On the difference between solutions of discrete tomography problems II

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Abstract

We consider the problem of reconstructing binary images from their horizontal and vertical projections. It is known that the projections do not necessarily determine the image uniquely. In a previous paper it was shown that the symmetric difference between two solutions (binary images that satisfy the projections) is at most $4\alpha\sqrt{2N}$. Here N is the sum of the projections in one direction (i.e. the size of the image) and α is a parameter depending on the projections. In this paper we give a lower bound: for each set of projections that has at least two solutions, we construct two solutions that have a symmetric difference of at least $2\alpha+2$. We also show that this is the best possible.

1 Introduction

An important problem in discrete tomography is to reconstruct a binary image on a lattice from given projections in lattice directions [6, 7]. Each point of a binary image has a value equal to zero or one. The line sum of a line through the image is the sum of the values of the points on this line. The projection of the image in a certain direction consists of all the line sums of the lines through the image in this direction.

For any set of more than two directions, the problem of reconstructing a binary image from its projections in those directions is NP-complete [5]. For exactly two directions, the horizontal and vertical ones, say, it is possible to reconstruct an image in polynomial time. Already in 1957, Ryser described an algorithm to do so [8]. He also characterised the set of projections that correspond to a unique binary image. Suppose F is uniquely determined and has row sums r_1, r_2, $\dots$, r_m. For each j with $1 \leq j \leq \max_i r_i$ we can count the number $\#\{l : r_l \geq j\}$ of row sums that are at least j. Then these numbers are exactly the non-zero column sums of F (in some order). See also [6, Theorem 1.7]).

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Alpers et al. [1, 2] studied the possible difference between a uniquely determined image and a second image with almost the same projections as the first one. Their results were generalised in [3] and the same ideas were used to study the difference between two solutions of the same set of projections in [4]. We give an overview of the main theorems here.

Consider given row sums $\mathcal{R} = (r_1, r_2, \dots, r_m)$ and column sums $\mathcal{C} = (c_1, c_2, \dots, c_n)$, and assume that there exists at least one binary image with exactly these line sums. Define $\mathcal{V} = (v_1, v_2, \dots, v_n)$ as $v_j = \#\{l : r_l \geq j\}$ for $1 \leq j \leq n$. Let F_1 the uniquely determined binary image with row sums $\mathcal{R}$ and column sums $\mathcal{V}$. Let $N = \sum_{i=1}^{m} r_i = \sum_{j=1}^{n} c_j$. Furthermore define the integer

$$\alpha(\mathcal{R}, \mathcal{C}) = \frac{1}{2} \sum_{j=1}^{n} |c_j - v_j|.$$

The parameter α indicates how close the line sums $(\mathcal{R}, \mathcal{C})$ are to line sums that uniquely determine an image. In particular, $\alpha = 0$ if and only if there is exactly one binary image with line sums $(\mathcal{R}, \mathcal{C})$. Intuitively, the larger α, the more possibilities there are for images that satisfy the line sums.

Alpers et al. [1, 2] proved that if F_2 is an image with line sums $(\mathcal{R}, \mathcal{C})$ and $\alpha(\mathcal{R}, \mathcal{C}) = 1$, then the size of the symmetric difference between F_2 and the uniquely determined set F_1 is bounded:

$$|F_1 \triangle F_2| \leq \sqrt{8N+1} - 1.$$

In [3] we generalised this to larger values of $\alpha(\mathcal{R}, \mathcal{C})$. For an image F_2 with line sums $(\mathcal{R}, \mathcal{C})$, write $\alpha = \alpha(\mathcal{R}, \mathcal{C})$ and let $p = |F_1 \cap F_2|$. Then

$$|F_1 \triangle F_2| \leq 2\alpha + 2(\alpha + p)\log(\alpha + p)$$

and

$$|F_1 \triangle F_2| \leq \alpha\sqrt{8N+1} - \alpha.$$

The first bound is asymptotically sharp when α is large compared to p. The second bound is better when α is small compared to p. We have a family of examples in which this bound is achieved up to a factor $\sqrt{\alpha}$.

