A poset structure for row convex permutominides

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Abstract

A permutominide *is a particular set of unit cells in the plane uniquely defined by pair of permutations. In this paper we study some combinatorial properties of these objects and we define a partial order on them which is isomorphic to the Bruhat order for permutations. This allows us to enumerate the class of row convex permutominides according to their size.*

1 Polyominides

In this paragraph we generalize the classical concept of *polyomino* admitting the connection of cells not only by edges but also by vertices.

In the plane $\mathbb{Z} \times \mathbb{Z}$ a *cell* is a unit square having integer coordinates. Two cells can be connected by one of their edges, see case a) of Fig.1, they can be connected by one of their vertices, see case b), or they can be disconnected, see case c).

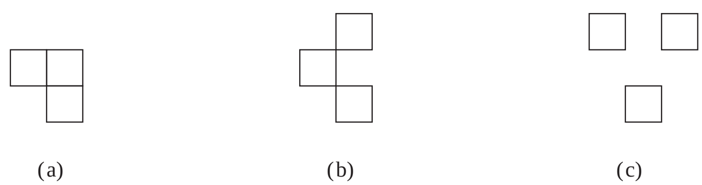


Figure 1: (a) Cells connected by edges; (b) Cells connected by vertices; (c) Disconnected cells.

A *polyominide* is a finite union of cells which are connected by vertices or edges. It is obvious that, requiring the connection only by edges, we obtain classical polyominoes as a particular subclass of this more general one of polyominides.
Polyominides are defined up to translations. We will consider only those having no "holes", i.e. their boundary must have exactly one component. In Fig. 2 we have some examples.

A polyominide C is said to be *row convex* (resp. *column convex*) if, for any two cells with the same ordinate (resp. abscissa), the row (resp. column) containing them is connected, see Fig 2. Finally C is said to be *convex* if is both row and column convex.

Looking at the boundary of a polyominide we define its vertices and sides. A *vertex* of a polyominide is simply a vertex of one of its cells having multiplicity one or three (where the multiplicity of a vertex V is the number of cells which share it as a vertex), see Fig. 3. A *side* of a polyominide C is a segment of its boundary which collects two vertices of C, see Fig. 3.

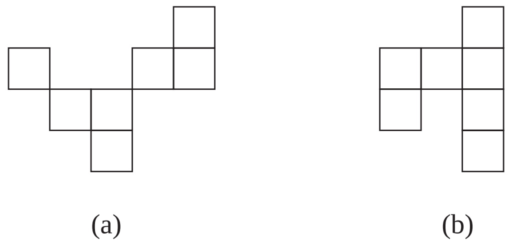


Figure 2: (a) A column convex (but not row convex) polyominide; (b) A column convex polyominide which is also a polyomino.

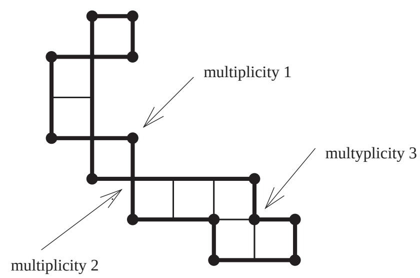


Figure 3: Vertices and sides in a polyominide. The vertices are highlighted.

2 Permutominides

In this paragraph we define a particular class of polyominides, which is a generalization of *permutominoes*.
Permutominoes were first introduced in [11], and then studied by F. Incitti in [10] while studying the problem of determining the $\widetilde{R}$-polynomials associated with a pair (π_1, π_2) of permutations.

A *permutominide* is a polyominide having exactly one side of its boundary for every abscissa and exactly one for every ordinate. It is easy to check that the minimal bounding rectangle containing a permutominide is always a square.
We refer to the convexity of a permutominide like for polyominides.

