On the exhaustive generation of convex permutominoes

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Abstract

A permutomino of size n is a polyomino determined by a pair (π_1, π_2) of permutations of size $n+1$, such that $\pi_1(i) \neq \pi_2(i)$, for $1 \leq i \leq n+1$. In this paper, after recalling some enumerative results about permutominoes, we give a first algorithm for the exhaustive generation of a particular class of permutominoes, the convex *permutominoes, proving that its cost is proportional to the number of generated objects.*

1 Introduction

A permutomino is a special polyomino, defined by two permutation matrices having the same size. The class of permutominoes was introduced in [11] while studying the problem of determining the $\widetilde{R}$-polynomials (related to the Kazhdan-Lusztig R-polynomials) associated with a pair (μ, ν) of permutations. In [10] a general definition of these combinatorial objects that uses some algebraic notions is given.

In this paper we use the definition of permutominoes given in [7] which does not use the algebraic notions and nevertheless, though different, it turns out to be equivalent to the one given in [10].

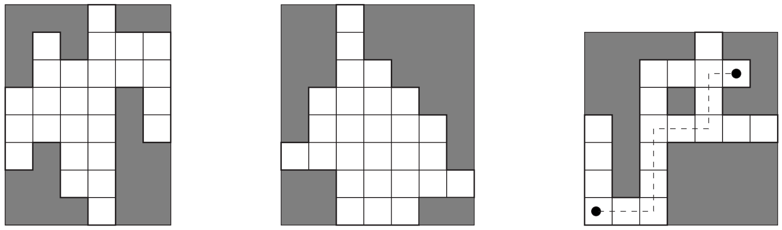
The main results about permutominoes concern the enumeration of various subclasses of permutominoes and the characterization for the permutations defining these subclasses [3, 6, 7, 8, 9], while at our knowledge nothing exists about their generation. On the other hand, exhaustive generation of combinatorial objects [1, 4, 5] is an area of increasing interest. In fact, many practical questions in diverse areas, such as hardware and software testing, and combinatorial chemistry, require for their solution the exhaustive search through all objects in the class.

Actually, in [7] a recursive generation of all convex permutominoes of size $(n+1)$ from the ones of size n, according to the ECO method [2], is presented. Section 2 contains basic definitions and some enumerative results of convex permutominoes of size n. Section 3 recalls the recursive generation of convex permutominoes presented in [7] and Section 4 illustrates the exhaustive generating algorithm based on the recursive construction recalled in Section 3.

2 Basics on permutominoes

2.1 Definitions and properties

In order to define permutominoes we need to introduce polyominoes. In the plane $\mathbb{Z} \times \mathbb{Z}$ a *cell* is a unit square and a *polyomino* is a finite connected union of cells having no cut point.



(a) (b) (c)

Figure 1: (a) a column convex polyomino; (b) a convex polyomino; (c) a directed (not convex) polyomino with a hole.

Polyominoes are defined up to translations. A *column* (*row*) of a polyomino is the intersection between the polyomino and an infinite strip of cells whose centers lie on a vertical (horizontal) line.

In order to simplify many problems which are still open on the class of polyominoes, several subclasses were defined by combining two notions: the geometrical notion of *convexity* and the notion of *directed growth.* A polyomino is said to be *column convex* [*row convex*] if its intersection with any vertical [horizontal] line is convex (Figure 1 (a)). A polyomino is *convex* if it is both column and row convex (Figure 1 (b)). In a convex polyomino the *semi-perimeter* is given by the sum of the number of rows and columns, while the *area* is the number of its cells.

A polyomino P is said to be *directed* when every cell of P can be reached from a distinguished cell, called *root* (the bottom leftmost cell), by a path which is contained in P and uses only north and east unit steps (Figure 1 (c)).

Let P be a polyomino without holes having n rows and n columns, $n \geq 1$; without loss of generality, we assume that the bottom leftmost vertex of the polyomino minimal bounding rectangle lies in $(1,1)$. Let $A_1, \ldots, A_{2(r+1)}$ be the sequence of the vertices of P obtained by visiting the boundary in clockwise sense, starting from its leftmost point with minimal ordinate.

