Reconstructing convex permutominoes

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Abstract

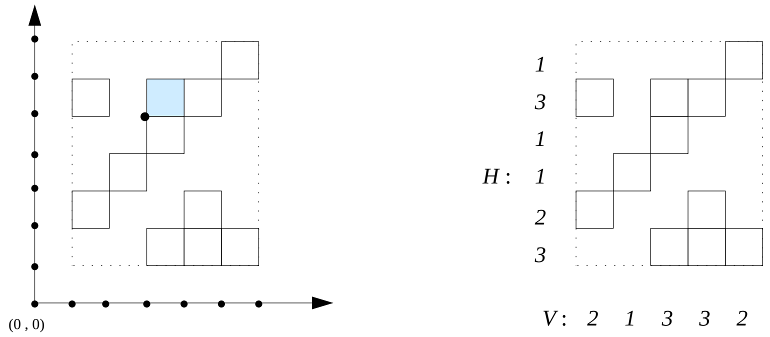
This paper studies the tomographical aspects of a new class of polyominoes, called permutominoes, which are defined by means of a pair of permutations. We inspect the classical problems of uniqueness and reconstruction on these objects, and in particular, using some combinatorial properties of their horizontal and vertical projections, we furnish a polynomial time reconstruction algorithm.

Keywords: Discrete Tomography, Reconstruction algorithm, Computational complexity.

1 Definitions and Preliminaries

Let our environment be the integer lattice $\mathbb{Z} \times \mathbb{Z}$. We define a *cell* to be a unit square whose vertices have integer coordinates, and a *set of cells* to be a finite subset S of them considered up to translations. In the sequel, without loss of generality, we consider the south-west corner of the minimal bounding rectangle of each set of cells to be placed in position $(1,1)$, we indicate each cell c with the position (i,j) of its south-west corner, say $c(i,j)$, and we number the columns and the rows of each set S with the common value of the abscissa and of the ordinate of its cells, respectively (see Fig. 1 (a)). Clearly the cell $c(i,j)$ of S will belong to its i-th column and to its j-th row, as expected.

To each set of cells S having m columns and n rows, we can also associate two integer vectors $V = (v_1, \ldots, v_m)$ and $H = (h_1, \ldots, h_n)$ such that, for each $1 \leq i \leq m$, $1 \leq j \leq n$, v_i and h_j are the number of cells of S which lie on column i and row j, respectively (see Fig. 1 (b)). The vectors H and V are called the horizontal and vertical projections of S, respectively. Given two vectors H and V, we will denote by $\mathcal{U}(H,V)$ the class of sets of cells having H and V as projections; if $\mathcal{U}(H,V)$ is non-void, then H and V are said to be *consistent.* A



(a) (b)

Figure 1: (a) A finite set of cells whose minimal bounding rectangle's south-west corner is in position $(1, 1)$. The highlighted cell is in position $(3, 5)$. (b) Its horizontal and vertical projections.

discrete set S is *unique* (with respect to H and V) if $\mathcal{U}(H, V) = \{S\}$. In such a case H and V are said to be *unique*.

Given a couple of projections, the following problems fit in the research area called *Discrete Tomography*:

i. deciding the existence of a discrete set which is consistent with the two projections (*consistency problem*),

ii. establishing if such a set is unique (*uniqueness problem*),

iii. furnishing an algorithm for its reconstruction (*reconstruction problem*).

Discrete Tomography is the part of the Computerized Tomography that deals, in general, with the retrieval of geometrical properties of finite sets of cells from the knowledge of a set of their projections.

However, in most cases a given couple of projections is common to a large class of discrete sets which may be very different in shape and which may present different geometrical properties. So, the problem of retrieving information about a particular discrete structure from its projections is ill-posed, which means that, in a strict mathematical setting, we are not able to solve it.

To handle this problem one can use some a priori information about the sets which have to be reconstructed in order to limit the class of possible solutions, and to guide towards them the reconstruction process. Usually the imposed constraints are related to connectivity or convexity properties. In particular, this last constraint has been deeply investigated in the past few years: efficient algorithms for reconstructing convex sets from two projections have been furnished which either rely on a coding of each convex set by means of 2-SAT formulas (cf. [4]), or which stress geometrical aspects strictly related to convexity (cf. [1]).