The name *permutominide* reflects a relation between these objects and permutations. In fact if C is a permutominide, having n rows and columns, $n \geq 1$, we can associate to C a couple of permutations of length $n+1$ called $\pi_1(C)$ and $\pi_2(C)$. To do this we assume, without loss of generality, that the south-west vertex of its minimal bounding rectangle is placed in $(1,1)$ and we consider $\mathcal{A} = \big(A_1, \ldots, A_{2(n+2)}\big)$ the list of its vertexes ordered clockwise starting from the lowest leftmost one. Now we have that $\pi_1(C)$ is represented by $(A_1, A_3, \ldots, A_{2n+1})$ and $\pi_2(C)$ by $(A_2, A_4, \ldots, A_{2n+2})$. The *size* of a permutominide is defined as the length of its associated permutations.

Polyominoes as a particular class of permutominides are simply called *permutominoes* (see Fig. 4). They have been widely studied during these last years; here we recall the main results: in [3] it was proved that the number of *parallelogram permutominoes* of size n is equal to c_{n-1}, where c_n is the n-th *Catalan number*, and that the number of *directed convex permutominoes* of size n is equal to $\frac{1}{2}\, b_{n-1}$, where b_n is the n-th binomial central number. Finally, the number of *convex permutominoes* of size $n+1$ is proved in [2] to be :

$$2\,(n+3)\,4^{n-2} - \frac{n}{2}\binom{2n}{n} \qquad n \geq 1, \tag{1}$$

the first few terms being $1, 4, 18, 84, 394, 1836, 8468, \ldots$ (sequence A126020 in [12]).

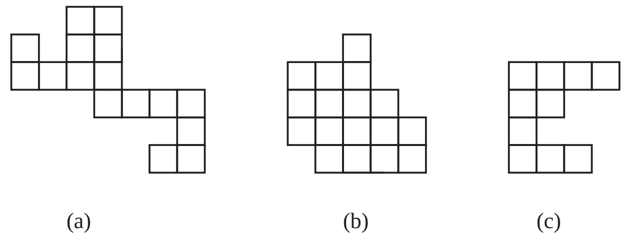


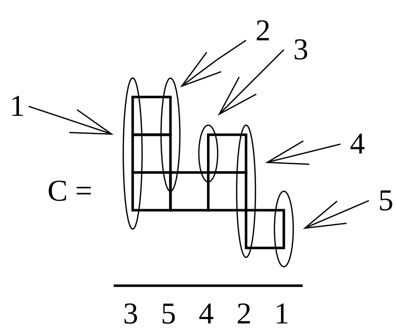
Figure 4: (a) A permutomino; (b) a convex permutomino ; (c) a row convex permutomino.

In this paper we associate a permutation to each permutominide to obtain a definition of a poset structure on the set of row convex permutominides, we will be able to enumerate these objects with respect to the size. Then we will consider also problems concerning the boundary of these objects.

Permutominides and permutations. First of all we give a rule to associate to each permutominide C a permutation $\pi(C)$ of length equal to the size of C.

Let C be a permutominide of size n.
There is an obvious way to identify each vertical side e of C with a number $x(e) \in \{1, \ldots, n\}$: we define $x(e)$ as the abscissa common to all the points of e. From now on we will refer to the vertical sides of a crossing permutomino in terms of this associated numbers.
It is now possible to associate to C a permutation $\pi(C)$ of length n by defining a total order on its vertical sides.
First of all, a couple of integers $(y(e), l(e))$ is associated to each vertical side e of C, where $y(e)$ is the ordinate of the lowest point of e and $l(e)$ is the length of the side e.
It is important to note that in any permutominide there are not two vertical sides $e \neq i$ such that $(y(e), l(e)) = (y(i), l(i))$.
Now, if e and i are two vertical sides of C, we say that $e \leq i$ if and only if $(y(e), l(e)) \leq (y(i), l(i))$ in the lexicographic order. It is clear that this is a total order on the set of the vertical sides of a permutominide and, for example, it is also true that the minimum side has always length one.
Now, for each vertical side e of C, writing in position $x(e)$ of a permutation the number of sides less than e in the defined order above we obtain a permutation of length n indicated by $\pi(C)$.
See the following example.

