Using Tomography in Digital Plane to solve problems of Geometric Tomography

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Abstract

We study the problem of determining in a constructive way a convex body in the plane from its tomographic projections. For this, we consider the similar problem in digital plane: reconstructing a lattice convex set from its discrete tomographic projection. We show that we can use a reconstruction algorithm for the discrete problem to solve the continuous reconstruction to any precision. The proof of this result uses stability properties of geometric tomography. An extension to point-source tomographic projections is also investigated.

Parallel sources case

If F is a convex body of $\mathbb{R}^2$ and $p = (a, b)$ is a direction then the (parallel source) tomographic projection of F with respect to p (denoted $\mathbf{X}_p F$) is defined by:

$$\mathbf{X}_p F(u) = \text{length}(\{(x, y) \in F \ : \ bx - ay = u\}).$$

Hammer's X-ray problem consists to reconstruct a convex body of $\mathbb{R}$ from a minimum number of tomographic projections. This is the most classical problem of Geometric Tomography ([5]).

In [8] the authors characterize completely the sets of directions which permit to reconstruct all the convex sets.(such a set is called a *Gardner-McMullen* set of directions. In particular it is proved that all the sets of directions which provide uniqueness have a cardinal not less than four. But this result is not constructive, it does not give a method to reconstruct the convex body from its projections. In the literature there are several descriptions of constructive methods which attempt to make this reconstruction: In [12] the authors are able to reconstruct and infinite set of points of the border of the convex body but until now we are not able to prove that this infinite set is dense in the border. In [11], the authors construct a subset and a superset of approximative solution by making some choices and use “filling operations”, but unfortunately there is no proof that the sequence of the obtained solution converges to the good set even when the directions provide uniqueness. In [7], the method reconstructs a polygon which tomographic projections has the least square distance with the projection. It is proven that the polygon tends to the set when the number of vertices tends to infinity but it lacks an efficient method to find the polygon.

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In this paper we use another strategy which works only for rational directions. This strategy uses algorithms of Discrete Tomography. The main aim of Discrete Tomography is to reconstruct a lattice set $E \subset \mathbb{Z}^2$ from discrete tomographic projections defined by:

$$X_pF(u) = \mathrm{card}(\{(x, y) \in F \;:\; bx - ay = u\}$$

where $p = (a, b)$ is the direction of the projection ([9, 10]).

To solve Hammer's X-ray problem, we will approximate the plane $\mathbb{R}^2$ by the discrete plane $r\mathbb{Z}^2 = \{(rx, ry) \;:\; x, y \in \mathbb{Z}\}$ for a certain resolution $r > 0$. The idea is to take r sufficiently small to make possible the reconstruction of the search set at the wanted precision.

A lattice set (subset of $\mathbb{Z}^2$) is said to be convex if it is the intersection of a convex subset of $\mathbb{R}^2$ with $\mathbb{Z}$. So if F is a convex body that the lattice set $F_r = \left(\frac{1}{r}F\right) \cap \mathbb{Z}^2$ is convex, moreover its tomographic projections satisfy:

$$\left\lfloor \frac{\mathbf{X}_pF(kr)}{r\|p\|} \right\rfloor \leq X_pF_r(k) \leq \left\lfloor \frac{\mathbf{X}_pF(kr)}{r\|p\|} \right\rfloor + 1 \qquad (1)$$

where $\lfloor x \rfloor$ denotes the largest integral value not greater that x. The following theorem shows that reciprocally if a sequence of sets satisfies Equation (1) then it tends to F.

Theorem 1 *Let $\mathcal{D}$ a Gardner-McMullen set of rational directions and $(E_r)_{r>0}$ a sequence of convex lattice sets such that for any $p \in \mathcal{D}$ and $k \in \mathbb{Z}$:*

$$\left\lfloor \frac{\mathbf{X}_pF(kr)}{r\|p\|} \right\rfloor \leq X_pE_r(k) \leq \left\lfloor \frac{\mathbf{X}_pF(kr)}{r\|p\|} \right\rfloor + 1 \qquad (2)$$

Then

$$\mathrm{area}(\mathrm{conv}(rE_r)\Delta F) \xrightarrow[r \to 0]{} 0.$$

(the notation $\|p\|$ denotes $\sqrt{a^2 + b^2}$ if $p = (a, b)$ and Δ denotes the symmetrical difference of sets). Notice that this theorem is a simplification of Proposition 20 of [2].

Sketch of proof : Suppose without loss of generality that $\mathcal{D} = \{p_1, p_2, p_3, p_4\}$. Let $\mathcal{X}$ be the function defined on the convex bodies by $\mathcal{X}(F) = (\mathbf{X}_{p_1}F, \mathbf{X}_{p_2}F, \mathbf{X}_{p_3}F, \mathbf{X}_{p_4}F)$.

In the following we suppose the the class of the convex bodies is embedded with Hausdorff distance: $d_H(E, F) = \inf\{\varepsilon \;:\; F_1 \subset F_2 + B(0, \varepsilon)$ and $F_2 \subset F_1 + B(0, \varepsilon)\}$ or Nikodym distance $d_N(E, F) = \mathrm{area}(E\Delta F)$ which anyway induce the same topology. Similarly, the topology on the image of $\mathcal{X}$ is induced by the product topology of Hausdorff/Nikodym distance of the Steiner symmetrals: $\{(x, y) : -\frac{\mathbf{X}_p(x)}{2} \leq y \leq +\frac{\mathbf{X}_p(x)}{2}\}$, it is also equivalent to the topology induced by the distance $d(\mathcal{X}(F), \mathcal{X}(F')) = \max_{1 \leq i \leq 4} \int_{-\infty}^{+\infty} |\mathbf{X}_{p_i}F(\alpha) - \mathbf{X}_{p_i}F'(\alpha)| \mathrm{d}\alpha$.