In this paper we concentrate on a particular class of discrete sets called *convex permutominoes*, recently introduced by F. Incitti in [10], while studying the $\tilde{R}$-polynomials (related with the Kazhdan-Lusztig R-polynomials) associated with a pair (π_1, π_2) of permutations, and which constitutes a subset of the well known class of *polyominoes* (cf. [8]). Recently, convex permutominoes have been studied from an algebraic and a combinatorial point of view, showing interesting connections with permutations (cf. [2]). In [5] it was shown that the the number of *convex permutominoes* of size $n+1$ is:

$$2\,(n+3)\,4^{n-2} \, - \, \frac{n}{2}\binom{2n}{n} \qquad n \geq 1.$$

Our aim here is to consider the tomographical aspects of convex permutominoes in order to furnish a polynomial time algorithm to reconstruct them from the horizontal and vertical projections, and, successively, to solve the related uniqueness problem.

The paper is organized as follows: in Section 2 we recall the main definitions on polyominoes and permutominoes, and provide a useful characterization of convex permutominoes by means of a property of their boundaries. In Section 3 we start the reconstructing process of a convex permutomino P by detecting the positions of its elements in columns 1 and n. We also show that only two different configurations of such elements are possible. Finally in Section 4 we conclude the reconstruction process, and we compute its computational complexity. A last remark about the number of different convex permutominoes consistent with the projections solves the related uniqueness problem and concludes the section.

2 Convex permutominoes

A polyomino is a set of cells whose interior is connected (see Fig. 2 (a)). A polyomino is said to be *horizontally convex*, briefly *h-convex* [resp. *vertically convex*, briefly *v-convex*] if every one of its rows [resp. columns] is connected (see Fig. 2 (b)). A polyomino is said to be *hv-convex*, or simply *convex*, if it is both h-convex and v-convex (see Fig. 2 (c)). From now on we will consider these polyominoes with "no holes", i.e. such that their boundary is made exactly of one component.

Let P be a polyomino and let $\mathcal{A} = \{A_1, \ldots, A_{2(r+1)}\}$ be the set of all its vertices (points in $\mathbb{Z} \times \mathbb{Z}$) ordered in a clockwise sense starting from the leftmost vertex having minimal ordinate, with $r > 0$. We say that P is a *permutomino* if:

$i)$ for each abscissa $i = 1, 2, \ldots, n+1$, there exists exactly one point of the set $\mathcal{P}_1 = \{A_1, A_3, \ldots, A_{(2r+1)}\}$ [resp. $\mathcal{P}_2 = \{A_2, A_4, \ldots, A_{2r+2}\}$] with abscissa i;

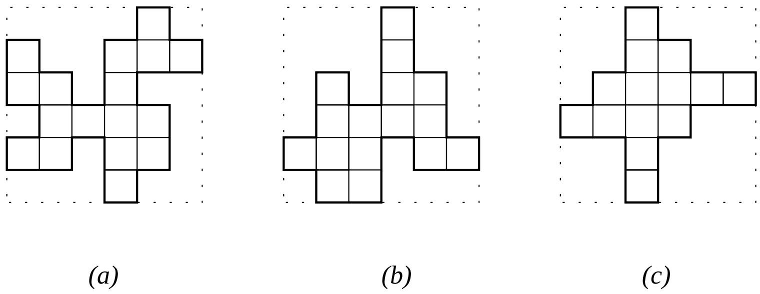


Figure 2: (a) a polyomino; (b) a vertically convex polyomino; (c) a convex polyomino.

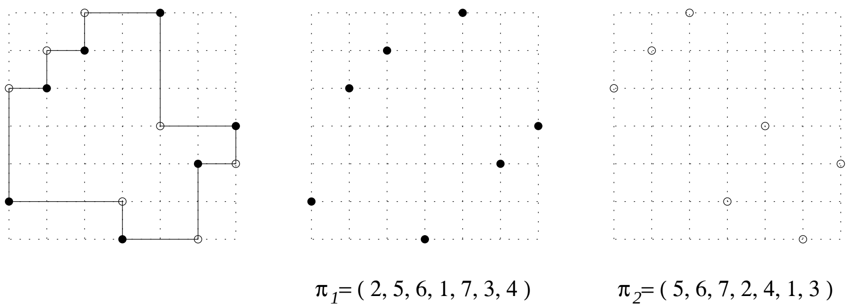


Figure 3: A permutomino and the two associated permutations. The dotted black vertices belong to the set $\mathcal{P}_1$, while the circled vertices belong to the set $\mathcal{P}_2$.