Using these results, we can also consider the difference between two images with the same line sums. This must be bounded by twice the upper bound for the difference between one of these images and a uniquely determined image. Hence we get the following result from [4].

Suppose F_2 and F_3 are two images with the same line sums $(\mathcal{R}, \mathcal{C})$. Write $\alpha = (\mathcal{R}, \mathcal{C})$. Then

$$|F_2 \triangle F_3| \leq 2\alpha\sqrt{8N+1} - 2\alpha.$$

As with the previous bound, there are examples for which this bound is only off by a factor $\sqrt{\alpha}$.

In this paper we consider the complementary problem: find the best lower bound for the symmetric difference between two solutions that you can at least achieve given a set of projections? For each set of projections that has at least two solutions, we construct two solutions that have a symmetric difference of at least $2\alpha + 2$. We also show that this bound is sharp.

2 Definitions and notation

Let F be a finite subset of $\mathbb{Z}^2$ with characteristic function χ. (That is, $\chi(x, y) = 1$ if $(x, y) \in F$ and $\chi(x, y) = 0$ otherwise.) For $i \in \mathbb{Z}$, we define *row* i as the set $\{(x, y) \in \mathbb{Z}^2 : x = i\}$. We call i the index of the row. For $j \in \mathbb{Z}$, we define *column* j as the set $\{(x, y) \in \mathbb{Z}^2 : y = j\}$. We call j the index of the column. Following matrix notation, we use row numbers that increase when going downwards and column numbers that increase when going to the right.

The *row sum* r_i is the number of elements of F in row i, that is $r_i = \sum_{j\in\mathbb{Z}} \chi(i, j)$. The *column sum* c_j of F is the number of elements of F in column j, that is $c_j = \sum_{i\in\mathbb{Z}} \chi(i, j)$. We refer to both row and column sums as the *line sums* of F. We will usually only consider finite sequences $\mathcal{R} = (r_1, r_2, \ldots, r_m)$ and $\mathcal{C} = (c_1, c_2, \ldots, c_n)$ of row and column sums that contain all the nonzero line sums. We may assume without loss of generality that $r_1 \geq r_2 \geq \ldots \geq r_m$ and $c_1 \geq c_2 \geq \ldots \geq c_n$.

Given sequences of integers $\mathcal{R} = (r_1, r_2, \ldots, r_m)$ and $\mathcal{C} = (c_1, c_2, \ldots, c_n)$, we say that $(\mathcal{R}, \mathcal{C})$ is *consistent* if there exists a set F with row sums $\mathcal{R}$ and column sums $\mathcal{C}$. We say that the line sums $(\mathcal{R}, \mathcal{C})$ uniquely determine such a set F if the following property holds: if F' is another subset of $\mathbb{Z}^2$ with line sums $(\mathcal{R}, \mathcal{C})$, then $F' = F$. In this case we call F *uniquely determined*.

We will now define a *uniquely determined neighbour* of a set F. This is a uniquely determined set that is in some sense the closest to F. See also [4, Section 4].

Definition 1. *Suppose F has row sums $r_1 \geq r_2 \geq \ldots \geq r_m$ and column sums $c_1 \geq c_2 \geq \ldots \geq c_n$. For $1 \leq j \leq n$, let $v_j = \#\{l : r_l \geq j\}$. Then the row sums r_1, r_2, $\ldots$, r_m and column sums v_1, v_2, $\ldots$, v_n uniquely determine a set F_1, which we will call the* uniquely determined neighbour of *F.*

Note that if F' is another set with row sums r_1, r_2, $\ldots$, r_m and column sums c_1, c_2, $\ldots$, c_n, then F_1 is a uniquely determined neighbour of F' if and only if it is a uniquely determined neighbour of F. Hence F_1 only depends on the row and column sums and not on the choice of the set F. We will therefore also speak about the *uniquely determined neighbour corresponding to the line sums* $(\mathcal{R}, \mathcal{C})$, without mentioning the set F.

