions $i+1$, $i+2, \ldots, n$, use $\mathcal{P}_5 = \{[|_k 1|_{i-k-1}2\}_{0\leq k\leq i-1}$.

(6) Suppose a set of patterns $\mathcal{P}$ gives $f(n)$ n-permutations avoiding it. We can obtain a set of patterns $\mathcal{P}^*$ that yields $kf(n)$ $(n+k)$-permutations avoiding it, where k is a constant. Here we use a result of [4] that the number of k-permutations simultaneously avoiding the patterns 123, 132, and 231 is k. To obtain $\mathcal{P}^*$ we can use (5) above with $i = k$ to make the first k letters of a $(n+k)$-permutation π be the largest letters in π. Then we add the prefix $[|_k$ to each pattern from $\mathcal{P}$ to make sure that for any fixed permutation of the first k letters, the letters in positions $k+1, k+2, \ldots, n+k$ give us exactly $f(n)$ different permutations. Finally, instead of each of the three patterns of the form xyz above, we add the set of prohibited patterns $\{[|_i xyz|_{k-i}]\}_{0\leq i\leq k}$ to get exactly k permutations for any fixed permutation in the positions $k+1, k+2, \ldots, n+k$.

We close this subsection by pointing out that the ideas described above suggest a way of *coding* combinatorial objects in terms of restricted permutations. This is a rather natural problem, taking into account that coding objects by words is a very useful approach under many different contexts.

3.2 Cartesian products of combinatorial objects and restricted permutations

In the following subsection we give other examples of applying our approach.

Let $\mathcal{O}_n$ be the set of all (natural) combinatorial objects of size associated with n and having interpretation in terms of restricted permutations. For instance, Motzkin paths $\mathcal{M}_n$ of length n, as well as Dyck paths $\mathcal{D}_n$ of length $2n$ are elements of $\mathcal{O}_n$ since the former is related to the n-permutations avoiding the GPs 1-23 and 13-2, whereas the latter is related to n-permutations avoiding the GP 2-13 (see [3]). There are many other combinatorial objects including different types of labeled trees appearing in $\mathcal{O}_n$. Let $\mathcal{O} = \cup_{n\geq 0}\mathcal{O}_n$.

Theorem 3.1. *Suppose $A_{n_i} \in \mathcal{O}_{n_i}$ for $i = 1, 2, \ldots, k$, and $n = \sum_i n_i$. Then*

$$A_{n_1} \times A_{n_2} \times \cdots \times A_{n_k} \in \mathcal{O}_n \subset \mathcal{O}.$$

In other words, the Cartesian product of combinatorial objects related to restricted permutations has an interpretation in terms of restricted permutations.

Proof: Suppose $n_0 = 0$. To create a structure for n-permutations related to the Cartesian product we first use the set of prohibited patterns

$$\cup_{0\leq j\leq (k-2)}\{[|_i 1|_{n_{j+1}-i-1}\text{-}2\}_{n_1+n_2+\cdots+n_j\leq i\leq n_{j+1}-1}$$

to subdivide each of our permutations into k parts of lengths n_1, $n_2, \ldots, n_k$ when reading the permutations from left to right, such that any letter of the s-th part is greater than any letter of the t-th part whenever $s < t$.

The fact that $A_{n_i} \in \mathcal{O}_{n_i}$ guarantees the existence of patterns (POPs) $p_{i,1}$, $p_{i,2}, \ldots$ such that the number of objects in A_{n_i} equals the number of n_i-permutations avoiding simultaneously all of the patterns from $P_i = \{p_{i,j}\}_{j=1,2,\ldots}$. We now can rewrite the patterns from P_i, creating in general a larger set of patterns (see the procedure below), to make them affect only the i-th part in the subdivision of our permutations described above. Treating all A_{n_i}'s this way

($i=1,2,\dots,k$), and joining all obtained sets of patterns into one set, we get a set of patterns P with the property that the number of n-permutations avoiding P is equal to the cardinality of the set $A_{n_1}\times A_{n_2}\times\cdots\times A_{n_k}$, and one can find an explicit bijection between those combinatorial objects using bijections corresponding to the k parts of our n-permutation subdivision.

Finally, we need to explain the rewriting process for P_i. Recall that if no opening bracket "[" (resp. closing bracket "]") occurs at the beginning (resp. end) of a pattern, the pattern is understood to have a dash at the beginning (resp. end). Suppose a pattern $p_i\in P_i$ has $\ell\geq 0$ dashes and if we omit the dashes and brackets, we get a word $w=a_1a_2\cdots a_h$ over a certain alphabet (where some of the letters of w may be incomparable). We keep the brackets if any, but instead of the j-th dash, if there is one, we put $|_{d_j}$. Suppose doing this way we get a pattern $\tau=\tau(d_1,d_2,\dots,d_\ell)$. Now, to the pattern p_i there corresponds the set

$$\{[|_{n_1+n_2+\cdots+n_{i-1}}\tau(d_1,d_2,\dots,d_\ell)\}_{d_1+d_2+\cdots+d_\ell=n_i-h},$$

which is a subset of the set of the patterns we are constructing. Note that if τ starts with a bracket, we just remove it; otherwise τ starts with $|_{d_1}$ and we can merge it with $|_{n_1+n_2+\cdots+n_{i-1}}$ to make the pattern start with $[|_{n_1+n_2+\cdots+n_{i-1}+d_1}a_1\cdots$; and if a bracket occurs at the end of p_i, we simply remove it.