ii) for each ordinate $j = 1, 2, \ldots, m+1$, there exists exactly one point of the set $\mathcal{P}_1 = \{A_1, A_3, \ldots, A_{(2r+1)}\}$ [resp. $\mathcal{P}_2 = \{A_2, A_4, \ldots, A_{2r+2}\}$] with ordinate j.

As an immediate consequence we have that $m = n$, and so a permutomino has n rows and n columns (n will be called its *size*), and the number of its vertices is $2(n+1)$. The two sets $\mathcal{P}_1$ and $\mathcal{P}_2$ can be regarded as two permutation matrices of $[n+1] = \{1, 2, \ldots, n+1\}$ having no common points; we indicate the permutations associated with them by π_1 and π_2, respectively (see Fig. 3).

The following simple characterization of permutominoes by means of a property of their boundary turns out to be the key point for the construction of a polynomial time reconstruction algorithm for convex permutominoes from their projections.

Our starting point is the lemma:

Lemma 1 *A polyomino P of size $n \times n$ is a permutomino if and only if for each i, with $1 \leq i \leq n$, there exist exactly one side of P lying along the abscissa* *i, and exactly one side of P lying along the ordinate i.*

The proof directly follows from the definition of permutomino since it allows exactly two vertices, and so a side, of P on each abscissa and on each ordinate. The following property is also straightforward:

Property 1 *If P is a permutomino, then its projection vector H [resp. V] has no two consecutive elements with the same value.*

3 Detecting the $w - foot$ and the $e - foot$ of a Convex Permutomino

Let us take into consideration the class of convex permutominoes. We first observe that there is a simple relationship between the couple (π_1, π_2) of permutations defining a convex permutomino and its horizontal and vertical projections. More precisely, it holds:

$$h_i = \sum_{1 \leq j \leq i} (\pi_2(j) - \pi_1(j)) \text{ and } v_i = \sum_{1 \leq k \leq i} \{(j - j') : \pi_1(j) = \pi_2(j') = n + 1 - k\}$$

with $1 \leq i \leq n$. For instance, the horizontal and vertical projections of the convex permutomino depicted in Fig. 3, can be easily computed from the permutations $\pi_1 = (2, 5, 6, 1, 7, 3, 4)$ and $\pi_2 = (5, 6, 7, 2, 4, 1, 3)$, giving $H = (2, 3, 4, 6, 5, 2)$, and $V = (3, 4, 5, 6, 3, 1)$. On the other hand, we will show in the sequel that a couple of projections does not uniquely determine a convex permutomino, and so its two permutations.

In order to define our reconstruction algorithm, we need to identify some points on the boundary square of a convex permutomino. We define the *north*, *south*, *east*, and *west foot* (briefly n, s, e, and $w-foot$) of a convex permutomino P to be the set of its (consecutive) cells that lie in row n, row 1, column n and column 1, respectively (see Fig. 4).

Referring to Figure 4, we can observe that each convex permutomino can be divided into three parts using the horizontal projections, i.e.

i. the longest increasing sequence at the beginning of the vector of projections;

ii. the longest decreasing sequence at the end of the vector of projections;

iii. the remaining (central) part of the projections.

These three parts resemble three parts of the permutomino, the first one going from the $s - foot$ up to the first encountered $e - foot$ or to the $w - foot$ (part *i*.), the next one which determines the central part of the permutomino, i.e. those columns between the $e - foot$ and the $w - foot$ (part *iii*.), and the last one going from the last encountered (*east* or *west*) $foot$, to the $n - foot$ (part *ii*.).

A similar division can be obtained by means of the vertical projections. Obviously at least one of the three parts of the convex permutomino has to be non empty. More formally, we can state the following lemma which turns out to be useful in the definition of our reconstruction algorithm.