Suppose line sums $\mathcal{R} = (r_1, r_2, \ldots, r_m)$ and $\mathcal{C} = (c_1, c_2, \ldots, c_n)$ are given, where $r_1 \geq r_2 \geq \ldots \geq r_m$ and $c_1 \geq c_2 \geq \ldots \geq c_n$. Let the uniquely determined neighbour corresponding to $(\mathcal{R}, \mathcal{C})$ have column sums $v_1 \geq v_2 \geq \ldots \geq v_n$. Then we define

$$\alpha(\mathcal{R},\mathcal{C}) = \frac{1}{2}\sum_{j=1}^{n}|c_j - v_j|.$$

Note that $\alpha(\mathcal{R},\mathcal{C})$ is always an integer, since $2\alpha(\mathcal{R},\mathcal{C})$ is congruent to

$$\sum_{j=1}^{n}(c_j + v_j) = \sum_{j=1}^{n} c_j + \sum_{j=1}^{n} v_j = 2\sum_{j=1}^{n} c_j \equiv 0 \mod 2.$$

Consider a set F with line sums $(\mathcal{R},\mathcal{C})$ and its uniquely determined neighbour F_1. Let $\alpha = \alpha(\mathcal{R},\mathcal{C})$. It was proved in [3, Lemma 4] that the symmetric difference $F \triangle F_1$ consists of α *staircases.* In this paper we will only use staircases of length 2, which we will define below. For the general definition of a staircase, see [3] or [4].

Definition 2. *A* staircase of length 2 *in $F \triangle F_1$ is a pair of points (p_1, p_2) in $\mathbb{Z}^2$ such that*

- *p_1 and p_2 are in the same row,*
- *p_1 is an element of $F \backslash F_1$,*
- *p_2 is an element of $F_1 \backslash F$.*

3 Main result

Suppose we are given row sums $\mathcal{R} = (r_1, r_2, \ldots, r_m)$ and column sums $\mathcal{C} = (c_1, c_2, \ldots, c_n)$, where $r_1 \geq r_2 \geq \ldots \geq r_m$ and $c_1 \geq c_2 \geq \ldots \geq c_n$. Assume that the line sums are consistent but do not uniquely determine a set F (hence at least two different sets with these line sums exist). Let $\alpha = \alpha(\mathcal{R},\mathcal{C})$.

In [4] it was shown that for all F_2 and F_3 satisfying these line sums, we have

$$|F_2 \triangle F_3| \leq 4\alpha\sqrt{2|F_2|}.$$

One may wonder how close we can get to achieving this bound. Our theorem shows that we can construct two sets that have a symmetric difference of size at least $2\alpha + 2$.

Theorem 1. *Let row sums $\mathcal{R} = (r_1, r_2, \ldots, r_m)$ and column sums $\mathcal{C} = (c_1, c_2, \ldots, c_n)$ be given, where $r_1 \geq r_2 \geq \ldots \geq r_m$ and $c_1 \geq c_2 \geq \ldots \geq c_n$. Assume that the line sums are consistent but do not uniquely determine a set F. Let $\alpha = \alpha(\mathcal{R},\mathcal{C})$. Then there exist sets F_2 and F_3 with these line sums such that*

$$|F_2 \triangle F_3| \geq 2\alpha + 2.$$

This bound is sharp: for each $\alpha \geq 1$ there are line sums $(\mathcal{R},\mathcal{C})$ with $\alpha = \alpha(\mathcal{R},\mathcal{C})$ such that for any F_2 and F_3 satisfying these line sums we have $|F_2 \triangle F_3| \leq 2\alpha + 2$.

4 Proof

In this entire section, the row sums $\mathcal{R} = (r_1, r_2, \ldots, r_m)$ and column sums $\mathcal{C} = (c_1, c_2, \ldots, c_n)$ with $r_1 \geq r_2 \geq \ldots \geq r_m$ and $c_1 \geq c_2 \geq \ldots \geq c_n$ are fixed. Furthermore, F_1 is the uniquely determined neighbour corresponding to $(\mathcal{R}, \mathcal{C})$, and $\alpha = \alpha(\mathcal{R}, \mathcal{C})$. We denote the column sums of F_1 by $v_1 \geq v_2 \geq \ldots \geq v_n$.

The proof is constructive. We will construct F_2 and F_3 such that they have the desired property. We will do this by changing a set F step by step. Only the final result of the construction will be called F_2 (or F_3); the intermediate sets will always be called F or F'. In Section 5 the construction is illustrated by an example.