We say that P is a *permutomino* if the sets $\mathcal{P}_1 = \{A_1, A_3, \ldots, A_{2r+1}\}$ and $\mathcal{P}_2 = \{A_2, A_4, \ldots, A_{2r+2}\}$ represent two permutations matrices of $[n+1] = \{1,2,\ldots, n+1\}$. Obviously, if P is a permutomino, then $r = n$, and n is called the *size* of the permutomino. The two permutations defined by $\mathcal{P}_1$ and $\mathcal{P}_2$ are indicated by $(\pi_1(P), \pi_2(P))$ (briefly, (π_1, π_2)), respectively (see Figure 2). Given a pair of permutations $\omega = (\mu, \nu)$ of $[n+1]$, we say that a permutomino P is *associated* with ω if $\mu = \pi_1(P)$ and $\nu = \pi_2(P)$.

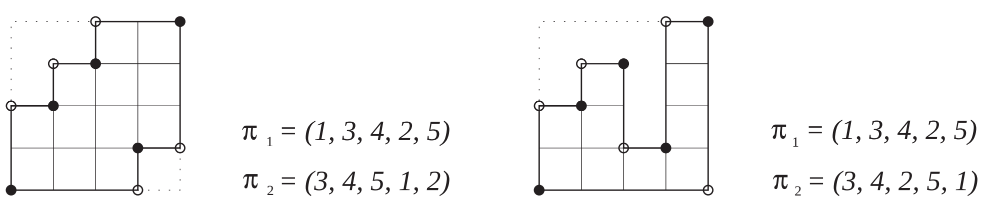


Figure 2: Two permutominoes and the associated permutations. The permutation π_1 (resp. π_2) is represented by black (resp. white) dots.

A permutomino is *convex* (*directed*) if it is a convex (directed) polyomino. A *parallelogram* permutomino is a directed and convex one having the $(1,1)$ and (n,n) vertices in common with its minimal bounding square; a *stack* permutomino is a directed and convex one in which the bottom side of its minimal bounding square belongs to the permutomino itself.

The definition of permutominoes leads to the following remarkable property:

Proposition 2.1. *Any permutomino P has the property that, for each abscissa (ordinate) there exists exactly one vertical (horizontal) side in the boundary of P having such coordinate. This property is also a sufficient condition for a polyomino to be a permutomino.*

Starting from the leftmost point having minimal ordinate, and moving in a clockwise sense, the boundary of a permutomino P can be encoded as a word in a four letter alphabet, $\{N, E, S, W\}$, where N (resp., E, S, W) represents a *north* (resp. *east*, *south*, *west*) unit step. Any occurence of a sequence NE, ES, SW or WN in the word encoding P defines a *salient point* of P, while any occurence of a sequence EN, SE, WS or NW defines a *reentrant point* of P (see Figure 3). For simplicity of notation and to clarify the definition of the construction recalled in Section 3, the reentrant points of a convex permutomino are grouped in four classes; in practice, the reentrant point determined by a sequence *EN* (resp. *SE*, *WS*, *NW*) is represented with the symbol α (resp. β, γ, δ).

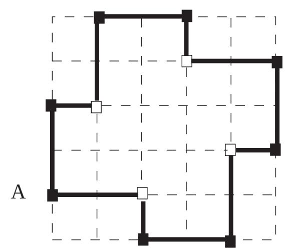


Figure 3: The coding of the boundary of a permutomino, starting from A and moving in clockwise sense; its salient (resp. reentrant) points are indicated by black (resp. white) squares.

