Generalized Rectangular Clifford Semiring

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Abstract. In the paper, the concepts of rectangular skew-ring and generalized rectangular Clifford semiring are firstly introduced. Secondly, some of their equivalence characterizations are given. And then, a spined product decomposition of generalized rectangular Clifford semiring under certain conditions is obtained. Also, a necessary and sufficient condition for a generalized rectangular Clifford semiring to be expressed into a strong b-lattice of rectangular skew-rings is finally shown. As a consequences, some results about Clifford semirings and rectangular Clifford semirings are strengthened and extended.

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1 Introduction

A semiring S is an algebraic structure $(S, +, \cdot)$ consisting of a non-empty set S together with two binary operations $+$ and $\cdot$ on S such that $(S, +)$ and $(S, \cdot)$ are semigroups connected by distributivity, that is, $a(b+c) = ab+ac$ and $(b+c)a = ba+ca$ for all $a, b, c \in S$. A skew-ring $(S, +, \cdot)$ is a semiring whose additive reduct $(S, +)$ is a group, not necessarily an abelian group.

A semiring $(S, +, \cdot)$ is said to be a b-lattice [9] if its additive reduct $(S, +)$ is a semilattice and its multiplication reduct $(S, \cdot)$ is a band.

It is well known that, regular semigroups play a very important role in the research of semigroups. Among of all the regular semigroups, there is a special and important one called Clifford semigroup which has a very beautiful structure theorem, i.e., a semigroup S is a Clifford semigroup if and only if S is a strong semilattice of groups.

In the last decades, generalizations of this class of semigroups have been extensively investigated by many authors and lots of interesting results have been obtained.

Recently, there are some experts and scholars who had generalized the Clifford semigroups to semirings with some semigroup techniques and methods. Meanwhile, they had also obtained a number of interesting results ([1]-[5],[8]-[11],[13],[14]).

In [9], the authors extended the important structure theorem of Clifford semigroups to the corresponding semirings, and showing that a semiring S is a Clifford semiring if and only if it is a strong distributive lattice of skew-rings. Also, as a further generalization, they presented a generalized Clifford semiring, and proved that a semiring S is a generalized Clifford semiring if and only if it is a strong b-lattice of skew-rings.

In [5], by using the concept of band semiring, the authors introduced the concepts of rectangular ring and rectangular Clifford semiring. After giving some of the two class of semirings' characterizations, they obtained the spined product decomposition of rectangular Clifford semiring under certain conditions. Finally, they obtained a necessary and sufficient condition for rectangular Clifford semiring to be a strong distributive lattice of rectangular rings. And then, some results in [9] are strengthened and extended.

Now, we will naturally quote such a question: whether can we give a new kind of semirings which can simultaneously strengthen and extend the main results of the generalized Clifford semirings and the rectangular Clifford semirings?

In the paper, we set about to discuss the above question and obtain the class of semirings which will be called generalized rectangular Clifford semirings.

We now describe the contents of this paper in some detail.

Firstly, we introduce the concepts of the rectangular skew-rings and generalized rectangular Clifford semirings. Secondly, we give some equivalence characterizations of them, and get the spined product decomposition of generalized rectangular Clifford semiring under certain conditions. Finally, we discuss a special case of generalized rectangular Clifford semiring and obtain the necessary and sufficient condition for the generalized rectangular Clifford semiring to be a strong b-lattice of rectangular skew-rings. As a consequence, some results in [5] and [9] are strengthened and extended.

For notations and terminologies not mentioned in this paper, the readers are referred to [6],[7],[9] or [12].

2 Generalized rectangular Clifford semirings

A semiring $S = (S, +, \cdot)$ is called an idempotent semiring, if for any $a \in S$,$a+a = a = a \cdot a$. An idempotent semiring $S = (S, +, \cdot)$ is called a band semiring, if S satisfies the conditions that for any $a, b \in S$, $a + ab + a = a$, $a + ba + a = a$.

DEFINITION 2.1 A band semiring $S = (S, +, \cdot)$ is called a $\mathcal{T}$ band semiring ,if the additive reduct $S = (S, +)$ of S is a $\mathcal{T}$ band, where $\mathcal{T}$ is, respectively, a rectangular, left(right) zero, left(right) regular, regular, normal, commutative band and so on.

From [5], a semiring S is said to be a rectangular ring, if S can be decomposed as a direct product of a rectangular band semiring and a ring. A semiring S is said to be a rectangular Clifford semiring if S is a distributive lattice of rectangular rings.

Actually, we can get the following more general definitions:

DEFINITION 2.2 A semiring S is said to be a rectangular skew-ring, if S can be decomposed as a direct product of a rectangular band semiring and a skew-ring, where a skew-ring means a semiring whose additive reduct is a group, not necessarily an abelian group.