Let the columns j for which $v_j > c_j$ have indices $j_1 \leq j_2 \leq \ldots \leq j_\alpha$, where each such j occurs $v_j - c_j$ times. Similarly, let the columns i for which $v_i < c_i$ have indices $i_1 \leq i_2 \leq \ldots \leq i_\alpha$, where each such i occurs $c_i - v_i$ times. Define a *column pair* as a pair (i_t, j_t). The consistency of the given line sums assures that $i_t > j_t$ for all t. For convenience, define $i_0 = j_0 = 0$ and $i_{\alpha+1} = j_{\alpha+1} = n + 1$.

We will construct both F_2 and F_3 by starting from $F = F_1$ and then for each t moving an element of F from column j_t to column i_t in the same row. After we have done that for $t = 1, 2, \ldots, \alpha$, the row sums of F have not changed, while the columns of F have changed from v_1, v_2, $\ldots$, v_n to c_1, c_2, $\ldots$, c_n. The symmetric difference $F_1 \triangle F$ then consists of α staircases of length 2. Each staircase is confined to a single row and corresponds to a column pair (i_t, j_t). We will show that we have a certain freedom in choosing the staircases.

Suppose we have moved an element for each of the column pairs (i_1, j_1), (i_2, j_2), $\ldots$, (i_{t-1}, j_{t-1}), where $t \geq 1$. The resulting set is called F and has column sums c'_1, c'_2, $\ldots$, c'_n. Now we want to move an element from column j_t to column i_t. For this we need a row l such that the point $(l, j_t) \in F$ and $(l, i_t) \notin F$. We have $c'_{j_t} > c_{j_t} \geq c_{i_t} > c'_{i_t}$, so $c'_{j_t} \geq c'_{i_t} + 2$. Hence there must be at least two rows that contain an element of F in column j_t but not in column i_t. This proves the existence of such a row l, and in fact at least two choices for l are possible. Now we move the element (l, j_t) to (l, i_t). The row sums of F do not change, while the column sum of column j_t decreases by one and the column sum of column i_t increases by one.

We construct both F_2 and F_3 using the construction above. First we construct F_2, making arbitrary choices for the rows in which we move elements. Then we will construct F_3. For this we let the choices in the construction depend on F_2, in a way we will describe below.

Let P_1, P_2, $\ldots$, P_r be the *distinct* column pairs, where P_h has multiplicity k_h: the column pair P_1 is equal to each of the pairs $(i_1, j_1), \ldots, (i_{k_1}, j_{k_1})$, the column pair P_2 is equal to each of the pairs (i_{k_1+1}, j_{k_1+1}), $\ldots$, $(i_{k_1+k_2}, j_{k_1+k_2})$, and so on. We have $k_1 + k_2 + \cdots + k_r = \alpha$. For two consecutive column pairs (i_t, j_t) and (i_{t+1}, j_{t+1}) that are not equal we have $i_{t+1} > i_t$, $j_{t+1} \geq j_t$ or $i_{t+1} \geq i_t$, $j_{t+1} > j_t$, so the second pair contains a column that did not occur in any of the previous pairs. This means that in $P_1, \ldots, P_r$ at least $r+1$ different columns are involved. For each P_h, we denote one of the columns in P_h as the *final* column of P_h in the following way.

- If one of the columns in P_h also occurs in P_{h+1}, then the other does not occur in $P_{h+1}, \ldots, P_r$. We call the latter the final column of the pair.
- If both columns in P_h do not occur in $P_{h+1}, \ldots, P_r$, and one of the columns occurs in P_{h-1}, then the other does not occur in $P_1, \ldots, P_{h-1}$. We call the former the final column of the pair.
- If both columns in P_h do not occur in $P_1, \ldots, P_{h-1}$ nor in $P_{h+1}, \ldots, P_r$, then we arbitrarily pick one of the columns in P_h and call it the final column of the pair.

By definition, we have the following properties: the final column of P_h does not occur in $P_{h+1}, \ldots, P_r$, and if the other column does not occur in $P_{h+1}, \ldots, P_r$ either, then the latter column only occurs in P_h.