2.2 Previous enumerative results

Let us recall the main enumerative results concerning convex permutominoes. In [8, 9], using bijective techniques, the authors provide enumeration of various classes of convex permutominoes, including the *parallelogram*, the *directed convex* and the *stack* ones; moreover, a characterization of the permutations associated with permutominoes of each class is given. Let $\mathcal{C}_n$ (resp. $\mathcal{P}_n$, $\mathcal{D}_n$, $\mathcal{S}_n$) be the set of convex (resp. parallelogram, directed convex, stack) permutominoes of size n and

$$\widetilde{\mathcal{C}}_n = \{\, \pi_1(P) \,:\, P \in \mathcal{C}_n \,\}, \quad \widetilde{\mathcal{P}}_n = \{\, \pi_1(P) \,:\, P \in \mathcal{P}_n \,\},$$

$$\widetilde{\mathcal{D}}_n = \{\, \pi_1(P) \,:\, P \in \mathcal{D}_n \,\}, \quad \widetilde{\mathcal{S}}_n = \{\, \pi_1(P) \,:\, P \in \mathcal{S}_n \,\}.$$

The enumeration results obtained in [8, 9] are shown in Table 3.
where c_n is the nth *Catalan number*,

$$c_n = \frac{1}{n+1}\binom{2n}{n}$$

and b_n are the *central binomial coefficients*

$$b_n = \binom{2n}{n}.$$

In [7] it was proved, using the ECO method [2], that the number of *convex permutominoes* of size n is:

Class of convex permutominoes	*Number of convex permutominoes with size n*	*Associated set of permutations*
Parallelogram	$\lvert\mathcal{P}_n\rvert = c_n$	$\lvert\widetilde{\mathcal{P}}_n\rvert = c_{n-1}$
Directed convex	$\lvert\mathcal{D}_n\rvert = \frac{1}{2}b_n$	$\lvert\widetilde{\mathcal{D}}_n\rvert = b_{n-1}$
Stack	$\lvert\mathcal{S}_n\rvert = 2^{n-1}$	$\lvert\widetilde{\mathcal{S}}_n\rvert = 2^{n-1}$

Table 3: Enumeration results.

$$2(n+3)4^{n-2} - \frac{n}{2}\binom{2n}{n}, \quad n \geq 1.$$

The first terms of the sequence are

$$1, 4, 18, 84, 394, 1836, 8468, \ldots$$

(sequence A126020 in [12]). The same enumerative result was obtained, using a different approach, by Boldi et al. [6].
An interesting study of combinatorial properties of the set $\widetilde{\mathcal{C}}_n$ is in [3].

3 ECO construction of convex permutominoes

In this section we recall the ECO construction of convex permutominoes as given in [7].

Let $\mathcal{C}_n$ be the set of convex permutominoes of size n and let $P \in \mathcal{C}_n$; the number of cells in the rightmost column of P is called the *degree* of P. Let us consider the following properties of a convex permutomino:

U1 : the uppermost cell of the rightmost column of P has the maximal ordinate among all the cells of the permutomino;

U2 : the lowest cell of the rightmost column of P has the minimal ordinate among all the cells of the permutomino.

According to the ECO method [2], it is necessary to define an operator $\vartheta : \mathcal{C}_n \rightarrow 2^{\mathcal{C}_{n+1}}$ which defines a recursive construction of all the convex permutominoes of size $(n+1)$ in a unique way from the objects of size n. The operator ϑ defined in [7] acts on a convex permutomino performing some local expansions on the cells of its rightmost column. Let $c_1, \ldots, c_n$ (resp. $r_1, \ldots, r_n$) be the columns (resp. rows) of a permutomino P of size n numbered from left to right (resp. bottom to top) and let $\ell(c_i)$ (resp. $\ell(r_i)$) be the number of cells in the ith column (resp. row), with $1 \leq i \leq n$. The four operations of ϑ, denoted by $(\alpha), (\beta), (\gamma)$ and (δ) are defined as follows:

(α) if P satisfies condition **U1**, then operation (α) adds a new column made of $\ell(c_n) + 1$ cells on the right of c_n, see Figure 4.

(β) it can be performed on each cell of c_n; so let d_i be the ith cell of c_n, from bottom to top, with $1 \leq i \leq \ell(c_n)$. Operation (β) adds a new row above the row containing d_i (of the same length), and add a new column on the right of c_n made of i cells, see Figure 5.

(γ) it can be performed on each cell of c_n; so let d_i be the ith cell of c_n, from bottom to top, with $1 \leq i \leq \ell(c_n)$. Operation (γ) adds a new row below the row containing d_i (of the same length), and add a new column on the right of c_n made of $n - i + 1$ cells, see Figure 6.