DEFINITION 2.3 A semiring S is said to be a quasi-generalized Clifford semiring if S is a b-lattice of skew-rings.

DEFINITION 2.4 A semiring S is said to be a generalized rectangular Clifford semiring if S is a b-lattice of rectangular skew-rings.

From the above definitions, we can see that a rectangular Clifford semiring is a generalized rectangular Clifford semiring, since the rectangular ring is the rectangular skew-ring and the distributive lattice is the b-lattice. Thus, the generalized rectangular Clifford semirings are the generalization of the rectangular Clifford semirings.

In the following, we will firstly discuss the equivalence characterizations of rectangular skew-rings and generalized rectangular Clifford semirings, and then give the spined product decomposition of the generalized rectangular Clifford semirings under certain conditions.

THEOREM 2.5 *A semiring S is a rectangular skew-ring, if and only if:*
(i) The additive reduct $(S,+)$ of S is a rectangular group, that is, it is a direct product of a rectangular band and a group;
(ii)$E^+(S) \subseteq E^{\cdot}(S)$, where $E^+(S)(E^{\cdot}(S))$ is the set of all additive (multiplicative) idempotents of S.

Proof. The proofs are similar with the corresponding ones of Theorem 1 in [14].

($\Rightarrow$) If the semiring S is a rectangular skew-ring, then S can be decomposed as a direct product of a rectangular band semiring R_1 and a skew-ring R_2, that is, $S = R_1 \times R_2$. It is clear that $(S,+) = (R_1,+)\times(R_2,+)$ is a rectangular group, and also $E^+(S) = \{(i,0)|i \in R_1\} \subseteq E^{\cdot}(S)$ is an idempotent subsemiring of S, where 0 is the zero element of R_2.

($\Leftarrow$) If the additive reduct $(S,+)$ of S is a rectangular group, then for any $e \in E^+(S)$, $(e+S+e,+)$ is a group, $(E^+(S),+)$ is a rectangular band, and then $e+S+e$ is a subskew-ring of S. Also, notice that $E^+(S) \subseteq E^{\cdot}(S)$, then $E^+(S)$ a rectangular band semiring.

Now, for any fixed $e \in E^+(S)$, define a mapping

$$\varphi : (E^+(S),+) \times (e+S+e,+) \to (S,+), (f, e+s+e) \mapsto f+e+s+e+f.$$

Then it is a routine way to check that φ is a semiring isomorphism. Hence, S can be decomposed as a direct product of a rectangular band semiring and a skew-ring, i.e., S is a rectangular skew-ring. □

LEMMA 2.6 *([5]) If S is a semiring whose additive reduct is a completely regular semigroup, then for any $a, b \in S$, $a \overset{+}{\mathcal{L}} b$ if and only if $V^+(a) + a = V^+(b) + b$.*

LEMMA 2.7 *([5]) If S is a semiring whose additive reduct is a completely regular semigroup, then for any $a, b \in S$, the following statements are equivalent:*

(1) $a\overset{+}{\mathcal{D}}b$;
(2) $V^+(a) + a = b + V^+(b)$;
(3) $(V^+(a) + a) \cap (b + V^+(b)) \neq \emptyset$.

LEMMA 2.8 *([12]) A semigroup S is an orthogroup if and only if it is a semilattice of rectangular groups, where an orthogroup means an orthodox completely regular semigroup.*

Now, we can obtain the following theorem.

THEOREM 2.9 *A semiring S is a generalized rectangular Clifford semiring if and only if the additive reduct $(S, +)$ of S is an orthogroup, $E^+(S) \subseteq E^{\cdot}(S)$ and S satisfies $a(a + V^+(a)) \subseteq V^+(a) + a$ for any $a \in S$.*

Proof. ($\Rightarrow$) Assume that S is a generalized rectangular Clifford semiring, i.e., S is a b-lattice B of rectangular skew-rings $S_\alpha(\alpha \in B)$. Then it is clear that $E^+(S) \subseteq E^{\cdot}(S)$ and the additive reduct $(S, +)$ of S is a semilattice B of rectangular groups $(S_\alpha, +)(\alpha \in B)$. And then $(S, +)$ is an orthogroup by Lemma 2.8, and also $S/\overset{+}{\mathcal{D}} \cong B$ is a b-lattice. Hence, for any $a \in S$, $a\ \overset{+}{\mathcal{D}}\ a^2$. By Lemma 2.7, $V^+(a) + a = a^2 + V^+(a^2)$. Since $aV^+(a) \subseteq V^+(a^2)$, then $a(a + V^+(a)) = a^2 + aV^+(a) \subseteq a^2 + V^+(a^2)$. Hence, for any $a \in S$, $a(a + V^+(a)) \subseteq V^+(a) + a$.