Our goal is to construct F_3 in such a way that, for all h, in the final column of P_h the symmetric difference between F_2 and F_3 is at least $2k_h$, while in any other column that occurs in one of the column pairs the symmetric difference between F_2 and F_3 is at least 2. (There is at least one such a column, since there are exactly r final columns, while at least $r+1$ columns are involved in the column pairs.) If we can achieve that, then we have

$$|F_2 \triangle F_3| \geq 2k_1 + 2k_2 + \ldots + 2k_r + 2 = 2\alpha + 2.$$

To achieve this, we choose the rows in which elements are moved for all equal column pairs at once. First we choose the rows for all pairs equal to P_1, then for all pairs equal to P_2, and so on.

Let t be the index of the last column pair in a sequence of k equal column pairs

$$(i_{t-k+1}, j_{t-k+1}) = (i_{t-k+2}, j_{t-k+2}) = \ldots = (i_t, j_t),$$

where $(i_{t-k}, j_{t-k}) \neq (i_{t-k+1}, j_{t-k+1})$ and $(i_t, j_t) \neq (i_{t+1}, j_{t+1})$. Suppose we have moved elements already for the column pairs $(i_1, j_1), \ldots, (i_{t-k}, j_{t-k})$. Call the resulting set F, with column sums $c'_1, \ldots, c'_n$. Assume that i_t is the final column of (i_t, j_t) (the case where j_t is the final column, is analogous). So we have $i_t \neq i_{t+1}$. Also, we have one of the following two properties:

(A) $j_t = j_{t+1}$,

(B) $j_t \neq j_{t+1}$, and $j_{t-k} \neq j_{t-k+1}$.

As this is the last time column i_t occurs, we need to choose the rows in such a way that by moving the elements of F the symmetric difference between F and

F_2 in this column becomes at least $2k$. Also, in case (B) we want the symmetric difference in column j_t to be at least 2.

Since we need to move k elements out of column j_t into column i_t, we have $c'_{j_t} \geq c_{j_t} + k \geq c_{i_t} + k \geq c'_{i_t} + 2k$, so there are at least $2k$ rows l such that $(l, j_t) \in F$ and $(l, i_t) \notin F$. Let R be the set of those $2k$ rows. (If there are more than $2k$ possible rows, then pick $2k$ of them.) We distinguish between two cases.

Case 1. Suppose there are k different rows l in R such that $(l, i_t) \notin F_2$. Then we move elements from column j_t to column i_t in each of those k rows. Call the resulting set F'. We have $(l, i_t) \in F' \backslash F_2$ for k different values of l. The number of elements of F' in column i_t must be equal to the number of elements of F_2 in column i_t, so there are also k different values of l for which $(l, i_t) \in F_2 \backslash F'$. Hence the symmetric difference between F' and F_2 in this column is at least $2k$.

In case (A) we are now done, as column j_t will be handled in a later column pair. Suppose we are in case (B). The column j_t only occurs in the column pairs (i_{t-k+1}, j_{t-k+1}), $\ldots$, (i_t, j_t), which are all equal. If for a row l we have $(l, i_t) \notin F_2$, then in the construction of F_2 this row was not used for a staircase corresponding to the column pair (i_t, j_t) (or one of the equal ones), so we must have $(l, j_t) \in F_2$. Hence after moving elements we have k different values of l for which $(l, j_t) \in F_2 \backslash F'$. So in column j_t the symmetric difference between F' and F_2 is at least $2k \geq 2$.

Case 2. Suppose there are at least $k+1$ different rows l in R such that $(l, i_t) \in F_2$. Let R' be a set of $k+1$ of those rows. Pick one of the rows in R' and call it l_0. Let R'' consist of l_0 and the $k-1$ other rows in $R \backslash R'$ (for which it may or may not hold that $(l, i_t) \in F_2$). Move elements from column j_t to column i_t in each of the k rows in R''. Call the resulting set F'. Then for all k rows l in $R \backslash R''$ we have $(l, i_t) \in F_2 \backslash F'$. Similarly to above, we find that the symmetric difference between F' and F_2 in column i_t is at least $2k$.

