A lagrangean approach to reconstruct bicolored images from discrete orthogonal projections

Fethi JARRAY*
Laboratoire CEDRIC, 292 rue Saint-Martin, 75003 Paris, France
Faculté des sciences de Gabes, 6072 Gabes, Tunisie

Abstract

The problem of reconstructing bicolored images from their orthogonal projections is studied. A binary integer programming model is formulated to build small size bicolored images with smoothness properties. Since the problem is NP-complete, we provide an iterative approximation based on lagrangean relaxation.

Keyword: Discrete Tomography; Images Reconstruction; Lagrangean Relaxation

1 Introduction

Discrete Tomography (DT) deals with the reconstruction of digital images from their horizontal and vertical line sums. The projections of an image consist of the number of pixels of each color on each line. Digital images are most commonly represented by integer matrices. A k-colored image is a digital image where each cell is either uncolored or colored by one from a given set of k colors [3, 6]. The digital reconstruction of images arises on a number of applications, including industrial nondestructive testing [2] and medical imaging [8, 12, 10]. The reconstruction of monocolored images (black and white) is well known to be a polynomial time problem and it is equivalent to the binary matrix reconstruction problem [11]. Gardner et al. [6] proved that reconstructing k-colored image is NP-complete for $k > 5$. Chrobak and Dürr [3] extended the NP-completness result to the case $k = 3$. Recently, Dürr et al. [4] proved that the reconstruction of two colors image is also NP-complete. By this result, the reconstruction of k-colored image is polynomially solvable for one color and NP-complete for two or more colors.

We will mainly focus in this paper on the problem of reconstructing bicolored images from their orthogonal projections. As the problem is usually highly underdetermined and a large number of solutions may exist, we add smoothness proprieties. This condition is often considered in images processing to enforce local coherence with homogenous regions. This work is motivated by the pioneering work of Dürr et al. [4] showing that this problem is NP-complete. As far as the authors know, no paper of the literature reports computational experiments for reconstructing such images.

*fethi.jarray@cnam.fr

The remainder of this paper is organized as follows. In Section 2, we introduce some definitions and notations. In Section 3, we propose an integer program to build bicolored images. In Section 4, we develop a lagrangean approach to reconstruct bicolored images. In the last section, we present and discuss the numerical results.

2 Definitions and notation

For a bicolored image, we suppose that each cell is either uncolored, colored by a or colored by b. We denote h_i^a the number of colored cells by a in row i and v_j^a the number of colored cells by a in column j. Similarly, we define the entries h_i^b and v_j^b. The vectors $H^a = (h_1^a, \ldots, h_m^a)$ and $V^a = (v_1^a, \ldots, v_n^a)$ are respectively called the horizontal and the vertical projection of color a. Analogously, the vectors $H^b = (h_1^b, \ldots, h_m^b)$ and $V^b = (v_1^b, \ldots, v_n^b)$ are respectively called the horizontal and the vertical projection of color b. The bicolored image reconstruction problem can be posed as follows:

Instance: Four integral vectors: $H^a \in N^m, V^a \in N^n$, $H^b \in N^m$ and $V^b \in N^n$.
Question: Construct a bicolored image respecting the projections (H^a, V^a) for color a and (H^b, V^b) for color b.

3 Integer programming model

The bicolored reconstruction can be reformulated as an optimization problem. We introduce the binary decision variables x_{ij} and y_{ij} such that $x_{ij} = 1$ if the cell (i, j) is colored by a and $y_{ij} = 1$ if the cell (i, j) is colored by b. This gives us the following model:

$$IP0 \quad \left\{ \begin{array}{lll} min \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} y_{ij} & & \\ s.t. & & \\ \sum_{j=1}^{n} x_{ij} = h_i^a & i = 1, \ldots, m & (1) \\ \sum_{i=1}^{m} x_{ij} = v_j^a & j = 1, \ldots, n & (2) \\ \sum_{j=1}^{n} y_{ij} = h_i^b & i = 1, \ldots, m & (3) \\ \sum_{i=1}^{m} y_{ij} = v_j^b & j = 1, \ldots, n & (4) \\ x_{ij},\, y_{ij} \in \{0, 1\} & & \end{array} \right.$$

The constraints (1) and (2) guarantee the satisfaction of the orthogonal projections of color a. The constraints (3) and (4) ensure that y respects the orthogonal projections of color b. We ignore the exclusiveness constraint (at the most one color on a cell) and we search for a solution (x_{ij}, y_{ij}) minimizing the number of overlapping 1's between x and y. It is obvious that the reconstruction problem admits a solution if and only if the optimum objective function is null. The program $IP0$ can be extended to consider multicolored images.