($\Leftarrow$) Assume that the additive reduct $(S, +)$ of a semiring S is an orthogroup, then by Lemma 2.8, $(S, +)$ is a semilattice $S/\overset{+}{\mathcal{D}}$ of rectangular groups $(S_\alpha, +)$. In view of the left and right distributive laws of multiplication over addition, it is clear that $\overset{+}{\mathcal{D}}$ is a congruence on $(S, \cdot)$. Also, since $a(a + V^+(a)) \subseteq V^+(a) + a$ holds, then by $a(a + V^+(a)) = a^2 + aV^+(a) \subseteq a^2 + V^+(a^2)$, we have $a(a + V^+(a)) \subseteq (a^2 + V^+(a^2)) \cap (V^+(a) + a)$. Thus, we have $(a^2 + V^+(a^2)) \cap (V^+(a) + a) \neq \emptyset$. By Lemma 2.7, $a\ \overset{+}{\mathcal{D}}\ a^2$. So we have $\overset{+}{\mathcal{D}}$ is a b-lattice congruence of S. Also, since $E^+(S) \subseteq E^{\cdot}(S)$ holds, by Theorem 2.5, every $\overset{+}{\mathcal{D}}-$ class of S is a rectangular skew-ring. Hence, S is a b-lattice of rectangular skew-rings, i.e., S is a generalized rectangular Clifford semiring. □

By the proof of the above theorem, we can immediately obtain the following corollary.

COROLLARY 2.10 *A semiring S is a generalized rectangular Clifford semiring if and only if $\overset{+}{\mathcal{D}}$ is a b-lattice congruence on S and every $\overset{+}{\mathcal{D}}-$ class is a rectangular skew-ring.*

DEFINITION 2.11 A semiring S is said to be a generalized band semiring if S is a b-lattice of rectangular band semirings.

Now, for some b-lattice skeleton B, let $\cup_{\alpha\in B}R_\alpha$ be the b-lattice $B-$decomposition of generalized band semiring R into rectangular band semirings R_α, $\cup_{\alpha\in B}T_\alpha$ be the b-lattice $B-$decomposition of quasi-generalized Clifford semiring T into skew-rings T_α. It is clear that the spined product $R\otimes_B T$ of R and T with respect to the same b-lattice B is a generalized rectangular Clifford semiring.

Next,we will give a theorem for a generalized rectangular Clifford semiring to be expressed a spined product of a generalized band semiring and a quasi-generalized Clifford semiring with respect to the same b-lattice.

THEOREM 2.12 *Let S be a generalized rectangular Clifford semiring and $\overset{+}{\mathcal{H}}$ be a congruence on $(S,+)$. Then S can be expressed a spined product of a generalized band semiring and a quasi-generalized Clifford semiring with respect to the same b-lattice .*

Proof. Assume that S is a generalized rectangular Clifford semiring, then (S,+) is an orthogroup. Since $\overset{+}{\mathcal{H}}$ is a congruence on (S,+), then by a result in [12] ,we have $(S,+)$ is an orthocryptogroup, which can be expressed a spined product of a band $(R,+)=\cup_{\alpha\in B^+}R_\alpha$ (a upper semilattice B^+ of rectangular bands $(R_\alpha,+)$) and a Clifford semigroup $(T,+)=\cup_{\alpha\in B^+}T_\alpha$ (a upper semilattice B^+ of groups $(T_\alpha,+)$). Thus, in S, if $(i,a)\in R_\alpha\times T_\alpha$,$(j,b)\in R_\beta\times T_\beta$, then we can get $(i,a)+(j,b)=(i+j,a+b)\in R_{\alpha+\beta}\times T_{\alpha+\beta}$, where $i+j(a+b)$ is the sum of i and $j(a$ and $b)$ in $(R,+)((T,+))$.

In the following, we will discuss the product of $(i,a)\in R_\alpha\times T_\alpha$ and $(j,b)\in R_\beta\times T_\beta$.

Let $(i,a)(j,b)=(k,c)\in R_\delta\times T_\delta \quad (*)$. Next, we show that $k(c)$ only depends on i and j (a and b).

In fact, if we assume that $(i,a),(i,a')\in R_\alpha\times T_\alpha$ and $(i,a)\overset{+}{\mathcal{H}}(i,a')$, then since $\overset{+}{\mathcal{H}}$ is a congruence on $(S,\cdot)$, we have $(i,a)(j,b)\overset{+}{\mathcal{H}}(i,a')(j,b)$. Now, let $(i,a')(j,b)=(k',c')\in R_\delta\times T_\delta$.Then we have $(k,c)\overset{+}{\mathcal{H}}(k',c')$, and then we can get $(k,c)\overset{+}{\mathcal{L}}(k',c')$ and $(k,c)\overset{+}{\mathcal{R}}(k',c')$. Hence, there exists $(u,v),(s,t),(u',v'),(s',t')\in R_\delta\times T_\delta$ such that $(u,v)+(k,c)=(k',c')$, $(s,t)+(k',c')=(k,c)$, $(k,c)+(u',v')=(k',c')$, $(k',c')+(s',t')=(k,c)$. And then we have $u+k=k'=k+u'$,$s+k'=k=k'+s'$. Since $(R_\delta,+)$ is a rectangular band, we have $k=(k+u')+k=k'+k$, $k'=k'+(s+k')=k'+k$, hence,$k=k'$, and k in $(*)$ is independent of a.