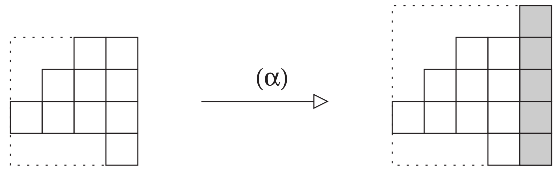


Figure 4: Operation (α); the added column has been highlighted.

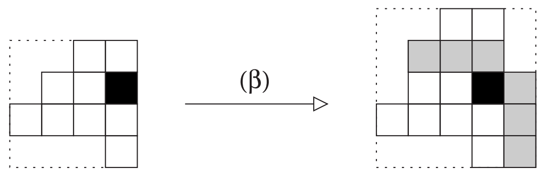


Figure 5: Operation (β); the cell d_i is filled in black , the added column and row have been highlighted.

(δ) if P satisfies condition **U2**, then operation (δ) adds a new column made of $\ell(c_n) + 1$ cells on the right of c_n, see Figure 7.

Obviously in any case the obtained permutomino is a convex permutomino of size $n + 1$.

We refer to [7] for further details and proofs.

4 The generating algorithm

Our aim is to illustrate an exhaustive generating algorithm for convex permutominoes, basing on the ECO construction recalled in previous section. In the sequel we will refer only to convex permutominoes, simply named "permutomino" .

First of all we define a subset of permutominoes of size n, the so called *active permutominoes*, then we will show the existence of a bijection between the active permutominoes of size n and the set of permutominoes of size $n - 1$. Finally, we will define the generating tree of permutominoes of size n.

4.1 Definition of active permutominoes

A permutomino P is *active* if the following conditions hold:

1. the leftmost column contains only one cell, ($\ell(c_1) = 1$);
2. the leftmost reentrant point α has abscissa 2.

That is, the word encoding the boundary of an active permutomino begins with "$NEN\dots$ and ends with $\dots WW$". In Figure 8 are depicted three active permutominoes of size 4, while in Figure 9 there are some permutominoes that, yet having only one cell in the leftmost column, are not active.

Let P be an active permutomino and let ϱ be the row of P containing the only cell in the leftmost column (in Figure 8 ϱ is highlighted); ϱ can be the bottom row but it is never the top row. From Proposition 2.1, the row ϱ ends at the same abscissa of either the above and the below row, if it exists; so a configuration as that depicted in Figure 10 is not admissible. Therefore, if in an active permutomino P of size n we remove ϱ, we obtain a convex permutomino P' of size $(n - 1)$.

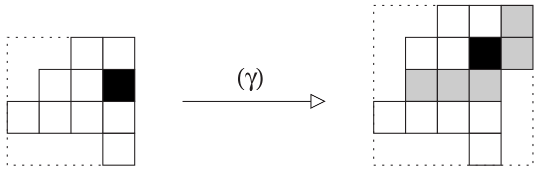


Figure 6: Operation (γ); the cell d_i is filled in black , the added column and row have been highlighted.

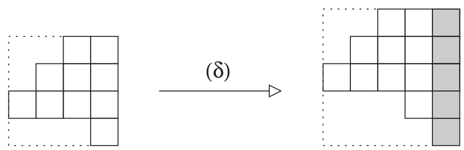


Figure 7: Operation (δ); the added column has been highlighted.

Viceversa, let P' be a permutomino of size $(n-1)$ and let τ the row in P' containing the leftmost salient point with minimal ordinate (point A in Figure 3). If in P' we add, below τ, a row ϱ one cell longer on the left and ending at the same abscissa of τ, we obtain an active permutomino P of size n:

i. if τ is the bottom row, the added row in P' is one cell longer on the left and therefore the new permutomino of size n is convex and active (see Figure 11 a));

ii. if $\tau = r_i$, with $1 < i < n$, the row r_{i-1} ends at the same abscissa of τ; so adding ϱ we obtain a convex and active permutomino of size n (see Figure 11 b)).