Example 2.1. *The permutation associated to the permutominide of size* 5 *in the figure is* $\pi(C) = (3, 5, 4, 2, 1)$*: indeed its vertical sides are totally ordered as* $5 < 4 < 1 < 3 < 2$*, as it is shown in the table.*



x(e)	*(y(e), l(e))*
1	*(2,3)*
2	*(3,2)*
3	*(3,1)*
4	*(1,3)*
5	*(1,1)*

It is easy to see that the same permutation can be associated to different permutominides as shown in Fig. 5, so it seems now natural to ask how many permutominides have the same permutation associated with. In the next section we will give the answer for a particular subclass of our objects.

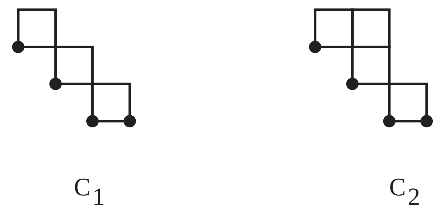


Figure 5: The same permutation $\pi = (4, 3, 2, 1)$ is associated to different permutominides $\pi = \pi(C_1) = \pi(C_2)$.

3 Row convex permutominides

We will focus on the class of row convex permutominides. We will give the answer to the question of how many permutominides of this type are associated to the same permutation by using the following immediate results.
Let us start with a particular type of permutations, we need the next Proposition:

Proposition 3.1. Let C be a permutominide, then C is row convex if and only if it has exactly two vertical sides with the same ordinate for their lowest points.

Now, for a given permutation π, we define the set of permutominides $\mathcal{P}(\pi) = \{P : P \text{ is row convex and } \pi(P) = \pi\}$. From Proposition 3.1 we have the next:

Proposition 3.2. Let $\pi \in S_n$ be the permutation $(n, n-1, \ldots, 2, 1)$. The cardinality of the set $\mathcal{P}(\pi)$ is equal to 2^{n-3}.

Proof: Let $C \in \mathcal{P}(\pi)$ be a generic row convex permutominide with permutation equal to π. Due to proposition 3.1 we have that the lowest points of the vertical sides of C are set as it is depicted in Fig. 6.

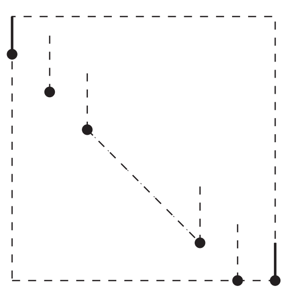


Figure 6: The positions of the lowest points of the vertical sides in a row convex permutominide having $(n, n-1, \ldots, 2, 1)$ as permutation.

In particular we focus our attention on those points having abscissa in $\{2, \ldots, n-2\}$ and we call them *free*. So we have $n-3$ free points.

To form the boundary of C these points can be collected between them and to the others lowest points of Fig. 6 in two ways: fix the points of the boundary of C, see Fig. 7, with coordinates $A = (1, n)$ and $B = (n-1, 2)$, then the free points can be part or not of the path which goes from A to B when the boundary of C is oriented clockwise. So we have 2^{n-3} possibilities to obtain the generic permutominide C, see Fig. 7 for the case $n = 5$.

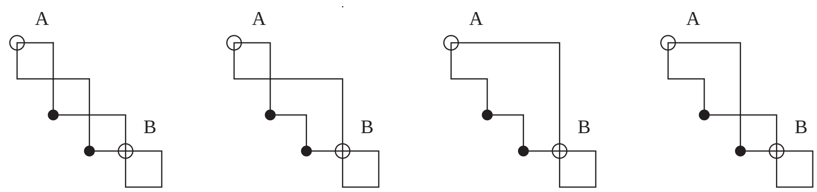


Figure 7: The four row convex permutominides with permutation $(5, 4, 3, 2, 1)$. The free points are highlighted.