Again, in case (A) we are done. Suppose we are in case (B). As column j_t only occurs in the column pairs (i_{t-k+1}, j_{t-k+1}), $\ldots$, (i_t, j_t), which are all equal, for at most k rows l in R we have $(l, j_t) \notin F_2$. This means that we can choose l_0 above in such a way that $(l_0, j_t) \in F_2$. After moving the elements, we then have $(l_0, j_t) \in F_2 \backslash F'$. So the symmetric difference between F' and F_2 in column j_t is at least 2.

At least one of Case 1 and Case 2 above must hold, since there are $2k$ rows in R. Therefore we have finished the construction of F_2 and F_3 such that $F_2 \triangle F_3 \geq 2\alpha + 2$.

We will now prove the second part of Theorem 1 by giving a family of examples for which the bound of $2\alpha + 2$ is sharp. Let $s \geq 1$ be an integer. Take $m = n = s + 1$ and let all row and column sums be equal to 1. These line sums are consistent. The uniquely determined neighbour F_1 has column sums $v_1 = s + 1$, $v_2 = v_3 = \ldots = v_{s+1} = 0$, so $\alpha = s$.

Suppose F_2 and F_3 satisfy the given row and column sums. We have $|F_2| =$

$|F_3| = s + 1$, hence

$$|F_2 \triangle F_3| \leq |F_2| + |F_3| = 2(s+1) = 2\alpha + 2.$$

This completes the proof of Theorem 1. □

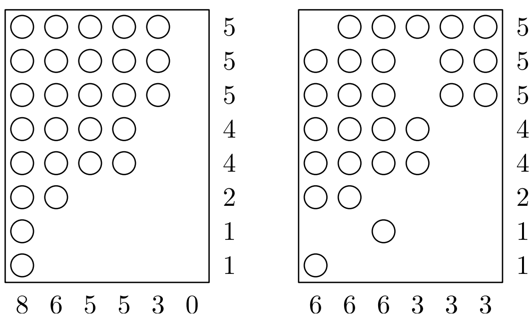
Remark 1. *There do not seem to be very many examples for which the bound of* $2\alpha+2$ *is sharp. In particular, they all seem to have* $m = n = \alpha+1$*. However, even in more general cases, when* α *is much larger than* n*, the bound is not very far off. Take for example* $m = n$ *and let all line sums be equal to* k*, where* $k \leq \frac{1}{2}n$*. The uniquely determined neighbour has* k *column sums equal to* n *and* $n-k$ *column sums equal to* 0*, so* $\alpha = k(n-k)$*. As* $n - k \geq \frac{1}{2}n$*, we have* $\alpha \geq \frac{1}{2}kn$*. Suppose* F_2 *and* F_3 *satisfy the given row and column sums, then* $|F_2| = |F_3| = kn$*, hence*

$$|F_2 \triangle F_3| \leq |F_2| + |F_3| = 2kn \leq 4\alpha.$$

5 Example

We illustrate the construction in the proof by an example. Let row sums $(5,5,5,4,4,2,1,1)$ and column sums $(6,6,6,3,3,3)$ be given. The uniquely determined neighbour F_1 has the same row sums, but column sums $(8,6,5,5,3,0)$ (see Figure 1(a)). From this we derive that $\alpha = 4$ and that the four column pairs are $(1,3)$, $(1,6)$, $(4,6)$ and $(4,6)$.

To construct F_2, we move one element from column 1 to column 3, one element from column 1 to column 6, and two elements from column 4 to column 6. We choose the rows to move elements in arbitrarily from the available rows. If we choose rows 7, 1, 2 and 3 respectively, we arrive at the set F_2 shown in Figure 1(b).