In general, there is not a unique solution to the problem of reconstructing bicolored images from orthogonal projections. One way to resolve this ambiguity is to incorporate smoothness into the program $IP0$. More precisely, we want that each cell (i, j) has the same color as its 4 nearest neighbors cells $(i + 1, j)$, $(i, j + 1)$, $(i - 1, j)$ and $(i, j - 1)$. By adding this prior knowledge in the program $IP0$, we get the following program:

$$IP \quad \begin{cases} min \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}y_{ij} - \alpha\sum_{i=1}^{m}\sum_{j=1}^{n-1} x_{ij}x_{i,j+1} - \beta\sum_{i=1}^{m-1}\sum_{j=1}^{n} x_{ij}x_{i+1,j} \\ -\alpha\sum_{i=1}^{m}\sum_{j=1}^{n-1} y_{ij}y_{i,j+1} - \beta\sum_{i=1}^{m-1}\sum_{j=1}^{n} y_{ij}y_{i+1,j} \\ s.t. \\ (1),(2),(3),(4) \\ x_{ij},\, y_{ij} \in \{0,1\} \end{cases}$$

The objective function is composed by 5 terms. The first term is as before the number of overlapping 1's between x and y. The second term is the number of pairs of horizontally adjacent cells having color a. The third term is the number of pairs of vertically adjacent cells having color a. The last two terms are defined analogously. The predefined parameters α and β give the weight of each term and reflect smoothness.

Consider the classical linearization of program IP obtained by replacing the quadratic terms $x_{ij}y_{ij}$ by the 0-1 variables zxy_{ij}, $x_{ij}x_{i,j+1}$ by zhx_{ij} (h for horizontally adjacent), $x_{ij}x_{i+1,j}$ by zvx_{ij}, $y_{ij}y_{i,j+1}$ by zhy_{ij} and $y_{ij}y_{i+1,j}$ by zvy_{ij}. We get the following equivalent integer linear program with five additional constraints to ensure the equivalence between IPL and IP:

$$IPL \quad \begin{cases} min \sum_{i=1}^{m}\sum_{j=1}^{n} zxy_{ij} - \alpha\sum_{i=1}^{m}\sum_{j=1}^{n-1} zhx_{ij} - \beta\sum_{i=1}^{m-1}\sum_{j=1}^{n} zvx_{ij} \\ -\alpha\sum_{i=1}^{m}\sum_{j=1}^{n-1} zhy_{ij} - \beta\sum_{i=1}^{m-1}\sum_{j=1}^{n} zvy_{ij} \\ s.t. \\ (1),(2),(3),(4) \\ x_{ij} + y_{ij} - 1 \leq zxy_{ij} \quad i=1,\ldots,m;\ j=1,\ldots,n & (5) \\ zhx_{ij} \leq \frac{x_{ij}+x_{i,j+1}}{2} \quad i=1,\ldots,m;\ j=1,\ldots,n-1 & (6) \\ zvx_{ij} \leq \frac{x_{ij}+x_{i+1,j}}{2} \quad i=1,\ldots,m-1;\ j=1,\ldots,n & (7) \\ zhy_{ij} \leq \frac{y_{ij}+y_{i,j+1}}{2} \quad i=1,\ldots,m;\ j=1,\ldots,n-1 & (8) \\ zvy_{ij} \leq \frac{y_{ij}+y_{i+1,j}}{2} \quad i=1,\ldots,m-1;\ j=1,\ldots,n & (9) \\ x_{ij},\, y_{ij}, zxy_{ij}, zhx_{ij}, zhy_{ij}, zvx_{ij}, zvy_{ij} \in \{0,1\} \end{cases}$$

4 Lagrangean relaxation

The lagrangean decomposition techniques was successfully used to reconstruct convex binary matrices [5].