Inversions and Bruhat order for permutominides. Here we will recall some basics on permutations. Remind that an *inversion* for a given permutation π is a couple of indexes (i, j) such that $i < j$ and $\pi(i) > \pi(j)$. The concept of inversion allows to define a partial order on the set S_n of permutations of a given length n. In fact, if π_1 and π_2 are two permutations of S_n, we have that π_1 covers π_2 in the *Bruhat* order if and only if there exists an inversion (i, j) of π_1 such that $\pi_2(i) = \pi_1(j)$, $\pi_2(j) = \pi_1(i)$ and for all $k \notin \{i, j\}$ $\pi_1(k) = \pi_2(k)$. We say that π_2 is obtained from π_1 *switching* an inversion. The Bruhat order $\leqslant_B$ is defined as the transitive closure of the covering relation defined above. As an example in Fig. 8 it is represented the Bruhat poset for S_4.

The concept of inversion of a permutation reflects on the associated permutominides. So an *inversion* in a permutominide C turns out to be a couple of its vertical sides which corresponds to an inversion of the associated permutation $\pi(C)$. More precisely an inversion is a couple of vertical sides (e, i) such that:

1. $x(e) < x(i)$.
2. $y(e) > y(i) \vee (\, y(e) = y(i) \wedge l(e) > l(i)\,)$.

As for permutations, in the particular case of row convex permutominides we can *switch* an inversion (or more in general two generic vertical sides) obtaining an object which is still in that class, as it is shown in proposition 3.3.

The result of a *switch* of two vertical sides $\{e, i\} \subseteq \mathbb{N}$ (remind that we identify a generic vertical side s with the number $x(s)$) of a row convex permutominide C of size n is defined as the permutominide C' of the same size such that :

1. $(y(i), l(i))$ in C is equal to $(y(e), l(e))$ in C'
2. $(y(e), l(e))$ in C is equal to $(y(i), l(i))$ in C'
3. If $k \notin \{e, i\}$ then $(y(k), l(k))$ has the same values in both C and C'.

The permutominide C' exists for proposition 3.3 and in particular we have that the permutation $\pi(C')$ associated to the result is equal to the permutation $\pi(C)$ except for the replacement of the elements with index e and i. As an example see Fig. 9.

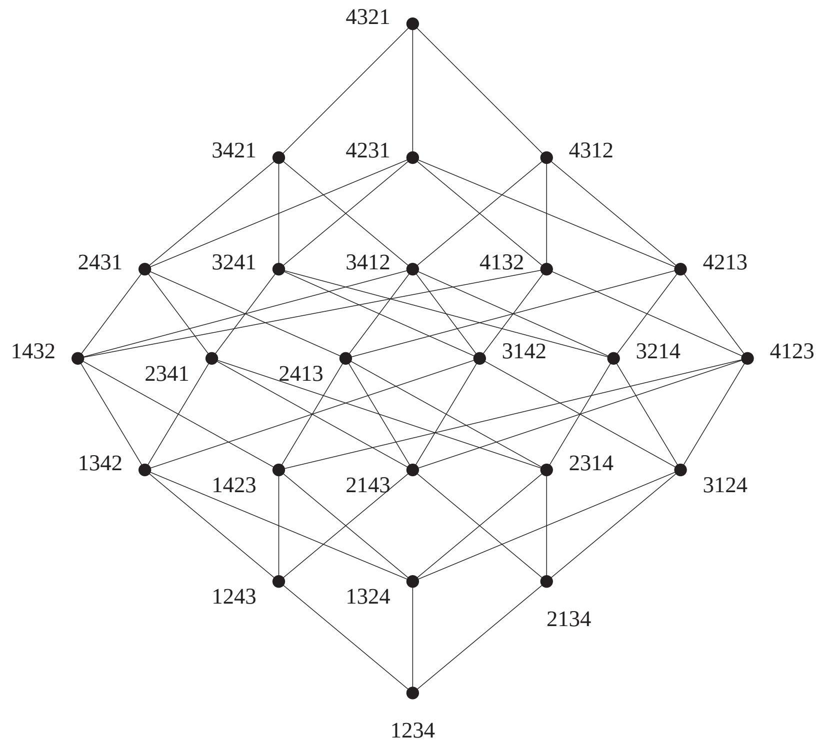


Figure 8: The Hasse diagram of the Bruhat order on the permutations of S_4.