For example, if $n_1+n_2+\cdots+n_{i-1}=10$, $n_i=5$ and $P_i=\{12],[13\text{-}2,21'\text{-}1\}$ then we rewrite the set P_i to be the set

$$\{[|_{13}12\}\cup\{[|_{10}13|_{d_1}2|_{d_2}\}_{d_1+d_2=2}\cup\{[|_{10+d_1}21'|_{d_2}1|_{d_3}\}_{d_1+d_2+d_3=2},$$

where the indicated subsets correspond to our three patterns.

■

The following corollary to theorem 3.1 gives a lower bound for the cardinality of the set of different combinatorial objects related to restricted permutations.

Corollary 3.1. *The cardinality of $\mathcal{O}$ is at least continuum.*

Proof: Since $\mathcal{O}=\cup_{n\geq 0}\mathcal{O}_n$, and $\mathcal{M}_n$ and $\mathcal{D}_n$ are from $\mathcal{O}$, we have according to theorem 3.1 that the Cartesian product involving an arbitrary number of terms each of which is either $\mathcal{M}_n$ or $\mathcal{D}_n$ gives an object belonging to $\mathcal{O}$. So the cardinality of $\mathcal{O}$ is at least as large as the cardinality of all binary strings, which is continuum.

■

We end up this subsection with listing some of known objects (but not all of them) related to restricted permutations. One can use these relations as building blocks in Cartesian products to discover new relations using theorem 3.1.

Sets of patterns to avoid	Related objects in case of n-permutations
none	Increasing binary trees on n vertices
1-2-3 (or 1-3-2)	Dyck paths of length $2n$
1-23	Partitions of $[n]$
1-23, 12-3	Non-overlapping partitions of $[n]$
1-23, 1-32	Involutions in S_n
1-23, 13-2	Motzkin paths of length n
132, [21	Increasing rooted trimmed trees with $n+1$ nodes
$12'21'$	Lattice walks of $n-1$ steps in N, S, E, W
$11'22'$, $22'11'$	Certain walks of $(n+1)$-steps (proposition 2.2)
$121'2'$, $212'1'$	Semi-alternating permutations (subsection 2.2)

3.3 Other examples of applying the approach

Let $\mathcal{S}_1(i,j,n+1)$ denote the set of $(n+1)$-permutations having the structure $AxB(n+1)C$, where A, B, and C are decreasing, x is the largest letter in AxB, $|AxB| = i$, and $|A| = j$. Clearly, $|\mathcal{S}_1(i,j,n+1)| = \binom{n}{i}\binom{i-1}{j}$. Moreover, there is a set of patterns so that the set of $(n+1)$-permutations avoiding it is exactly $\mathcal{S}_1(i,j)$ – we can take, for example, the union of sets of patterns given by (2)–(4) in subsection 3.1 with n replaced by $(n+1)$ and choosing appropriate values of ℓ and t in (4).

The set $\mathcal{S}_2(k,n)$ is obtained by prohibiting the patterns $|_k$-132 and $|_k$-231 in n-permutations. It is known [4] that the number of n-permutations avoiding simultaneously the patterns 132 and 231 is given by 2^{n-1}, and thus $|\mathcal{S}_2(k,n)| = \binom{n}{k}2^{n-k-1}$.

We define the set $\mathcal{S}_3(i,j,n+2)$ to be the set of $(n+2)$-permutations of the form $A1B(n+2)C$ where $|A| = i$, $|B| = j$, A and B are decreasing, C avoids 132 and 231. The number of such permutations (using the result the of previous paragraph) is $\binom{n}{i}\binom{n-i}{j}2^{n-i-j}$. Also, this is clear how to describe the set of prohibited patterns to create the structure using considerations above. We only point out that, to fix 1 in position $(i+1)$, one can use the same method as when fixing the largest element in a certain position, the only difference being that we switch 1 and 2 in the patterns.

Let $\mathcal{S}_4(k,n+k)$ be the set of $(n+k)$-permutations having the structure AB, where $|A| = k$, any letter in A is greater than any letter in B, A avoids 123, 132, and 231, and B avoids 1-2-3 and 2-3-1. Clearly, $\mathcal{S}_4(k,n+k)$ is just a particular case of the situation described in (6) in previous paragraph. Thus $|\mathcal{S}_4(k,n+k)| = k(\binom{n}{2}+1)$ since the number of n-permutations avoiding 1-2-3 and 2-3-1 is $\binom{n}{2}+1$ (see, e.g., [10]).