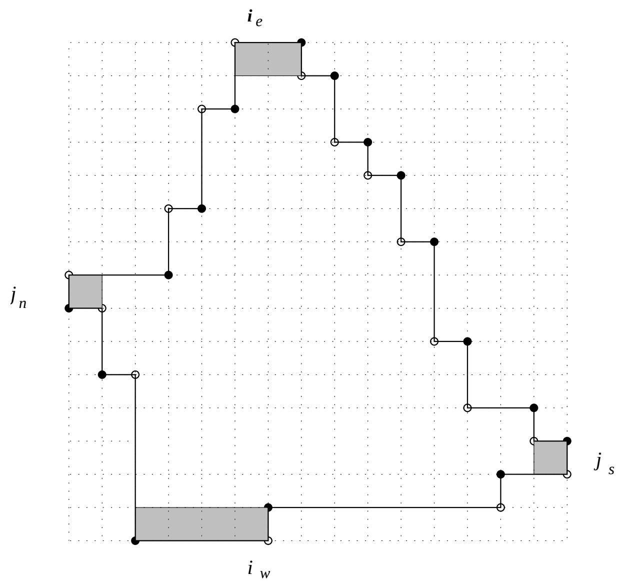


Figure 4: A convex permutomino and the indexes i_w, i_e, j_n and j_s. The highlighted cells are those of the four feet.

Lemma 2 *Let P be a convex permutomino of size n, and $H = (h_1, \ldots, h_n)$, $V = (v_1, \ldots, v_n)$ its horizontal and vertical projections. We consider the four indexes*

$$i_w = min\{\, i : v_i > v_{i+1}\} \,, \qquad i_e = max\{\, i : v_i > v_{i-1}\},$$

$$j_n = max\{j : h_j > h_{j-1}\} \quad and \quad j_s = min\{j : h_j > h_{j+1}\},$$

with $1 < i, j < n$ (see Fig. 4). It holds that

i) if the index i_w [resp. i_e] exists, then the cell $c(i_w, 1)$ or the cell $c(i_w, n)$ [resp. the cell $c(i_e, 1)$ or the cell $c(i_e, n)$] belongs to P;

ii) if the index j_n [resp. j_s] exists, then the cell $c(1, j_n)$ or the cell $c(n, j_n)$ [resp. the cell $c(1, j_s)$ or the cell $c(n, j_s)$] belongs to P.

Proof We prove assertion *i)* related to the index i_w, letting the reader to infer the other cases by symmetry.

So, let us proceed by contradiction assuming the existence of the column i_w in P which extends from row j_w to row j'_w, with $j_w \leq j'_w$, $j_w \neq 1$ and $j'_w \neq n$. By definition of i_w, and since each abscissa $1, \ldots, n+1$ contains exactly one side of P, then it holds that $c(i,1)$ and $c(i,n)$ do not belong to P, for each i, $1 \leq i < i_w$ (see Fig. 5, (a)).

Now, analyzing column $i_w + 1$ of P two cases may arise: since $v_{i_w+1} < v_{i_w}$, and only one side of P has to lie on the abscissa $i_w + 1$, then either $c(i_w + 1, j_w)$ or $c(i_w + 1, j'_w)$, but not both, does not belong to P. In the first case the permutomino P lacks the $s - foot$ (Fig. 5, (b)), while in the second it lacks the $n - foot$ (Fig. 5, (c)), reaching a contradiction in both cases. □

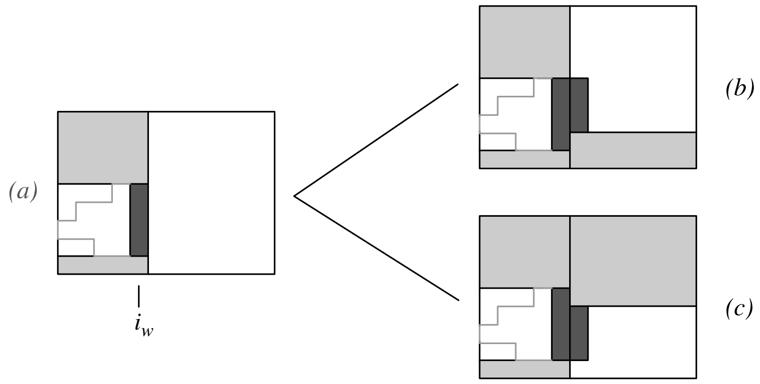


Figure 5: The two cases of Lemma 2. The areas shaded in light grey represent regions where there cannot be placed cells.