Similarly, by the distributive laws of S, we can show that k in $(*)$ is independent of b.And these also show that k in $(*)$ depends only on i and j.

Also, let $(i^{'},a)(j,b) = (k^{''},c^{''})$. Then $(k,c) = (i,a)(j,b) = [(i,0) + (i^{'},a)+(i,0)](j,b) = (i,0)(j,b)+(i^{'},a)(j,b)+(i,0)(j,b) = (k,0)+(k^{''},c^{''})+(k,0) = (k,c^{''})$. Thus, $c = c^{''}$, that is, c in $(*)$ is independent of i.

Similarly,by the distributive laws of S, we can show that c in $(*)$ is independent of j. And these also show that c in $(*)$ depends only on a and b.

Now,we can define a multiplication on $R(T)$ as follows:

for any $i \in R_\alpha, j \in R_\beta[(a \in T_\alpha, b \in T_\beta)]$, $ij = k \Leftrightarrow (i,0)(j,0) = (k,0)$; $[ab = c \Leftrightarrow (i,a)(j,b) = (ij,c)]$.

It is easy to see that $(R,+,\cdot)[(T,+,\cdot)]$ is a semiring, and also a generalized band semiring [a quasi-generalized Clifford semiring], such that $R(T)$ is a b-lattice B of rectangular band semirings $(R_\alpha,+,\cdot)$ [of skew-rings $(T_\alpha,+,\cdot)$].

Hence,we have shown that the semiring S is a spined product of a generalized band semiring $\cup_{\alpha\in B}R_\alpha$ and a quasi-generalized Clifford semiring $\cup_{\alpha\in B}T_\alpha$with respect to the same b-lattice B. □

Remark. The above theorem generalize the corresponding results about the Clifford semirings in [9] and the rectangular Clifford semirings in [5].

3 A special case

In this section, we will discuss a special case of the generalized rectangular Clifford semirings, and will give a sufficient and necessity condition for the generalized rectangular Clifford semirings to be the strong b-lattice of rectangular skew-rings.

Definition 3.1 ([9]) Let B be a b-lattice and $\{S_\alpha : \alpha \in B\}$ be a family of pairwise disjoint semirings which are indexed by the elements of B. For each $\alpha \leq \beta$ in B, we now embed $S_\alpha$in S_β via a semiring monomorphism $\phi_{\alpha,\beta}$ satisfying the following conditions :

(1.1) $\phi_{\alpha,\alpha} = I_{S_\alpha}$, the identity mapping on S_α;

(1.2) $\phi_{\alpha,\beta}\phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$ if $\alpha \leq \beta \leq \gamma$;

(1.3)$S_\alpha\phi_{\alpha,\beta}S_\beta\phi_{\beta,\gamma} \subseteq S_{\alpha\beta}\phi_{\alpha\beta,\gamma}$ if $\alpha+\beta \leq \gamma$,i.e.,$\alpha+\beta+\alpha\beta \leq \gamma$.

On $S = \cup_{\alpha\in Y}S_\alpha$, we define addition $+$ and multiplication $\cdot$ for $a \in S_\alpha$,$b \in S_\beta$, as follows:

(1.4) $a+b=a\phi_{\alpha,\alpha+\beta}+b\phi_{\beta,\alpha+\beta}$
and
$a\cdot b=c\in S_{\alpha\beta}$ such that $c\phi_{\alpha\beta,\alpha+\beta}=a\phi_{\alpha,\alpha+\beta}\cdot b\phi_{\beta,\alpha+\beta}$. Same as the notation of strong semilattice of semigroups , we denote the above system by $S=<B,S_{\alpha},\phi_{\alpha,\beta}>$ and call it the strong b-lattice B of the semirings $S_{\alpha},\alpha\in B$.

In an obvious way, we may replace b-lattice B in the above definition by distributive lattice D, and $S=\langle D,S_{\alpha},\phi_{\alpha,\beta}\rangle$ is called the strong distributive lattice D of the semirings $S_{\alpha}(\alpha\in D)$.

LEMMA 3.2 *([9]) The system $S=\langle B,S_{\alpha},\phi_{\alpha,\beta}\rangle$ defined above is a semiring.*

Recall that a nonempty subset A of a semigroup S is left (respectively, right) unitary if for any $a,b\in S$, $a,ab\in A$ (respectively, $a,ba\in A$) implies that $b\in A$; A is unitary if it is both left and right unitary. A regular semigroup S is E-unitary if $E(S)$ is a unitary subset of S.