Proposition 3.3. The application of a switch on two generic vertical sides of a row convex permutominide produces a row convex permutominide.

Proof sketch. When we switch two vertical sides we simply replace their abscissas in the starting permutominide and then collect their vertices to the others as it was at the beginning, see Fig. 9.
It is easy to check that replacing, in the sense explained previously, the abscissas of two vertical sides starting from a row convex, both the properties of being a permutominide and of being row convex are preserved in the result.

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Using, as it is done in the Bruhat order for permutations, the concept of inversion for

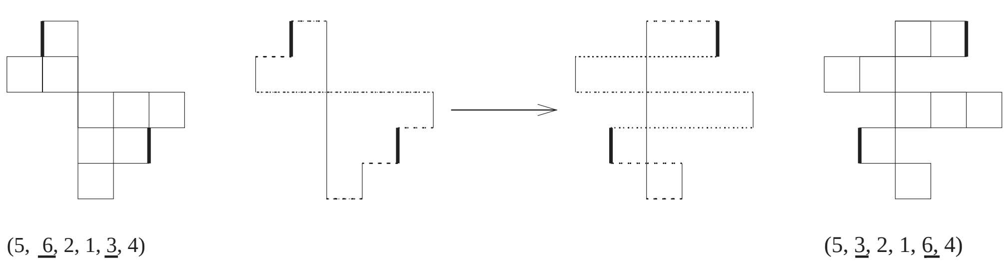

Figure 9: Switching an inversion of a row convex permutominide.

row convex permutominides we obtain a partial order $\preceq$ on this class. Taken two row convex permutominides of the same size, C and C', we say that C covers C' if and only if C' is the result of a switching of an inversion in C. Then we define the relation $\preceq$ as the transitive closure of the covering relation defined above.
It is now obvious that, for example, the maximal (resp. minimal) elements of our poset are those row convex permutominides associated to the maximum (resp. minimum) permutation in the Bruhat order of the same size.

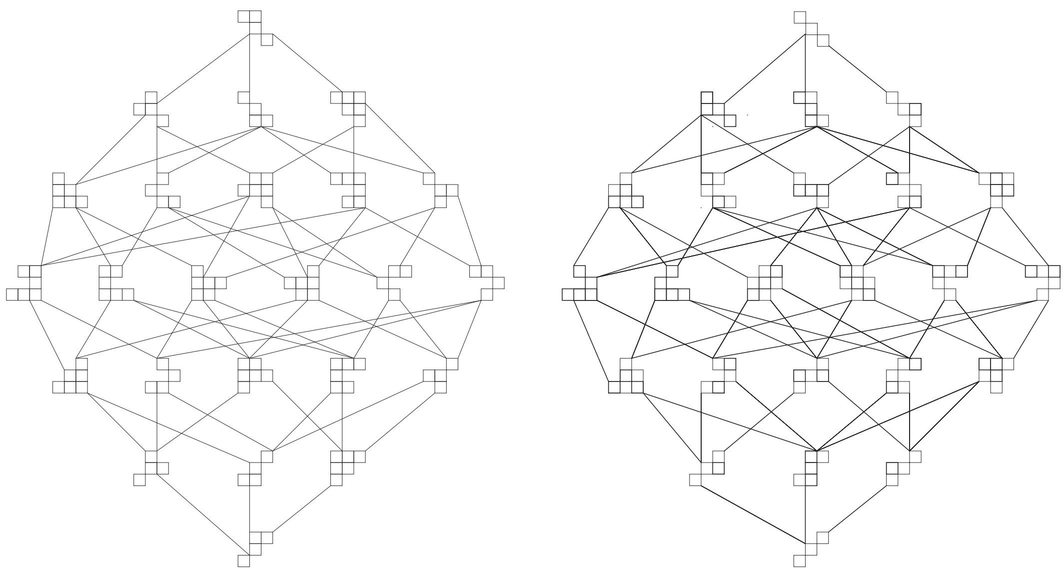

Figure 10: The Hasse diagram of the partial order on row convex permutominides of size 4.