Lemma 3 *Let $H = (h_1, \ldots, h_n)$ be an integer vector. All the elements between the rows j_s and j_n of a convex permutomino P consistent with H are uniquely determined up to the choice of $c(1, j_n)$ or $c(n, j_n)$ in P.*

Proof Let us assume that $j_n > j_s$, and $c(1, j_n) \in P$ (if $c(n, j_n) \in P$ a similar reasoning holds). By Lemma 2, it holds that $c(n, j_s) \in P$, so all the cells $c(1, j_n), \ldots, c(h_{j_n}, j_n)$ and $c(n - h_{j_s} + 1, j_s), \ldots, c(n, j_s)$ are in P. Obviously if one of h_{j_n} or h_{j_s} is greater than n, then no permutomino P consistent with H exists.

Now we assume that the elements of P in row j, with $j_s + 1 < j \leq j_n$, are determined, in particular $c(i, j), \ldots, c(i', j) \in P$, with $i' = i + h_j - 1$, and we compute those in row $j + 1$: by Lemma 1 and by vertical convexity we immediately get

- if $h_j > h_{j+1}$, then it holds $c(i' - h_{j-1} + 1, j - 1), \ldots, c(i', j - 1) \in P$;
- if $h_j < h_{j+1}$, then it holds $c(i, j - 1), \ldots, c(i + h_{j-1} - 1, j - 1) \in P$. Again if $i + h_{j-1} - 1 > n$ then no permutomino P exists.

Since h_j and h_{j-1} can not be equal, the proof is complete. □

The previous two lemmas allow us to define a procedure that computes all the possible positions of the $w-foot$ and $e-foot$, and the cells between them, if any, of a convex permutomino P consistent with a couple of projections H and V. This will be a useful starting point for our final reconstruction algorithm:

Procedure: *Compute_feet*

Input: two integer vectors $H = (h_1, \ldots, h_n)$ and $V = (v_1, \ldots, v_n)$.

Step 1: compute the indexes j_s and j_n from H.

Step 2: if $(j_n > j_s)$ or $(j_n = j_s$ and $v_1 \neq 1$ and $v_n \neq 1)$ then

Step 2.1:
- for $j_n \leq j \leq j_n + v_1 - 1$, let $c(1, j), \ldots, c(h_j, j)$ belong to P;
- for $j_s - v_n + 1 \leq j \leq j_s$, let $c(n - h_j + 1, j), \ldots, c(n, j)$ belong to P;
- fill the rows between j_s and j_n as in the proof of Lemma 3, and return P as output.

Step 2.2:
- for $j_n \leq j \leq j_n + v_1 - 1$, let $c(n - h_j + 1, j), \ldots, c(n, j)$ belong to P;
- for $j_s - v_n + 1 \leq j \leq j_s$, let $c(1, j), \ldots, c(h_j, j)$ belong to P;
- fill the rows between j_s and j_n as in the proof of Lemma 3, and return P as output.

Step 3: if $j_n = j_s$ then, for $j_n - 1 \leq j \leq j_n + 1$

Step 3.1:
- if $v_1 = 1$ then $c(n - h_j + 1, j), \ldots, c(n, j)$ belong to P;
- return P as output.

Step 3.2:
- if $v_n = 1$ then $c(1, j), \ldots, c(h_j, j)$ belong to P;
- return P as output.

Step 3 of *Compute_feet* considers the case of centered convex permutominoes, i.e. those permutominoes containing all the n cells of (at least) one row. In such a situation, it is easy to check that by definition, exactly one row of length n has to belong to each permutomino, and so exactly one of the two *west* and *east feet* contains one single cell.