LEMMA 3.3 *([12]) Let S be a regular semigroup in which $E(S)$ is a left (or right) unitary subset. Then S is E-unitary and orthodox.*

DEFINITION 3.4 A semiring S is said to be a generalized normal band semiring if S is a strong b-lattice of rectangular band semirings.

THEOREM 3.5 *A generalized rectangular Clifford semiring S is a strong b-lattice of rectangular skew-rings if and only if $E^{+}(S)$ is a generalized normal band semiring and is unitary in the regular additive reduct $(S,+)$ of S.*

Proof. ($\Rightarrow$) Assume that a generalized rectangular Clifford semiring S is a strong b-lattice B of rectangular skew-rings $S_{\alpha}(\alpha\in B)$, that is, $S=\langle B,S_{\alpha},\phi_{\alpha,\beta}\rangle$. Then it is clear that $E^{+}(S)$ is a generalized normal band semiring.

In the following,we will show that $E^{+}(S)$ is unitary in $(S,+)$. Since (S,+) is an orthogroup, and $E^{+}(S)\subseteq S$,by the Lemma 3.3, we only need to show that $E^{+}(S)$ is left unitary in $(S,+)$.

Let $S_{\alpha}=R_{\alpha}\times T_{\alpha}$, where $R_{\alpha}(T_{\alpha})$ is a rectangular band semiring (skew-ring), $\alpha\in B$. If $(i,a)\in R_{\alpha}$, $(j,0)\in E^{+}(S_{\beta})$ and $(k,0)\in E^{+}(S_{\alpha+\beta})$ such that $(i,a)+(j,0)=(k,0)$, then we have $(i,a)\phi_{\alpha,\alpha+\beta}+(j,0)\phi_{\beta,\alpha+\beta}=(k,0)$.

Now, denote $(i,a)\phi_{\alpha,\alpha+\beta}=(k^{'},a^{'})\in S_{\alpha+\beta}$,$(j,0)\phi_{\alpha,\alpha+\beta}=(k^{''},0)\in S_{\alpha+\beta}$. We have $(k^{'},a^{'})=(k^{'},a^{'})+(k^{''},0)=(k,0)$,and then $a^{'}=0$, that is, $(i,a)\phi_{\alpha,\alpha+\beta}=(k^{'},0)\in E^{+}(S_{\alpha+\beta})$.Since $\phi_{\alpha,\alpha+\beta}$ is injective, we have $a=0$, i.e., $(i,a)\in E^{+}(S_{\alpha})$. Thus, $E^{+}(S)$ is left unitary in $(S,+)$.

($\Leftarrow$) Assume that S is a generalized rectangular Clifford semiring, i.e., S is a b-lattice B of rectangular skew-rings $S_\alpha = R_\alpha \times T_\alpha$, where $R_\alpha(T_\alpha)$ is a rectangular band semiring (skew-ring), $(\alpha \in B)$. For any $\alpha, \beta \in B$ with $\alpha \leq \beta$ and a fixed $(j, 0) \in E^+(S_\beta)$, we define

$$\phi_{\alpha,\beta} : S_\alpha \to S_\beta, (i, a)\phi_{\alpha,\alpha+\beta} = (i, a) + (j, 0) + (i, 0)(\forall (i, a) \in S_\alpha).$$

If $(j^{'}, 0) \in S_\beta$,then since $E^+(S_\beta)$ is a rectangular band semiring and $E^+(S)$ is a generalized normal band semiring, we immediately have $(i, a) + (j, 0) + (i, 0) = (i, a) + (i, 0) + (j, 0) + (j^{'}, 0) + (j, 0) + (i, 0) = (i, a) + (i, 0) + (j^{'}, 0) + (j, 0) + (j^{'}, 0) + (i, 0) = (i, a) + (i, 0) + (j^{'}, 0) + (i, 0) = (i, a) + (j^{'}, 0) + (i, 0)$. And this shows that the definition of $\phi_{\alpha,\beta}$ is not dependent on the choice of the elements in $E^+(S_\beta)$.

With a similar method in the Theorem 3.8 of [14], it is a routine way for us to check that $\phi_{\alpha,\beta}$ is a semiring monomorphism, and satisfies the conditions $(1.1) - (1.4)$.