With these topologies, we know by [13] that the inverse function of $\mathcal{X}$ is continuous.

Let $E'_r = \mathrm{conv}(rE_r)$, we have:

$$\frac{X_pE'_r(kr)}{r\|p\|} - 1 \leq X_pE_r(k) \leq \frac{\mathbf{X}_pE'_r(kr)}{r\|p\|} + 1 \qquad (3)$$

So, by combining (2) and (3) we have:

$$\mathbf{X}_p F(kr) - 2r\|p\| \le \mathbf{X}_p E'_r(kr) \le \mathbf{X}_p F(kr) + 2r\|p\|$$

so by uniform continuity of the function $x \mapsto \mathbf{X}_p F(x)$ we have: $d(\mathcal{X}(E'_r), \mathcal{X}(F)) \xrightarrow[r \to 0]{} 0$ so $d_N(E'_r, F) \xrightarrow[r \to 0]{} 0$ by continuity of the inverse of $\mathcal{X}$. $\square$

So to reconstruct a sequence of sets which approximate the searched convex body it is sufficient to solve, for a sequence of resolutions r which tends to zero, the following problem of Discrete Tomography:
AppRec($\mathcal{B}, \mathcal{D}$)
Input: A map $f : \mathcal{D} \times \mathbb{Z} \to \mathbb{N}_0$ with finite support.
Output: A convex lattice set $E \subset \mathbb{Z}^2$, if it exists, which satisfies

$$f(p,k) \le X_p E(k) \le f(p,k) + 1, \text{ for all } (p,k) \in \mathcal{D} \times \mathbb{Z} .$$

Theorem 1 shows that the construction of a solution to Hammer's X-problem can be made by solving AppRec($\mathcal{C}, \mathcal{D}$) for the class $\mathcal{C}$ of convex lattice sets and the discrete tomographic projections:

$$f(p,k) = \begin{cases} 0 & \text{if } kr \text{ is outside the closure of } \{x \ : \ \mathbf{X}_p F(x) = 0\} \\ \left\lfloor \frac{\mathbf{X}_p F(kr)}{r\|p\|} \right\rfloor & \text{otherwise.} \end{cases}$$

Unfortunately we do not know if there are efficient (for example polynomial-time) algorithms which solve the problem AppRec($\mathcal{C}, \mathcal{D}$).

However we can use filling operations and a recursive procedure to reconstruct the set in a time which can be exponential in theory but looks experimentally reasonable (see [3, section 7.4] for the extension of the filling operations to the approximative case and for example [4, page 248] for the recursive procedure).

Point sources case

Now we investigate the point-source extensions of the previous results. In Geometric Tomography we define the projection $\mathbf{X}_C(F)$ of the body $F \subset \mathbb{R}^2$ with respect to the center C by:

$$\mathbf{X}_C F(\alpha) = \text{length}(\{M \in F \ : \ \widehat{CM} = \alpha\})$$

where $\widehat{CM}$ denotes the angle between the unit vector $(1,0)$ and the vector $\overrightarrow{CM}$. Notice that in this paper, the angle α is up to 2π: the length is taken on a semi-line, the tomographic projections are *directed*. It is proved that any convex body is completely determined by three tomographic projections with respect to three non-collinear sources. ([14])

Similarly in Discrete Tomography, if C is an integer point the projection $X_C F$ of a lattice set F with respect to the center C is defined by

$$X_C F(\alpha) = \text{card}(\{M \in F \ : \ \widehat{CM} = \alpha\}).$$

Notice that the support of the projection (set of α for which $X_C F(\alpha) \neq 0$) is finite.

We now extend Theorem 1 to the point-source case. For this extension we will suppose that the point sources have integral coordinates and that the resolution is of the form $r = \frac{1}{n}$ where n is an integer. Consequently the point sources are also always in $r\mathbb{Z}^2$. We denote by $\mathcal{RA}$ the set of "rational" angles $\{\alpha \,:\, \alpha = 0 \mod \pi \text{ or } \tan\alpha \in \mathbb{Q}\}$ and for any $\alpha \in \mathcal{RA}$ we denote by $\|\alpha\|$ the quantity:

$$\|\alpha\| = \begin{cases} 1 & \text{if } \alpha = 0 \mod \pi \\ \sqrt{p^2+q^2} & \text{where } \tan\alpha = \frac{p}{q} \text{ such that } p \text{ and } q \text{ are coprime integers.} \end{cases}$$

We can now formulate the theorem:

Theorem 2 *Let $\mathcal{C} = \{C_1, C_2, C_3\}$ be a set of three non-collinear points in $\mathbb{Z}^2$ and $(E_n)_{n\in\mathbb{N}^*}$ a sequence of lattice sets such that for any $C \in \mathcal{C}$ and any $\alpha \in \mathcal{RA}$ we have:*

$$\left\lfloor \frac{n\mathbf{X}_C F(\alpha)}{\|\alpha\|} \right\rfloor \le X_{nC} E_n(\alpha) \le \left\lfloor \frac{n\mathbf{X}_C F(\alpha)}{\|\alpha\|} \right\rfloor + 1 \tag{4}$$

then

$$\text{area}(\text{conv}(rE_n)\Delta F) \xrightarrow[r\to 0]{} 0.$$

Sketch of proof : The proof is similar to the proof of Theorem 1. The main differences are the consideration of the property (4) only for $\|\alpha\| \le \sqrt{n}$ and the use of the stability result of [1] instead the one of [13]. □

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