Proposition 3.4. Let C and C' be two different row convex permutominides such that $\pi = \pi(C) = \pi(C')$, take an inversion of π and switch it in C and C'. The results are always different permutominides.

Proof. Observe that a permutominide is completely determined by the sequence of the lowest points of its vertical sides and by the length of its vertical sides. Starting from different permutominides associated to the same permutation these characteristics are different also in the results of the switch of the inversion.

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In the case $n = 4$, shown in Fig. 10, we have a poset consisting of two incomparable subposets isomorphic to the Bruhat order on S_4 and each row convex permutominide of size 4 is in exactly one of these subposets. This is true in general, see the next Corollary, and it is due to Proposition 3.4 and to Proposition 3.2. Corollary 3.1 gives also the enumeration for the class of row convex permutominides with respect to the size.

Corollary 3.1. Let RCP_n be the set of row convex permutominides of size n, then the following holds:

1. The poset $(RCP_n, \preceq)$ consists of exactly 2^{n-3} disconnected copies isomorphic to the Bruhat order on S_n.
2. Each element in RCP_n is an element of exactly one of these copies.
3. The number of row convex permutominides of size n is equal to $2^{n-3}n!$

Proof. The statements are obvious consequences of the previous Propositions.
In particular we have that each row convex permutominides $P \in RCP_n$ is obtained, by a sequence of switches of inversions, from a row convex permutominide $\widetilde{P}$ having $\tau = (n, n-1, n-2, \ldots, 2, 1)$ as permutation.
In fact, from P we can find $\widetilde{P}$ simply considering $\pi(P)$ in the Bruhat order on S_n and choosing, in its Hasse diagram, a path which goes from $\pi(P)$ to the maximum permutation that is τ. Then, starting from $\pi(P)$, each time we find in the chosen path a permutation which covers the previous by a switch of an inversion (j, k), we perform on the permutominide related to the covered permutation the switch of the vertical sides having abscissas in $\{j, k\}$.
In this way we always obtain a permutominide, see Proposition 3.3, and it is clear that the obtained permutominide is greater, in the order defined on these objects, than the starting one. The procedure stops when we find the permutominide $\widetilde{P}$.

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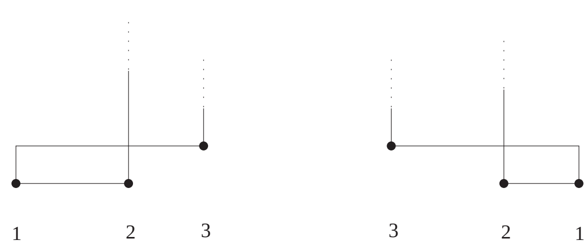
4 Other combinatorial properties of row convex permutominides

For any given $\pi \in S_n$ we have defined $\mathcal{P}(\pi) = \{P : \pi(P) = P\}$. By Corollary 3.1 we have that $|\mathcal{P}(\pi)| = 2^{n-3}$. It seems now natural to investigate the properties of the set $\mathcal{P}(\pi)$. For instance we would like to know if it contains at least a permutomino.

Proposition 4.1. If $\pi^{-1}(1) < \pi^{-1}(2) < \pi^{-1}(3)$ or $\pi^{-1}(3) < \pi^{-1}(2) < \pi^{-1}(1)$ then $\mathcal{P}(\pi)$ cannot contain a row convex permutomino.