Finally, we underline that by Lemma 2, the procedure *Compute_feet* detects all the possible positions of the $w - foot$ and $e - foot$ of each permutomino consistent with H, so the following remarkable property holds:

Property 2 *The convex permutominoes consistent with an integer vector H have only two admissible positions for the west and east feet.*

4 Reconstructing a Convex Permutomino

Before proceeding we observe that the convexity constraint on a permutomino imposes that the cells which lie on each row and on each column are all consecutive. Consequently we have

Property 3 *All the cells on a row [resp. on a column] of a convex permutomino are completely determined once we know two consecutive of them, one lying in P and the other external to it, together with the projection of the row [resp. of the column] itself.*

So, starting with two integer vectors $H = (h_1, \dots, h_n)$ and $V = (v_1, \dots, v_n)$, we define a series of steps that lead to the reconstruction of a convex permutomino P consistent with them, or gives failure if such a permutomino does not exist. Using the two different placements of the *east* and *west feet* of P given by $Compute\_feet$, it proceeds in reconstructing P row by row, according to the values of the vectors H and V, and taking into account the convexity constraints. Two integer variables $urow$ and $lrow$ which store the indexes of the uppermost and of the lowermost reconstructed row, respectively, will support the computation; for each of them we also consider the columns of the leftmost and rightmost cells of P lying on it.

Reconstruction

Input: Two integer vectors $H = (h_1, \dots, h_n)$ and $V = (v_1, \dots, v_n)$.

Step 1: run $Compute\_feet$; for each output P do:

Step 2: let $urow = max\{j_n, j_s\}$, $lrow = min\{j_s, j_n\}$, u_1 and u_2 be the indexes of the columns containing respectively the leftmost and the rightmost cells of P in row $urow$, and l_1 and l_2 be the indexes of the columns containing the leftmost and the rightmost cells of the row $lrow$ (see Fig. 6).

Step 3: while $urow \neq n$ and $lrow \neq 1$ do

Step 3.1: if $u_2 < l_2$, then in column l_2 there exist two cells $c(l_2, j) \in P$ and $c(l_2, j+1) \notin P$, with $urow < j \leq lrow$:

- if $lrow > j - v_{l_2} + 1$, then $c(l_2, lrow-1) \in P$ and $c(l_2+1, lrow-1) \notin P$, else $c(l_1, lrow-1) \in P$ and $c(l_1-1, lrow-1) \notin P$ (see Fig. 7);

- detect the remaining elements of row $lrow - 1$ of P using Property 3 and update $lrow$, l_1, and l_2.

Step 3.2: if $u_2 > l_2$, act symmetrically as in Step 3.1 to determine row $urow + 1$.

Step 3.3: if $u_2 = l_2$, then act symmetrically as in Steps 3.1 and 3.2 using columns of index u_1 and l_1.

Observe that, by definition of permutomino, $u_1 = l_1$ and $u_2 = l_2$ can not simultaneously occur.

Step 4:

- complete the reconstruction of P according to Property 3;

- if P is a convex permutomino consistent with H and V gives P as output and stop the reconstruction process, otherwise return to Step 1 considering the other output of $Compute\_feet$

Step 5: if no valid P has been found, give FAILURE as output.

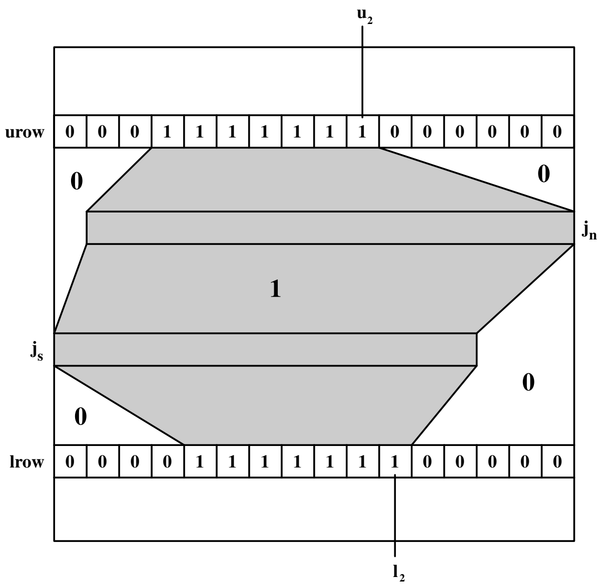


Figure 6: The action of a generic instance of the loop in step 3, when $u_1 < l_1$ and $u_2 < l_2$

Theorem 1 *Given a couple of horizontal and vertical projections H and V, the procedure Reconstruction finds a convex permutomino consistent with them, if it exists, otherwise it returns failure.*

Proof Let P be a convex permutomino consistent with H and V. By Property 2 the positions of its $w - foot$ and $e - foot$ and those of the elements between them are detected by $Compute\_feet$.