This means that it exists a bijection ψ between the set $\mathcal{C}_{a,n}$ of active permutominoes of size n and the set $\mathcal{C}_{n-1}$:

$$\psi : \mathcal{C}_{a,n} \rightarrow \mathcal{C}_{n-1}$$

such that $\psi(P)$ is the permutomino of size $n-1$ obtained by removing ϱ from the active permutomino P of size n.

4.2 The exhaustive generating algorithm

The algorithm we propose for the exhaustive generation of convex permutominoes of size n is based on the bijection ψ, defined in Section 4.1, and on the ECO construction of permutominoes recalled in Section 3.

The generating process is described by an operator ϕ so defined:

ϕ **Operator:**

1. The first permutomino of the generating process is P_n^r, that is the permutomino of size n associated with the pair of permutations (π_1, π_2) of $[n+1]$:

$$\pi_1 = (1, 2, 3, \dots, n, n+1) \qquad \pi_2 = (2, 3, 4, \dots, n+1, 1)$$

In Figure 12 is depicted P_n^r with $n = 4$.

2. P_n^r is an active permutomino; let $\bar{P}^r$ the permutomino of size $(n-1)$ such that $\bar{P}^r = \psi(P_n^r)$ ($\bar{P}^r$ *is obtained by removing the bottom row of* P_n^r).

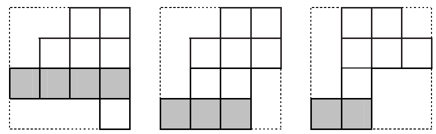


Figure 8: Three active permutominoes of size 4.

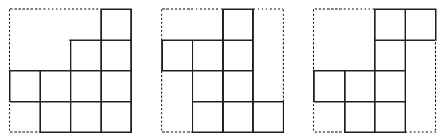


Figure 9: Some not active permutominoes of size 4.

3. Apply operations (β), (γ) and (δ) of ECO construction to $\bar{P}^r$. Every application generates a new convex permutomino of size n.
4. For each new active permutomino Q repeat the following actions until active permutominoes are generated:

   4.1 remove the ϱ row from Q obtaining $\bar{Q} = \psi(Q)$;

   4.2 apply all the possible operations of the ECO construction to $\bar{Q}$. Every application generates a new convex permutomino of size n.

Our strategy can be represented using a rooted tree, say $\mathcal{C}_n$*-tree*, so defined:

1. the root is P_n^r and it is at level 0;
2. if $Q \in \mathcal{C}_n$*-tree* is an active permutomino at level $k \geq 0$, then $\phi(Q) = \vartheta(\psi(Q))$ (ϑ is the operator defined in the ECO construction) and every $P \in \phi(Q)$ is a son of Q and it is at level $(k+1)$. For the sake of simplicity, we say that $P \in \phi^{k+1}(P_n^r)$.

In Figure 13 $\mathcal{C}_3$*-tree* is illustrated.

Proposition 4.1. $\mathcal{C}_n$-tree *contains all and only the convex permutominoes of size n, i.e* $\mathcal{C}_n$-tree $= \mathcal{C}_n$.

Proof: [Only] The permutominoes in $\mathcal{C}_n$*-tree* are obtained from permutominoes of size $(n-1)$ by applying the ECO construction; so, as proved in [7], we generate permutominoes of size n. Therefore, since the permutominoes of size $(n-1)$ are each other different, the ones of size n are different too.
[All] We must prove that for each permutomino $Q \in \mathcal{C}_n$ there exists a path from P_n^r to Q, that is it exists a finite sequence $P_0, P_1, \ldots, P_k$ with $k \in \mathbb{N}$ and $P_k = Q$ such that:

- $P_0 = P_r^n$;
- $P_{i+1} \in \phi(P_i), \qquad 0 \leq \mathrm{i} \leq k-1.$

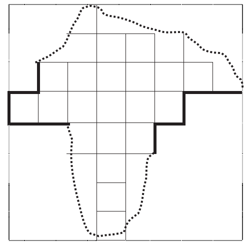


Figure 10: A not admissible configuration.