The set $\mathcal{S}_5(k,n+k)$ is essentially the same as $\mathcal{S}_4(k,n+k)$ with the only exception that B avoids different patterns, namely 1-2-3, 1-3-2, and 2-1-3. Since the number of n-permutations avoiding these three patterns is F_n (see, e.g., [10]), where F_n is the n-th Fibonacci number with $F_0 = 1$ and $F_1 = 1$, we have $|\mathcal{S}_5(k,n+k)| = kF_n$.

In Table 1 we provide some of the connections related to the structures described above for certain choices of i, j, and k. Note, that we meet non-trivial connections between different classes of restricted permutations. However, the connections from Table 1 beg direct bijections to indicate the actual coding of the objects in terms of restricted permutations, which we leave as open problems.

Closing this subsection, we indicate that restricted permutations might be used to give combinatorial proofs of certain identities. For example, let us prove that

$$3\binom{n}{4} = \binom{\binom{n-1}{2}}{2}$$

using $\mathcal{S}_1(4,1,n+1)$. As it mentioned above, the left-hand side gives the number of permutations of the form $AxB(n+1)C$, where $|A| = 1$, $|B| = 2$, x is the largest letter in AxB, and A, B, C are decreasing. On the other hand, a permutation having the same structure can be obtained as follows. Choose two different pairs of numbers from $[n-1] = \{1,2,\dots,n-1\}$. If the pairs do not have a common element, then without loss of generality the pairs are $\{a,b\}$, $\{c,d\}$, and $a < b$, $d = \max\{a,b,c,d\}$. Then $AxB = cdba$. Otherwise, the pairs have exactly one common element. Thus, we have $\{a,b\}$, $\{b,c\}$, and without loss the generality $a < c$. Then $x = n$ and $AxB = bnca$.

4 POPs in words and other combinatorial structures

In case of words, similar ideas can be used as those in section 3 when we consider permutations. In particular, an analogue of theorem 3.1 can be proved. We record it as theorem 4.1, where $\Omega = \cup_{n\geq 0}\Omega_n$ and Ω_n is the set of all objects (of size depending on n) related to restricted words of length n over some alphabet having more than one letter. We want to point out that when proving theorem 4.1 in the way proposed in the proof of theorem 3.1, we skip the step that follows after subdivision of a word of length n into parts. The relative order of the elements from two different parts is irrelevant.

Set of permutations	Formula	Sequence in [12]	Combinatorial interpretation
$\mathcal{S}_1(2,0,n+1)$	$\binom{n}{2}$	A000292	$(n+3)$-permutations avoiding 1-3-2-4 and having exactly one descent (a descent in a permutation π is an i such that $\pi(i) > \pi(i+1)$).
$\mathcal{S}_1(3,1,n+1)$	$2\binom{n}{3}$	A007290	Acute triangles made from the vertices of a regular n-polygon.
$\mathcal{S}_1(4,1,n+1)$	$3\binom{n}{4}$	A050534	Lines drawn through the points of intersections of n straight lines in a plane, no two of which are parallel, and no three of which are concurrent.
$\mathcal{S}_2(1,n)$	$n2^{n-2}$	A057711	Number of 1's in all palindromic compositions of $N = 2(n-1)$. E.g., there are 5 palindromic compositions of 6, namely 111111, 11211, 2112, 1221, and 141, containing a total of 16 1's.
$\mathcal{S}_3(1,1,n+2)$	$\binom{n}{2}2^{n-1}$	A001815	$(n+3)$-permutations containing 1-3-2 and 1-2-3 exactly once.
$S_4(2,n+2)$	n^2-n+2	A014206	Binary bitonic sequences of length n (a bitonic sequence is $a_1 \leq a_2 \leq \cdots \leq a_h \geq a_{h+1} \geq \cdots \geq a_{n-1} \geq a_n$ or $a_1 \geq a_2 \geq \cdots \geq a_h \leq a_{h+1} \leq \cdots \leq a_{n-1} \leq a_n$).
$S_5(2,n+2)$	$2F_n$	A006355	Binary vectors of length $n+2$ with no singletons.

Table 1: Examples of relations between restricted permutations and other combinatorial objects

Theorem 4.1. *Suppose* $W_{n_i} \in \Omega_{n_i}$ *for* $i = 1, 2, \ldots, k$, *and* $n = \sum_i n_i$. *Then*

$$W_{n_1} \times W_{n_2} \times \cdots \times W_{n_k} \in \Omega_n \subset \Omega.$$

In other words, the Cartesian product of combinatorial objects related to restricted words has an interpretation in terms of restricted words.

However, this paper is primarily oriented to permutations in the sense of different connections to other structures using our approach; thus, we do not provide here examples related to words and their relations to other combinatorial objects leaving this as a direction for future research.

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