By using the lagrangean relaxation method, we transform the primal problem (IPL) into the following lagrangean relaxation problem, LR, where constraint (5) is relaxed using lagrangean multipliers λ_{ij}, (6) is relaxed using μhx_{ij}, (7) is relaxed using μvx_{ij}, (8) is relaxed using μhy_{ij} and (9) is relaxed using μvy_{ij}.

$$LR \quad \begin{cases} min \sum_{i=1}^{m}\sum_{j=1}^{n} C_{ij}x_{ij} + \sum_{i=1}^{m}\sum_{j=1}^{n} D_{ij}y_{ij} + \sum_{i=1}^{m}\sum_{j=1}^{n}(1-\lambda_{ij})zxyij \\ +\sum_{i=1}^{m}\sum_{j=1}^{n-1}(\mu hx_{ij} - \alpha)zhx_{ij} + \sum_{i=1}^{m-1}\sum_{j=1}^{n}(\mu vx_{ij} - \beta)zvx_{ij} \\ +\sum_{i=1}^{m}\sum_{j=1}^{n-1}(\mu hy_{ij} - \alpha)zhy_{ij} + \sum_{i=1}^{m-1}\sum_{j=1}^{n}(\mu vy_{ij} - \beta)zvy_{ij} \\ s.t. \\ (1),(2),(3),(4) \\ x_{ij},\, y_{ij}, zxy_{ij}, zhx_{ij}, zhy_{ij}, zvx_{ij}, zvy_{ij} \in \{0,1\} \end{cases}$$

All the lagrangean multipliers are nonnegative because the relaxed constraints are inequality constraints. The cost matrix C is defined by the following:

$$C \begin{cases} c_{1,1} = \lambda_{1,1} - 0.5\alpha\mu hx_{1,1} - 0.5\beta\mu vx_{1,1} \\ c_{1,n} = \lambda_{1,n} - 0.5\alpha\mu hx_{1,n-1} - 0.5\beta\mu vx_{1,n} \\ c_{m,1} = \lambda_{m,1} - 0.5\alpha\mu hx_{m,1} - 0.5\beta\mu vx_{m-1,1} \\ c_{m,n} = \lambda_{m,n} - 0.5\alpha\mu hx_{m,n-1} - 0.5\beta\mu vx_{m-1,n} \\ c_{1,j} = \lambda_{1,j} - 0.5\alpha\mu hx_{1,j} - 0.5\beta\mu vx_{1,j} - 0.5\alpha\mu hx_{1,j-1} \\ c_{m,j} = \lambda_{m,j} - 0.5\alpha\mu hx_{m,j} - 0.5\beta\mu vx_{m-1,j} - 0.5\alpha\mu hx_{m,j-1} \\ c_{i,1} = \lambda_{i,1} - 0.5\beta\mu vx_{i,1} - 0.5\alpha\mu hx_{i,1} - 0.5\beta\mu vx_{i-1,1} \\ c_{i,n} = \lambda_{i,n} - 0.5\beta\mu vx_{i,n} - 0.5\beta\mu vx_{i-1,n} - 0.5\alpha\mu hx_{i,n-1} \\ c_{i,j} = \lambda_{i,j} - 0.5\alpha\mu hx_{i,j} - 0.5\beta\mu vx_{i,j} - 0.5\beta\mu vx_{i-1,j} - 0.5\alpha\mu hx_{i,j-1} \end{cases}$$

where $i = 2, \ldots, m-1$ and $j = 2, \ldots, n-1$.

Analogously, we define the matrix cost D using the lagrangean parameters μhy and μvy instead of μhx and μvx.

4.1 Computation of the dual function

We establish the following theoretical lower bound:

Proposition 1 *A lower bound on the optimal value of IP is*

$$\underline{LB} = -\alpha \sum_{j=1}^{n-1} [min(v_j^a, v_{j+1}^a) + min(v_j^b, v_{j+1}^b)] - \beta \sum_{i=1}^{m-1} [min(h_i^a, h_{i+1}^a) + min(h_i^b, h_{i+1}^b)].$$

Proof 1 *The objective function of IP is* $\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}y_{ij} - \alpha\sum_{i=1}^{m}\sum_{j=1}^{n-1} x_{ij}x_{i,j+1} - \beta\sum_{i=1}^{m-1}\sum_{j=1}^{n} x_{ij}x_{i+1,j} - \alpha\sum_{i=1}^{m}\sum_{j=1}^{n-1} y_{ij}y_{i,j+1} - \beta\sum_{i=1}^{m-1}\sum_{j=1}^{n} yx_{ij}y_{i+1,j}$.