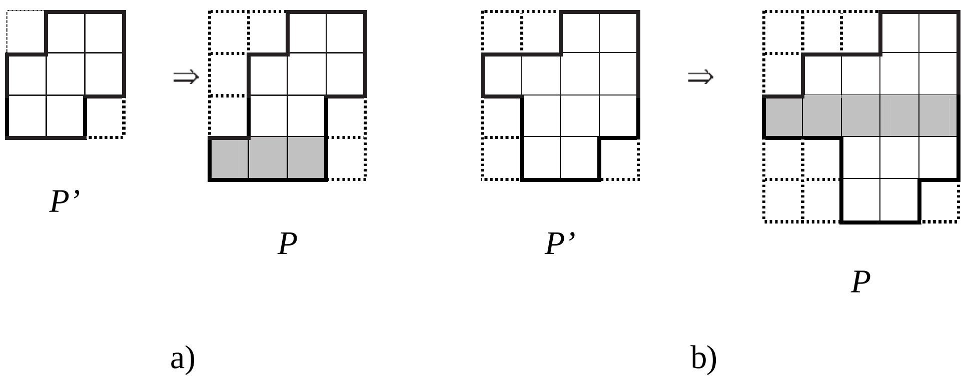


Figure 11: A permutomino P' of size $(n-1)$ and the corresponding active permutomino P of size n.

In other words, we must prove that there exists $k \geq 1$ such that $Q \in \phi^k(P_n^r)$. We know that the active permutominoes of size n are as many as the permutominoes of size $(n-1)$. If the algorithm generate all active permutominoes of size n, since we apply ψ to all these permutominoes during the algorithm, we apply the ECO-construction to all permutominoes of size $(n-1)$ and thus it means that we generate all permutominoes of size n. So it is sufficient to prove the following:

Proposition 4.2. *All the active permutominoes of size n are generated.*

Proof: By induction on the size n.
Base. For $n = 1$ there is only the permutomino containing one cell; if $n = 2$ the unique active permutomino is P_2^r, (see Figure 14). So for $n \leq 2$ Proposition 4.2 yields.
Inductive hypothesis. Let us assume that all the permutominoes of size $(n-1)$ are generated and let $\mathcal{C}_{n-1}$*-tree* be the associated tree. Then, starting from the root P_{n-1}^r it is possible to reach any permutomino of size $(n-1)$. So, for each permutomino $P_{n-1} \in \mathcal{C}_{n-1}$ there exists a k such that:

$$P_{n-1} \in \phi^k(P_{n-1}^r)$$

Inductive step. Let $\bar{P}_n$ be an active permutomino of size n and let $Q_{n-1} = \psi(\bar{P}_n)$ (Q_{n-1} is obtained from $\bar{P}_n$ removing ϱ). So there exists $\bar{k}$ such that

$$Q_{n-1} \in \phi^{\bar{k}}(P_{n-1}^r).$$

But

$$P_{n-1}^r = \psi(P_n^r)$$

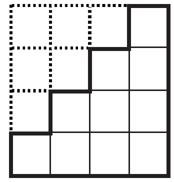


Figure 12: P_n^r with $n = 4$.

so it follows that:

$$Q_{n-1} \in \phi^{\bar{k}}(P_n^r)$$

that is, there is a path from the root P_n^r to any permutomino of size $(n-1)$. Therefore, each permutomino of size n is reachable from P_n^r.

$\jmath$

4.3 Algorithm cost analysis

First of all, we will prove that the height of $\mathcal{C}_n$-*tree* is n. The proof is based on the following propositions.

Proposition 4.3. *Using the ECO construction of Section 3, an active permutomino of size n is generated by one and only one active permutomino of size $(n-1)$.*

*Proof:*It follows straightforward from the ECO construction which never adds a column on the left containing one cell to the permutoninoes.