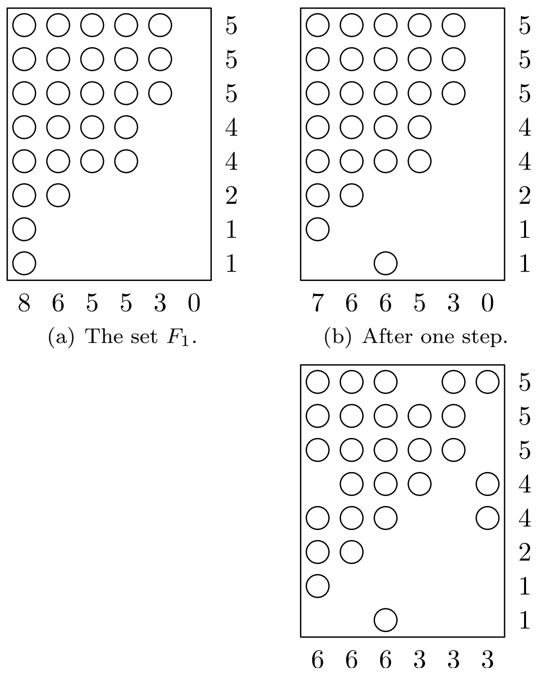


(a) The set F_1 with its row and column sums.

(b) The set F_2 with its row and column sums.

Figure 1

Now we construct the set F_3 step-by-step, following the proof of the theorem. We start with F_1, shown again in Figure 2(a). For the first column pair, we need to move an element from column 1 to column 3. The available rows are 6, 7 and 8. We need only two of them, so let us take $R = \{7, 8\}$. Column 3 is the final column in this column pair, so in this column we need to make sure that we achieve a symmetric difference of at least 2 with F_2. We have $(8,3) \notin F_2$, so we are in case 1 and we pick row 8 for our staircase. Hence we delete the element $(8,1)$ and add the element $(8,3)$. The new situation is shown in Figure 2(b).



(a) The set F_1. (b) After one step. (c) After two steps. (d) The set F_3.

Figure 2: The construction of the set F_3.

The next column pair is $(1, 6)$. Now column 1 is the final column pair, and all rows except row 8 are available. We are again in case 1 and pick row 4. Figure 2(c) shows the new situation, after deleting $(4, 1)$ and adding $(4, 6)$.

Finally, we need to move two elements at once for the column pair $(4, 6)$, which occurs twice. Column 6 is the final column, so we need to achieve a symmetric difference of at least 4 with F_2 in this column. We also need a symmetric difference of at least 2 in column 4 (case (B)). We have $R = \{1, 2, 3, 5\}$. As $(1, 8)$, $(2, 8)$ and $(3, 8)$ are all elements of F_2, we are in case 2. We have $R' = \{1, 2, 3\}$ and we need to find an $l_0 \in R'$ such that $(l_0, 4) \in F_2$. The only possible choice is $l_0 = 1$. We find $R'' = \{1, 5\}$, so we delete $(1, 4)$ and $(5, 4)$, and we add $(1, 6)$ and $(5, 6)$. This completes the construction of F_3. The resulting set is shown in Figure 2(d).

The construction guarantees that the symmetric difference between F_2 and F_3 is at least $2\alpha + 2 = 10$, but we have in fact constructed two sets with symmetric difference 14.

References

[1] A. Alpers, *Instability and stability in discrete tomography*, Ph.D. thesis, Technische Universität München, ISBN 3-8322-2355-X, Shaker Verlag, Aachen (2003).

[2] A. Alpers, S. Brunetti, Stability results for the reconstruction of binary pictures from two projections, *Image and Vision Computing* 25 (2007) 1599-1608.

[3] B.E. van Dalen, Stability results for two directions in discrete tomography, *Discrete Math.* 309 (2009) 3905-3916.

[4] B.E. van Dalen, On the difference between solutions of discrete tomography problems, to be published in *Journal of Combinatorics and Number Theory* 1 (2009) 16 pp.

[5] R.J. Gardner, P. Gritzmann, D. Prangenberg, On the computational complexity of reconstructing lattice sets from their X-rays, *Discrete Math.* 202 (1999) 45-71.

[6] G.T. Herman, A. Kuba, editors, *Discrete Tomography: Foundations, Algorithms and Applications*, Birkhäuser, Boston (1999).

[7] G.T. Herman, A. Kuba, editors, *Advances in Discrete Tomography and Its Applications*, Birkhäuser, Boston (2007).

[8] H.J. Ryser, Combinatorial properties of matrices of zeros and ones, *Canad. J. Math.* 9 (1957) 371-377.