Each cycle of Step 3 uniquely reconstructs a row of P, in fact, in Step 3.1, the placement of the cells $c(l_2, lrow - 1)$ or $c(l_1, lrow - 1)$ inside or outside P is determined by the projection v_{l_2}, while the fact that the cells $c(l_2 + 1, lrow - 1)$

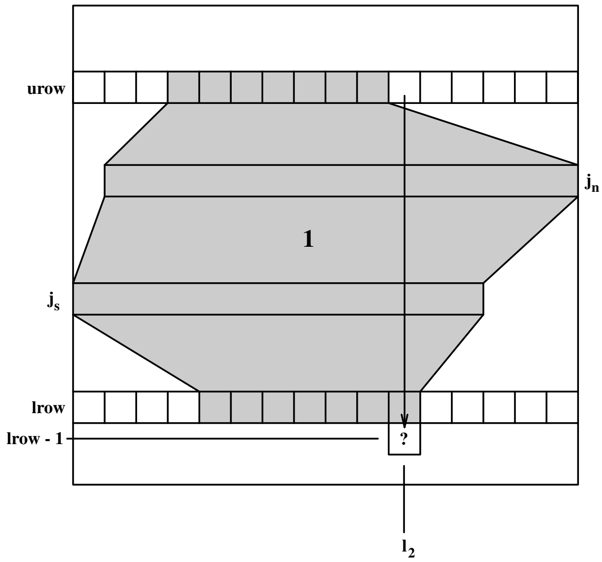


Figure 7: The way of performing Step 3 of the reconstruction process when $u_2 < l_2$.

or $c(l_1 - 1, lrow - 1)$ is not in P follows from the vertical convexity of P. These arguments hold for the cases in Steps 3.2 and 3.3 too.

The correctness of Step 4 is still guaranteed by Property 3, so the permutomino P is uniquely reconstructed once detected its $w - foot$ and $e - foot$. □

Theorem 2 *The algorithm* Reconstruction *reconstructs a convex permutomino from two vectors of projections in $O(n^2)$ time complexity, where n is the dimension of the permutomino.*

Proof We will analyze the complexity of the algorithm step by step:

Step 1: the procedure *Compute_feet* requires $O(n)$ time to compute j_n and j_s, and $O(n^2)$ to reconstruct the elements of permutominoes obtained as output (which are at most two).

Step 2 is clearly performed in $O(n)$ time.

Step 3: each run of the main cycle reconstructs a row of P, and requires $O(n)$ time. So, the total amount of time is $O(n^2)$.

Step 4: performs a deterministic reconstruction of the remaining rows of P, again in $O(n^2)$ time.

We conclude that the total amount of time for the reconstruction process is $O(n^2)$. □

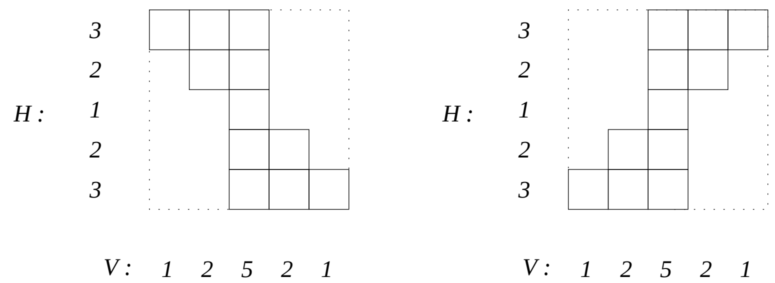


Figure 8: Two convex permutominoes having the same horizontal and vertical projections

Finally, we stress once more the remarkable consequence about the number of convex permutominoes consistent with two projections. The procedure *Reconstruction* acts deterministically to complete the reconstruction of each P obtained from *Compute_feet*, so each initial configuration of the $w - foot$ and of the $e - foot$ leads to at most one single feasible solution. As a consequence it the following unusual result holds

Theorem 3 *Given a couple of integer vectors H and V there exist at most two convex permutominoes consistent with them.*

Figure 8 shows two different permutominoes consistent with the same couple of projections. Notice that one permutomino is the symmetric image of the other; we could conjecture that this result holds in every case where we have two solutions.

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