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Some Properties of Quotient Topology on Residuated Lattices

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Abstract. In this paper, we consider uniform topology on a residuated lattice. We use this to define the quotient topology and the uniform topology on a quotient residuated lattice. We investigate relationship between these topologies on a quotient residuated lattice and study the properties of these topologies.

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1 Introduction

The concept of residuated lattices were firstly introduced by M. Ward and R.P. Dilworth [9] as generalization of ideal of rings. The properties of these structures were presented in [1], [2] and [8].
The quotient residuated lattice induced by a filter was defined and studied in [3]. We used the cocept of filters to define uniform topology on a residuated lattice and obtained some properties of this topology in [4]. In this paper, we use uniform topology to define uniform topology and quotient topology on a quotient residuated lattice and prove that these topologies coincide. Then we investigate some properties of this topological space such as compactness, connectness, Hausdorff property, etc.

2 Preliminaries

DEFINITION 2.1 ([1]) A *residuated lattice* is an algebraic structure $L = (L, \wedge, \vee, \rightarrow, *, 0, 1)$ such that
(1) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice with the least element 0 and the greatest element 1,
(2) $(L, *, 1)$ is a commutative monoid where 1 is a unit element,
(3) $x * y \leq z$ iff $x \leq y \rightarrow z$, for all $x, y, z \in L$.

Let $(L, \wedge, \vee, *, \rightarrow, 0, 1)$ be a residuated lattice. From now on, let L denote a residuated lattice.

PROPOSITION 2.2 *([7], [8]) Let L be a residuated lattice. Then we have the following properties:*
(1) $1 \rightarrow x = x$, $x \rightarrow x = 1$, $x \rightarrow 1 = 1$,
(2) $x \leq y$ *if and only if* $x \rightarrow y = 1$,
(3) if $x \leq y$, *then* $z \rightarrow x \leq z \rightarrow y$ *and* $y \rightarrow z \leq x \rightarrow z$,
(4) $y \leq x \rightarrow y$,
(5) $x * (x \rightarrow y) \leq x, y$,
(6) $x \rightarrow (y \rightarrow z) = (x * y) \rightarrow z$,
(7) $(x \rightarrow y) * (y \rightarrow z) \leq x \rightarrow z$,
(8) $(x \rightarrow y) \vee (x \rightarrow z) \leq x \rightarrow (y \vee z)$,
(9) $(y \vee z) \rightarrow x = (y \rightarrow x) \wedge (z \rightarrow x)$,
for all $x, y, z \in L$.

DEFINITION 2.3 ([7], [8]) Let F be a non-empty subset of a residuated lattice L. F is called a *filter* if
(1) $1 \in F$,
(2) if $x, x \rightarrow y \in F$, then $y \in F$.

THEOREM 2.4 *([7], [8]) A non-empty subset F of a residuated lattice L is a filter if and only if*
(1) $x * y \in F$ *for all* $x, y \in F$,
(2) if $x \leq y$ *and* $x \in F$, *then* $y \in F$.

DEFINITION 2.5 ([7], [8]) For every subset $X \subseteq L$, the smallest filter of L contaning X (i.e., the intersection of all filters F of L such that $X \subseteq F$) is called the *filter generated* by X and will be denoted by $< X >$.

THEOREM 2.6 *([7], [8]) Let F_1, F_2 be non-empty filters of L, then*

$$< F_1 \cup F_2 >= \{x \in L : x \geq a * b, \quad for \quad some \quad a \in F_1, b \in F_2\}.$$

THEOREM 2.7 *([3]) Let J be a filter of L. Define*

$$x \equiv_J y \text{ if and only if } x \rightarrow y \in J \text{ and } y \rightarrow x \in J.$$

Then $\equiv_J$ is a congruence relation on L.

THEOREM 2.8 *([3]) Let J be a filter of L and $\equiv_J$ be the congruence relation which is defined in Theorem 2.7. The set of all congruence classes $\bar{x} = \{y \in L : x \equiv_J y\}$ is denoted by L/J. On this set define*

$$\bar{x} \wedge \bar{y} = \overline{x \wedge y} \quad , \quad \bar{x} \vee \bar{y} = \overline{x \vee y},$$
$$\bar{x} * \bar{y} = \overline{x * y} \quad , \quad \bar{x} \rightarrow \bar{y} = \overline{x \rightarrow y}.$$

Then $(L/J, \wedge, \vee, \rightarrow, *, \bar{0}, \bar{1})$ is a residuated lattice which is called the quotient residuated lattice with respect to the filter J.