By constraint (2), we have $\sum_{i=1}^{m} x_{ij} = v_j^a$ *and* $\sum_{i=1}^{m} x_{i,j+1} = v_{j+1}^a$ *for columns j and j+1. Since $x_{ij} \leq 1$, we get* $\sum_{i=1}^{m} x_{ij}x_{i,j+1} \leq v_j^a$ *and* $\sum_{i=1}^{m} x_{ij}x_{i,j+1} \leq v_{j+1}^a$. *Thus* $\sum_{i=1}^{m} x_{ij}x_{i,j+1} \leq min(v_j^a, v_{j+1}^a)$ *and* $\sum_{i=1}^{m}\sum_{j=1}^{n-1} x_{ij}x_{i,j+1} \leq \sum_{j=1}^{n-1} min(v_j^a, v_{j+1}^a)$. *Hence the second term of the objective function is not less than* $-\alpha\sum_{j=1}^{n-1} min(v_j^a, v_{j+1}^a)$. *By considering pairs of consecutive rows, we deduce that the third term is not less than* $-\beta\sum_{i=1}^{m-1} min(h_i^a, h_{i+1}^a)$. *Analogously for color b, we deduce that the fourth term is not less than* $-\alpha\sum_{j=1}^{n-1} min(v_j^b, v_{j+1}^b)$ *and the last term is not less than* $-\beta\sum_{i=1}^{m-1} min(h_i^b, h_{i+1}^b)$.

Finally, since the first term is not negative, we deduce the lower bound: $\underline{LB} = -\alpha\sum_{j=1}^{n-1}[min(v_j^a, v_{j+1}^a) + min(v_j^b, v_{j+1}^b)] - \beta\sum_{i=1}^{m-1}[min(h_i^a, h_{i+1}^a) + min(h_i^b, h_{i+1}^b)]$.

Since LR is a minimization problem and the binary variables zxy, zhx, zvx, zhy and zvy do not appear in the constraints, we establish the following result:

Proposition 2 *For every optimal solution of LR, we have*
$zxy_{ij} = 1$ *if* $1 < \lambda_{ij}$, *otherwise* 0;
$zhx_{ij} = 1$ *if* $\mu hx_{ij} < \alpha$, *otherwise* 0;

$zvx_{ij} = 1$ *if* $\mu vx_{ij} < \beta$, *otherwise* 0;
$zhy_{ij} = 1$ *if* $\mu hy_{ij} < \alpha$, *otherwise* 0;
$zvy_{ij} = 1$ *if* $\mu vy_{ij} < \beta$, *otherwise* 0.

We denote by $w(LR)$ the optimal value of LR. We note that the colors are decoupled and the problem LR can be decomposed into two subproblems, one per color. For each color, the associate subproblem to solve has the following general form:

$$Q \quad \begin{cases} min \sum_{i=1}^{m} \sum_{j=1}^{n} w_{ij} z_{ij} \\ s.t. \\ \sum_{j=1}^{n} z_{ij} = h_i \quad i = 1, \ldots, m \\ \sum_{i=1}^{m} z_{ij} = v_j \quad j = 1, \ldots, n \\ z_{ij} \in \{0, 1\} \end{cases}$$

where $w_{ij} = C_{ij}, h_i = h_i^a, v_j = v_j^a$ for color a and $w_{ij} = D_{ij}, h_i = h_i^b, v_j = v_j^b$ for color b.

We note that the program Q is equivalent to a min-cost max-flow problem in a bipartite graph $G(R, C, E)$ with $|R| = m$, $|C| = n$ and $|E| = m \times n$ (see Figure 1). Each vertex $r_i \in R$ corresponds to row i and each vertex $c_j \in C$ corresponds to column j. Also, $E = \{(r_i, c_j) \forall r_i \in R, c_j \in C\}$. Each arc (r_i, c_j) has a unitary capacity and a cost equals w_{ij}. We add to $G(R, C, E)$ two nodes: a source s and a sink t. There is an arc from s to every row node r_i with capacity h_i and null cost. Similarly, there is an arc from every column node c_j to t with capacity v_j and null cost.