Firstly, we show that $\phi_{\alpha,\beta}$ is a semiring homomorphism. For any $(i, a), (i', a') \in S_\alpha$, we have

$$\begin{aligned}
& [(i,a) + (i',a')]\phi_{\alpha,\beta} = (i,a) + (i',a') + (j,0) + (i,0) + (i',0) \\
= \; & (i,a) + (i',a') + (i',0) + (j,0) + (i,0) + (i',0) + (i',0) \\
= \; & (i,a) + (i',a') + (i',0) + (i,0) + (i',0) + (j,0) + (i',0) \\
= \; & (i,a) + (i',a') + (i',0) + (j,0) + (i',0) \\
= \; & (i,a) + (i,0) + (i',a') + (j,0) + (i',0) \quad (\text{put}(i',a') + (j,0) = (j',b) \in S_\beta) \\
= \; & (i,a) + (i,0) + (j',0) + (i',a') + (j,0) + (i',0) \\
= \; & (i,a) + (i,0) + (i,0) + (j',0) + (i',0) + (i',a') + (j,0) + (i',0) \\
= \; & (i,a) + (i,0) + (j',0) + (i,0) + (i',0) + (i',a') + (j,0) + (i',0) \\
= \; & (i,a) + (j',0) + (i,0) + (i',a') + (j,0) + (i',0) \\
= \; & (i,a)\phi_{\alpha,\beta} + (i',a')\phi_{\alpha,\beta}, \\
& (i,a)\phi_{\alpha,\beta} \cdot (i',a')\phi_{\alpha,\beta} = [(i,a) + (j,0) + (i,0)][(i',a') + (j,0) + (i',0)] \\
= \; & (i,a)(i',a') + (i,a)(j,0) + (i,a)(i',0) + (j,0)(i',a') + (j,0)(j,0) \\
& +(j,0)(i',0) + (i,0)(i',a') + (i,0)(j,0) + (i,0)(i',0) \\
= \; & [(i,a)(i',a') + (i,a)(i',0) + (i,a)(i',0)] + (i,a)(j,0) + \cdots + (j,0)(j,0) \\
& +(j,0)(i',0) + (i,0)(i',a') + (i,0)(j,0) + (i,0)(i',0) \\
& (\text{since} \quad E^+(S) \quad \text{is generalized normal band semiring}) \\
= \; & [(i,a)(i',a') + (i,a)(i',0) + (i,0)(i',a')] + \cdots + (j,0) + (i,0)(i',0) \\
& (\text{since} \quad E^+(S_\alpha) \quad \text{is a rectangular band semiring}) \\
= \; & (i,a)(i',a') + (j,0) + (i,0)(i',0) \\
= \; & [(i,a)(i',a')]\phi_{\alpha,\beta}.
\end{aligned}$$

This shows that $\phi_{\alpha,\beta}$ is a semiring homomorphism.

Secondly,it is clear that $\phi_{\alpha,\beta}$ satisfies (1.1) − (1.2).

Thirdly,we show that $\phi_{\alpha,\beta}$ is a semiring monomorphism. If $(i,a),(i',a') \in S_\alpha$, we have $(i,a)\phi_{\alpha,\beta} = (i',a')\phi_{\alpha,\beta}$,that is,

$$(i,a)+(j,0)+(i,0) = (i',a')+(j,0)+(i',0). \qquad (1)$$

Then by left-adding$(i,0)$ on both sides of the above formula, we get

$$(i,a)+(j,0)+(i,0) = (i+i',a')+(j,0)+(i',0).$$

Left-adding $(i,-a')$ on both sides of the above formula, we get

$$(i,-a'+a)+(j,0)+(i,0) = (i+i',0)+(j,0)+(i',0) \in E^+(S_\beta).$$

Since $(E^+(S),+)$is unitary in $(S,+)$,$(i,-a'+a) \in E^+(S_\alpha)$,then $a = a'$, hence,

$$(i,a)+(j,0)+(i,0) = (i',a)+(j,0)+(i',0). \qquad (2)$$

By right-adding $(i,-a)+(j,0)+(i,0)$ on both sides of the above formula,we have

$$\begin{aligned}
(i,0)\phi_{\alpha,\beta} &= [(i,a)+(i,-a)]\phi_{\alpha,\beta} = (i,a)\phi_{\alpha,\beta} + (i,-a)\phi_{\alpha,\beta} \\
&= [(i,a)+(j,0)+(i,0)] + [(i,-a)+(j,0)+(i,0)] \\
&= [(i',a)+(j,0)+(i',0)] + [(i,-a)+(j,0)+(i,0)] \\
&= (i',a)\phi_{\alpha,\beta} + (i,-a)\phi_{\alpha,\beta} \\
&= [(i',a)+(i,-a)]\phi_{\alpha,\beta} = (i'+i,0)\phi_{\alpha,\beta},
\end{aligned}$$

and then

$$\begin{aligned}
(i,0)+(j,0)+(i,0) &= (i'+i,0)+(j,0)+(i'+i,0) \\
&= (i',0)+(i,0)+(j,0)+(i',0)+(i,0) \\
&= (i',0)+(j,0)+(i,0)+(i',0)+(i,0) \\
&= (i',0)+(j,0)+(i,0),
\end{aligned}$$