Proof. Suppose $C \in \mathcal{P}(\pi)$ and take the vertical side of C having abscissa equal to $\pi^{-1}(1)$, remind that it has length one. So it must be connected to the side with abscissa equal to $\pi^{-1}(3)$. Then the hypothesis corresponds to one of the two configurations sketched below:



which clearly cannot lead to a row convex permutomino.

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The following result is less obvious.

Theorem 4.1. The set $\mathcal{P}(\pi)$ contains at least one row convex permutomino if and only if $\pi^{-1}(2)$ is not between $\pi^{-1}(1)$ and $\pi^{-1}(3)$.

Proof. The necessary condition is equivalent to Proposition 4.1. The sufficient condition states that if $\pi^{-1}(2)$ is not between $\pi^{-1}(1)$ and $\pi^{-1}(3)$, then $\mathcal{P}(\pi)$ contains at least one row convex permutomino. Let us prove the statement assuming that $\pi^{-1}(2) > \pi^{-1}(1)$ and $\pi^{-1}(2) > \pi^{-1}(3)$; the case $\pi^{-1}(2) < \pi^{-1}(1)$ and $\pi^{-1}(2) < \pi^{-1}(3)$ is symmetric. We give a procedure to build a row convex permutomino P such that $\pi(P) = \pi$. Let Π be defined as

$$\Pi(k) = \begin{cases} \pi(k) - 1 & \text{if } \pi(k) > 1 \\ \\ 1 & \text{otherwise.} \end{cases}$$

Let

$$i = \max\{\pi^{-1}(1), \pi^{-1}(3)\} < \pi^{-1}(2).$$

The index i partitions the points $(j, \Pi(j))$ into two non-empty subsets, called the *left* and the *right side* of Π, precisely

$$\mathcal{L} = \{\, (j, \Pi(j)) \,:\, j \leq i \,\}, \qquad \mathcal{R} = \{\, (j, \Pi(j)) \,:\, j > i \,\}.$$

Let us consider the following two cases:

i. $\pi^{-1}(n) < i$.

Let us consider the index $m > i$ such that $\Pi(m) = \max\{\Pi(j) : j > i\}$. Clearly $(m, \Pi(m)) \in \mathcal{R}$. We say that two points $(r, \Pi(r))$ and $(s, \Pi(s))$ in $\mathcal{L}$ (resp. $\mathcal{R}$) are *consecutive* if $\Pi(r) < \Pi(s)$ (resp. $\Pi(r) > \Pi(s)$) and there is no index l such that $(l, \Pi(l) \in \mathcal{L}$ (resp. $\mathcal{R}$) and $\Pi(r) < \pi(l) < \Pi(s)$ (resp. $\Pi(r) > \Pi(l) > \Pi(s)$). Now, any two consecutive points $(r, \Pi(r))$ and $(s, \Pi(s))$ in $\mathcal{L}$ can be connected by means of a path $N^{\Pi(s)-\Pi(r)}W^{r-s}$ if $r > s$, or a path $N^{\Pi(s)-\Pi(r)}E^{s-r}$ otherwise, where, as usual, N (resp. S, E, W) represents a north (resp. south, east, west) step of length 1. Then, the points $(\Pi^{-1}(n), n)$ and $(m, \Pi(m))$ are connected by means of a path $NE^{m-\Pi^{-1}(n)}S^{n-m}$. Now, any two consecutive points $(r, \Pi(r))$ and $(s, \Pi(s))$ in $\mathcal{R}$ can be connected by means of a path $S^{\Pi(s)-\Pi(r)}W^{r-s}$ if $r > s$, or $N^{\Pi(s)-\Pi(r)}E^{s-r}$ otherwise. Finally, the point $(\pi^{-1}(2), 1)$ is connected to $(\pi^{-1}(1), 1)$ by means of an horizontal path of length $\pi^{-1}(2) - \pi^{-1}(1)$. It should be clear that the obtained path does not cross, and then it is the boundary of a row convex permutomino P such that $\pi(P) = \pi$ (see Figure 11 (a)).