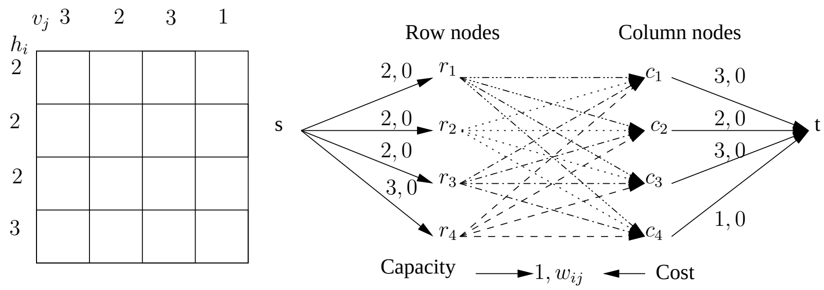

Figure 1: Min-cost max-flow equivalent problem to Q

We see that the computation of the dual function can be decomposed into reduced easy problems. It is well known that a min-cost max-flow problem in a bipartite graph can be solved in a polynomial time[1]. Since the dual function gives a lower bound of the optimal value of the initial problem IPL, the dual lagrangean problem is now

$$LD \quad \begin{cases} max\, w(LR) \\ s.t. \\ \mu xy_{ij}, \mu hx_{ij}, \mu hy_{ij}, \mu vx_{ij}, \mu vy_{ij} \geq 0 \end{cases}$$

We use the subgradient method to solve the problem LD since it is a non differentiable optimization problem. More details about the subgradient method can be found in [9]. In each iteration of the subgradient procedure, the lagrangean multipliers are updated by

$\lambda_{ij} = max(0, \lambda_{ij} + t(x_{ij} + y_{ij} - zxy_{ij} - 1));$
$\mu hx_{ij} = max(0, \mu hx_{ij} + t(zhx_{ij} - \frac{x_{ij}+x_{i,j+1}}{2}));$
$\mu vx_{ij} = max(0, \mu vx_{ij} + t(zhx_{ij} - \frac{x_{ij}+x_{i+1,j}}{2}));$
$\mu hy_{ij} = max(0, \mu hy_{ij} + t(zhy_{ij} - \frac{y_{ij}+y_{i,j+1}}{2}));$
$\mu vy_{ij} = max(0, \mu vy_{ij} + t(zhy_{ij} - \frac{y_{ij}+y_{i+1,j}}{2}))$;
where t is the step length.

4.2 Computation of primal solution

We use the dual solution to develop a heuristics to get a primal feasible solution. Let $(x, y, zhx, zvx, zhy, zvy)$ be a solution of the problem LR then (x, y) is a primal feasible solution for the program IP. Thus an upper bounds on the minimum value of IP is

$$LB = \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}y_{ij} - \alpha \sum_{i=1}^{m}\sum_{j=1}^{n-1}(x_{ij}x_{i,j+1} + y_{ij}y_{i,j+1}) - \beta \sum_{i=1}^{m-1}\sum_{j=1}^{n}(x_{ij}x_{i+1,j} + y_{ij}y_{i+1,j})$$

As a summary we can describe the reconstruction method as follows:

Reconstruction algorithm
Set $LB = -5m \times n$, $UB = m \times n$, $\rho = 0.55$, and $t = 1$.
Initialize the lagrangean multipliers.
Repeat
1. Solve the lagrangean problem LR and update LB.
2. Compute a primal solution and update UB.
3. Set $t = \rho t$
4. Update the lagrangean multipliers
Until Stop criteria is verified.

As in parametric algorithms, many experiments have been done to determine the best values of the parameters ρ and t. The best values are those given the least CPU time.

5 Computational results

We have implemented our algorithm in language C. The min-cost/max-flow models are solved by the $CS2$ network flow library developed by Andrew Goldberg [7]. All results are obtained using a PC with 3.8 GHz processor and 512 MBs of RAM.