that is,

$$(i,0)+(j,0)+(i,0) = (i',0)+(j,0)+(i,0). \qquad (3)$$

By right-adding $(i,-a)+(j,0)+(i,0)$ on both sides of the formula (2),we have

$$\begin{aligned}
(i,0)\phi_{\alpha,\beta} &= [(i,-a)+(i,a)]\phi_{\alpha,\beta} = (i,-a)\phi_{\alpha,\beta} + (i,a)\phi_{\alpha,\beta} \\
&= [(i,-a)+(j,0)+(i,0)] + [(i,a)+(j,0)+(i,0)]
\end{aligned}$$

$$\begin{aligned} &= [(i,-a)+(j,0)+(i,0)]+[(i',a)+(j,0)+(i',0)] \\ &= (i,-a)\phi_{\alpha,\beta}+(i',a)\phi_{\alpha,\beta} \\ &= [(i,-a)+(i',a)]\phi_{\alpha,\beta}=(i+i',0)\phi_{\alpha,\beta}, \end{aligned}$$

and then

$$\begin{aligned} (i,0)+(j,0)+(i,0) &= (i+i',0)+(j,0)+(i+i',0) \\ &= (i,0)+(i',0)+(j,0)+(i,0)+(i',0) \\ &= (i,0)+(j,0)+(i',0)+(i,0)+(i',0) \\ &= (i,0)+(j,0)+(i',0), \end{aligned}$$

that is,

$$(i,0)+(j,0)+(i,0)=(i,0)+(j,0)+(i',0). \qquad (4)$$

By left-adding $(i',0)$ on both sides of the formula (4),we have

$$\begin{aligned} (i',0)+(j,0)+(i,0) &= (i',0)+(j,0)+(i,0)+(i,0) \\ &= (i',0)+(i,0)+(j,0)+(i,0) \\ &= (i',0)+(i,0)+(j,0)+(i',0) \\ &= (i',0)+(i,0)+(j,0)+(i',0)+(i',0) \\ &= (i',0)+(i,0)+(i',0)+(j,0)+(i',0) \\ &= (i',0)+(j,0)+(i',0), \end{aligned}$$

that is,

$$(i',0)+(j,0)+(i,0)=(i',0)+(j,0)+(i',0). \qquad (5)$$

Now,by formulas (3) and (5),we have $(i,0)+(j,0)+(i,0)=(i',0)+(j,0)+(i',0)$. (6)

By left-multiplying $(i,0)$ on both sides of the formula (6),we have

$$(i,0)(i,0)+(i,0)(j,0)+(i,0)(i,0)=(i,0)(i',0)+(i,0)(j,0)+(i,0)(i',0),$$

that is, $(i,0)+(k,0)+(i,0)=(ii',0)+(k,0)+(ii',0)$, where$(k,0)\in E^+(S_{\alpha\beta})=E^+(S_\alpha)$. Hence,$i=ii'$.

Similarly,by right-multiplying $(i',0)$ on both sides of the formula (6),we have $(i,0)(i',0)+(j,0)(i',0)+(i,0)(i',0)=(i',0)(i',0)+(j,0)(i',0)+(i',0)(i',0)$, that is,$(ii',0)+(k_1,0)+(ii',0)=(i',0)+(k_1,0)+(i',0)$, where $(k_1,0)\in E^+(S_{\alpha\beta})=E^+(S_\alpha)$. Hence, $ii'=i'$, and then, $i=i'$. Thus, $(i,a)=(i',a')$. This shows that $\phi_{\alpha,\beta}$ is a semiring monomorphism.

Fourthly, we show that $\phi_{\alpha,\beta}$ satisfies (1.3).

Let$(i,a)\in S_\alpha,(j,b)\in S_\beta$ and $(k,0)\in E^+(S_\gamma)$,where $\alpha+\beta\leq\gamma$.Then