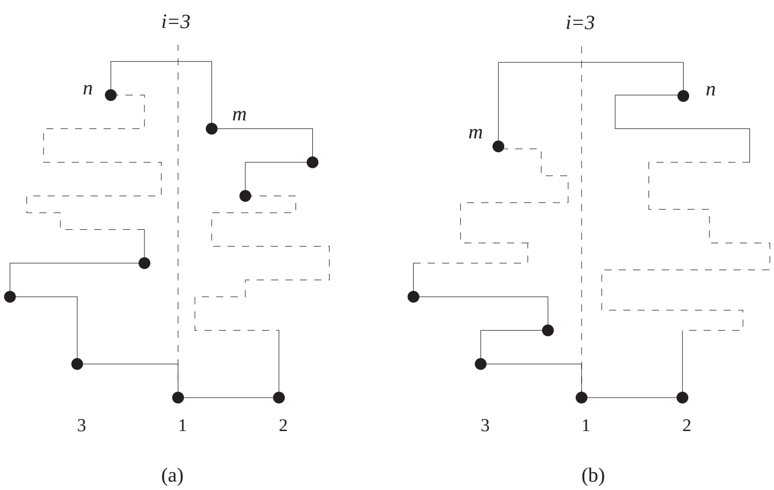

Figure 11: The construction of a row convex permutomino, with: (a) $\pi^{-1}(n) < i$, (b) $\pi^{-1}(n) > i$.

ii. $\pi^{-1}(n) > i$.

The procedure to build P is essentially the same as the previous one, with the only remark that here the highest point of the set $\mathcal{L}$ is $(m, \pi(m))$, being m the index such that $\pi(m) = \max\{\pi(j) : j < i\}$ (see Figure 11 (b)).

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We observe that the row convex permutomino built using the above described method is always *centered*, i.e. there is at least one column running from the upper to the lower side of its minimal bounding square. It should also be clear that it is not necessarily the unique row convex permutomino P such that $\pi(P) = P$. For instance Figure 12 depicts the four row convex permutominides in $\mathcal{P}(\pi)$, with $\pi = (3, 4, 1, 2, 5)$. Two of them are permutominoes: the first one is obtained applying our procedure, and it is centered, while the second is not centered.

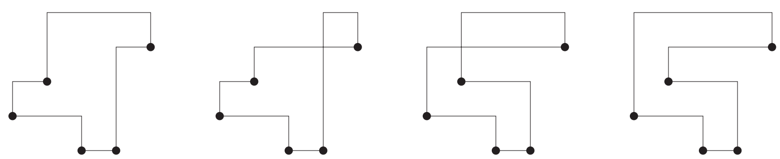

Figure 12: The set $\mathcal{P}(\pi)$, with $\pi = (3, 4, 1, 2, 5)$.

5 Further works

In this paragraph we present some problems related to the class of objects defined in this paper.

1. We have shown how to associate a permutation to a permutominide, and then we have enumerated the class of row convex permutominides. Our technique is based on the next facts:

   (a) The class is closed under the operation of switch.

   (b) Starting from different permutominides associated to the same permutations π and switching an inversion of π we reach different permutominides associated to a same permutation π'.

   (c) We have been able to count the number of different row convex permutominides associated to the permutation having the maximum number of inversions. We have seen that from this permutation we obtain all the others by switching inversions and that is true also for the associated permutominides.

   Is it possible to do this for a more general class of permutominides which contains the class of row convex?

2. Another problem which may can be studied is related to the correspondence between parameters which are typical of the objects we have defined as permutominides and properties of the poset we have defined on them. For example one could investigate the relations between parameters of these particular diagrams and inversions of the associated permutations.

3. We have seen that permutominides are also uniquely determined by particular pairs of permutations with the same size, if P is a permutominide we have defined the couple of permutations $(\pi_1(P), \pi_2(P))$.

   Looking at $\pi_1(P)$ and $\pi_2(P)$ in the classical Bruhat poset for permutations, which are their characteristics ?

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