To test our algorithm, we have used two sets of images. The first one consists of some created images having smoothness properties. The second set consists of random images of various sizes.

5.1 Images with smoothness properties

The images of this set are manually generated and are composed of small squares. Figure 2 illustrates an image exactly reconstructed by the algorithm for $\alpha = 0.5$ and $\beta = 0.5$.

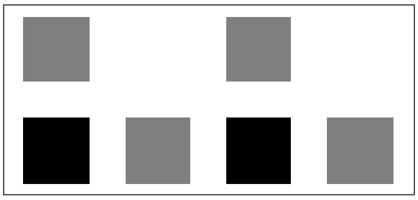

Figure 2: A 80×175 image exactly reconstructed with smoothness criteria; $(\alpha, \beta) = (0.5, 0.5)$

The step length of the subgradient method was reduced with a factor of 0.55 after each iteration. The stopping criteria is when $(UB - LB)/LB \leq 0.05$. The algorithm also terminates when there is no improvement in the gap in the last 10 iterations. We carry out several experiments to find the best initial values of the lagrangean multipliers. The best found values are $\lambda_{ij} = 1$, $\mu hx_{ij} = \alpha$, $\mu hy_{ij} = \alpha$ and $\mu vx_{ij} == \beta$ and $\mu vy_{ij} = \beta$. We remind that the subgradient method is very sensitive to the initial values of the multipliers and to the rule applied to control the step size.

The results of computational experiments are summarized in Table 1. The first column contains the size of the problem. The second column gives the relative gap between the upper bound and the value of the feasible solution. The third column gives the relative gap between the bound $\underline{LB}$ and the value of the feasible solution. CPU time is the time spent to compute the feasible solution and the upper bound. Note that the CPU time required for computing LB is negligible.

Table 1: Average of computational results

Size	$(UB - LB)/LB$	(UB-$\underline{\text{LB}}$)/$\underline{\text{LB}}$	CPU time
(80,80)	18	13	24
(85,85)	13.5	7	35
(80,83)	17	13.4	22
(96,96)	37.8	36.4	46
(175,80)	2	0	2
(175,80)	4.5	2.3	0.76
(175,80)	10.8	8.1	33
(175,80)	5.5	3.5	0.4

For the image number 5, the heuristics finds the optimal solution and the lower bound is reached. For other images, the algorithm finds an approximative solution and the gap between the upper bound and the lower bound is about 15%. The overall execution time of the algorithm is always small. We note that the bound $\underline{LB}$ is always better than the lower bound provided by the dual program. We notice also that the execution time depends on both the size of the images and the number of switching components.

5.2 Random images without smoothness properties

We have also implemented a program that randomly generates bicolored images. Each pixel has a uniform probability to be colored by color a and if it is not colored by a it has then a uniform probability to be colored by color b. As a summary we can describe the generation of bicolored images as follows:

Randomly generating $m \times n$ bicolored images

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cell (i, j) uncolored for $i = 1, \ldots, m$ and $j = 1, \ldots, n$

For $i = 1, \ldots, m$ **do**

For $j = 1, \ldots, n$ **do**

p= rand(0,1)

If $p = 1$ then cell (i, j) colored by color a

else q= rand(0,1)

If $q = 1$ then cell (i, j) colored by color b

end If

end do

end do

The function rand(0,1) randomly selects 0 or 1. For this class of images, we set $\alpha = 0$ and $\beta = 0$ since random images rarely have smoothness properties. For $\lambda_{ij} = 0$, by Proposition 2, we have $zxy_{ij} = 0$ and the optimal objective value of LR is null. Hence the algorithm find the tightest lower bound in one iteration. However, the upper bound is generally far from the lower bound and the approximative solution is not of good quality.

6 Conclusion

In this paper, we have provided an iterative algorithm to approximate bicolored images from orthogonal projections taking into account a smoothness criteria. Our approach is based on lagrangean relaxation and uses min-cost max-flows algorithms for solving the reduced problems.

Although we have used a few number of test images, we believe that the computational results are very encouraging as they show that the problem can be solved to near optimality very fast.

Promising future research directions include colored images with more than two colors and more than two directions of projections. Also investigating the problem under more constraints and automatically generating smooth colored images are possible research directions.

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