$$\begin{aligned}
& [(i,a)\phi_{\alpha,\gamma}][(j,b)\phi_{\beta,\gamma}] = [(i,a)+(k,0)+(i,0)][(j,b)+(k,0)+(j,0)] \\
= & (i,a)(j,b)+(i,a)(k,0)+(i,a)(j,0)+(k,0)(j,b)+(k,0)(k,0) \\
& +(k,0)(j,0)+(i,0)(j,b)+(i,0)(k,0)+(i,0)(j,0) \\
= & (i,a)(j,b)+(i,a)(j,0)+(i,a)(k,0)+(i,a)(j,0)+(k,0)(j,b) \\
& +(k,0)+(k,0)(j,0)+(i,0)(j,b)+(i,0)(k,0)+(i,0)(j,0) \\
= & (i,a)(j,b)+(i,a)(j,0)+(i,a)(j,0)+(i,0)(j,b)+(i,a)(k,0) \\
& +(k,0)(j,b)+(k,0)+(k,0)(j,0)+(i,0)(k,0)+(i,0)(j,0) \\
= & (i,a)(j,b)+(i,a)(k,0)+(k,0)(j,b)+(k,0)+(k,0)(j,0)+(i,0)(k,0) \\
& +(i,0)(j,0) \\
= & (i,a)(j,b)+(k_1,0)+(i,0)(j,0) \\
& [\text{put } (i,a)(k,0)+(k,0)(j,b)+(k,0)+(k,0)(j,0)+(i,0)(k,0)=(k_1,0)\in \\
& \qquad E^+(S_{\alpha\gamma+\beta\gamma+\gamma})=E^+(S_\gamma)] \\
= & [(i,a)(j,b)]\phi_{\alpha\beta,\gamma}.
\end{aligned}$$

Hence,(1.3) holds.

Finally,we show that $\phi_{\alpha,\beta}$ satisfies (1.4). For $(i,a)\in S_\alpha,(j,b)\in S_\beta$,if $(i,a)+(j,b)=(k,c)\in S_{\alpha+\beta}$, then

$$(i,a)+(j,b)=(i,a)+(j,b)+(k,0)=(i,a)+[(j,b)+(k,0)].$$

Now,we let $(j,b)+(k,0)=(l,d)\in S_{\alpha+\beta}$,then,we have

$$\begin{aligned}
& (i,a)+(j,b)=(i,a)+(j,b)+(j,0) \\
= & (i,a)+(j,b)+(k,0)+(j,0) \\
= & (i,a)+(l,0)+(j,b)+(k,0)+(j,0) \\
= & (i,a)+(i,0)+(i,0)+(l,0)+(j,0)+(j,b)+(k,0)+(j,0) \\
= & (i,a)+(i,0)+(l,0)+(i,0)+(j,0)+(j,b)+(k,0)+(j,0) \\
= & [(i,a)+(l,0)+(i,0)]+[(j,b)+(k,0)+(j,0)] \\
= & (i,a)\phi_{\alpha,\alpha+\beta}+(j,b)\phi_{\beta,\alpha+\beta}.
\end{aligned}$$

Also,we have

$$\begin{aligned}
& [(i,a)\phi_{\alpha,\alpha+\beta}][(j,b)\phi_{\beta,\alpha+\beta}] = [(i,a)+(k,0)+(i,0)][(j,b)+(k,0)+(j,0)] \\
= & (i,a)(j,b)+(i,a)(k,0)+(i,a)(j,0)+(k,0)(j,b)+(k,0)(k,0) \\
& +(k,0)(j,0)+(i,0)(j,b)+(i,0)(k,0)+(i,0)(j,0) \\
= & (i,a)(j,b)+(i,a)(k,0)+(k,0)(j,b)+(k,0)+(k,0)(j,0) \\
& +(i,0)(k,0)+(i,0)(j,0) \\
= & (i,a)(j,b)+(k_1,0)+(i,0)(j,0) \\
& [\text{put } (i,a)(k,0)+(k,0)(j,b)+(k,0)+(k,0)(j,0)+(i,0)(k,0)=(k_1,0)\in
\end{aligned}$$

$$\begin{aligned} & E^+(S_{\alpha(\alpha+\beta)+\beta(\alpha+\beta)+(\alpha+\beta)}) = E^+(S_{\alpha+\beta})] \\ = \quad & [(i,a)(j,b)\phi_{\alpha\beta,\alpha+\beta}], \end{aligned}$$

Since $\phi_{\alpha\beta,\alpha+\beta}$ in injective,we obtain

$$(i,a)(j,b) = [(i,a)\phi_{\alpha,\alpha+\beta} \cdot (j,b)\phi_{\beta,\alpha+\beta}]\phi^{-1}_{\alpha\beta,\alpha+\beta},$$

thus,(1.4) holds.

Hence, S is a strong b-lattice of rectangular skew-rings. □

In the above theorem, if the regular additive reduct $(S,+)$ of S is commutative, then we immediately have:

COROLLARY 3.6 *A generalized rectangular Clifford semiring S is a strong b-lattice of rectangular rings if and only if $E^+(S)$ is a generalized normal band semiring and is unitary in the commutative regular additive reduct $(S,+)$ of S.*

In the above corollary, it is clear that any strong b-lattice $\langle B, S_\alpha, \phi_{\alpha,\beta}\rangle$ of rectangular rings S_α is a generalized rectangular Clifford semiring. However, the converse may not be true. The counterexample can be also seen in the example 3.10 in [14].

Remark. Theorem 3.5 and Corollary 3.6 actually strengthen and extend the corresponding results about Clifford semirings in [9] and rectangular Clifford semirings in [5].

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