$\jmath$

Proposition 4.4. *Given an active permutomino Q, the longest path in the generating tree starting from Q has length j if its α points lie in $(2,h),(3,h+1),\ldots,(j+1,h+j-1)$, h being the ordinate of the lefmost α point.*

Proof: The permutomino $\psi(Q)$ of size $(n-1)$ to which the ECO construction is applied, is obtained removing from Q the row ϱ, so the leftmost α point of Q is removed in $\psi(Q)$. Therefore, $\psi(Q)$ will be active, and then, from Proposition 4.3, it can generate new active permutominoes, only if it has an α point at abscissa 2; thus, Q must have an α point at abscissa 3. In the same way, a permutomino generated from $\psi(Q)$ will be active only if $\psi(Q)$ has an α point at abscissa 3, that is if Q has an α point at abscissa 4, and so on up to $(j+1)$.

$\jmath$

The permutomino P_n^r has $(n-1)$ consecutive α points in $(2,2),\ldots,(n,n)$, so, from Proposition 4.4, the longest path starting from P_n^r has length $(n-1)$. Therefore, the height of $\mathcal{C}_n$-*tree* is n.

From the generating algorithm of Section 4.2 it follows that the generation of a permutomino P at level k in the $\mathcal{C}_n$-*tree* depends only on a permutomino at level $(k-1)$, that is, $\mathcal{C}_n$-*tree* is generated level by level. Therefore, since the generation of a permutomino of size n from one of size $(n-1)$ and operator ψ have a constant cost, we may conclude that the cost of the exhaustive generating algorithm is proportional to the number of permutominoes of size n.

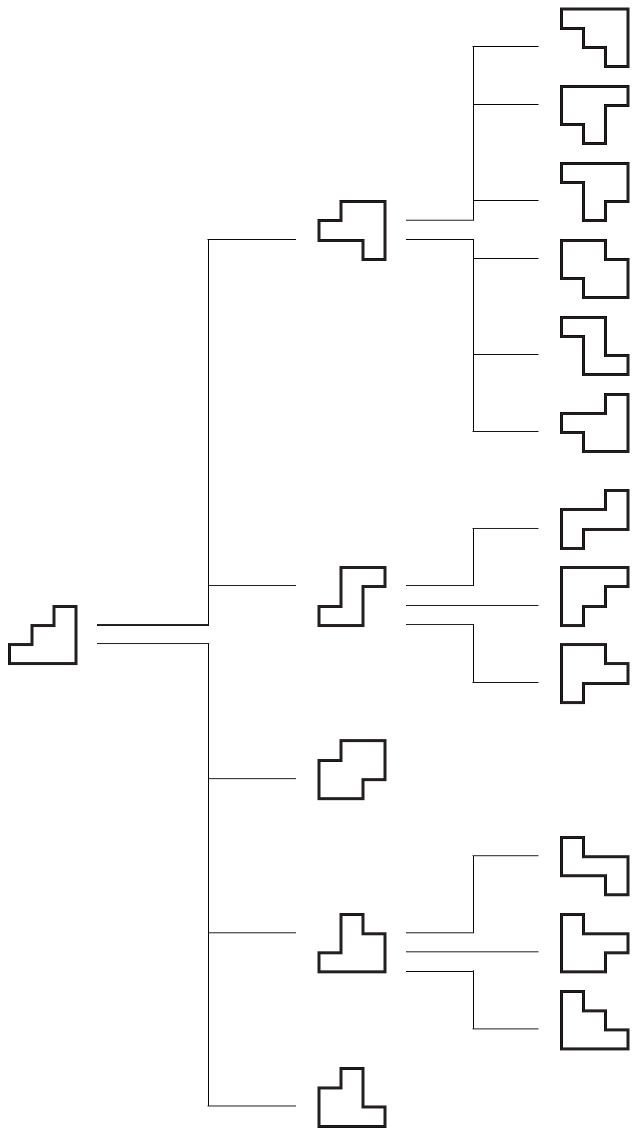


Figure 13: $\mathcal{C}_3$-*tree*.

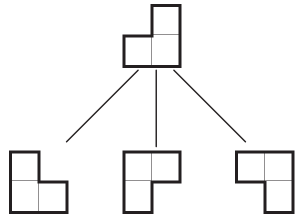


Figure 14: $\mathcal{C}_2$-*tree*.

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