Notation. Let X be a non-empty set, U and V be subsets of $X \times X$. Then we define
$U \circ V = \{(x, y) \in X \times X : (z, y) \in U \text{ and } (x, z) \in V \text{ for some } z \in X\}$,
$U^{-1} = \{(x, y) \in X \times X : (y, x) \in U\}$,
$\Delta = \{(x, x) \in X \times X : x \in X\}$.

DEFINITION 2.9 ([5], [6]) Let X be a non-empty set. A uniformity on X is a non-empty collection K of subsets of $X \times X$ which satisfy the following conditions:
(U1) $\Delta \subseteq U$ for any $U \in K$,
(U2) if $U \in K$, then $U^{-1} \in K$,
(U3) if $U \in K$, then there exists a $V \in K$ such that $V \circ V \subseteq U$,
(U4) if $U, V \in K$, then $U \cap V \in K$,
(U5) if $U \in K$ and $U \subseteq V \subseteq X \times X$, then $V \in K$.
Then pair (X, K) is called a *uniform structure.*

THEOREM 2.10 *([4]) Let Λ be a family of filters of a residuated lattice L which is closed under intersection. Suppose that $U_F = \{(x, y) \in L \times L : x \equiv_F y\}$ for every $F \in \Lambda$ and $\mathcal{K}^* = \{U_F : F \in \Lambda\}$. Then $\mathcal{K} = \{U \subseteq L \times L : U_F \subseteq U \text{ for some } U_F \in \mathcal{K}^*\}$ is a uniformity on L. Let $U \in \mathcal{K}$, define $U[x] = \{y \in L : (x, y) \in U\}$. Then $\tau_\Lambda = \{O \subseteq L : \forall x \in O, \exists U \in \mathcal{K}$ such that $U[x] \subseteq O\}$ is a topology on L and is called the uniform topology on L induced by Λ.*

3 Quotient topology on a residuated lattice

Let Λ be a family of filters of a residuated lattice L which is closed under intersection. In this section, L is a topological space with uniform topology τ_Λ induced by Λ.

THEOREM 3.1 *Let J be a filter of L. For each $F \in \Lambda$, let $\bar{F} = < F \cup J > /J$. Then $\Lambda^* = \{\bar{F} : F \in \Lambda\}$ gives a uniform topology on L/J, where $U_{\bar{F}} = \{(\bar{x}, \bar{y}) \in L/J \times L/J : \bar{x} \equiv_{\bar{F}} \bar{y}\}$.*

Proof. It follows from Theorem 2.10. □

From now on, let τ be the quotient topology on L/J ($\tau = \{O \subseteq L/J : \pi^{-1}(O) \in \tau_\Lambda\}$ where $\pi : L \to L/J$ is the canonical epimorphism) and τ_{Λ^*} be the uniform topology induced by Λ^* on L/J.

THEOREM 3.2 *For each $x \in L$ and $F \in \Lambda$, $\pi(U_F[x]) = U_{\bar{F}}[\bar{x}]$. Hence each $U_{\bar{F}}[\bar{x}]$ is open in the quotient topology and $\tau_{\Lambda^*} \subseteq \tau$.*

Proof. Let $\bar{y} \in \pi(U_F[x])$. There exists $z \in U_F[x]$ such that $\pi(z) = \bar{y}$, that is $\bar{z} = \bar{y}$. By Theorem 2.10 and Theorem 2.7, we have

$$z \to x \in F \subseteq < F \cup J > \quad , \quad x \to z \in F \subseteq < F \cup J >,$$
$$z \to y \in J \subseteq < F \cup J > \quad , \quad y \to z \in J \subseteq < F \cup J >.$$

By Proposition 2.2 part (7),

$$(y \to z) * (z \to x) \leq y \to x,$$
$$(x \to z) * (z \to y) \leq x \to y.$$

Since $< F \cup J >$ is a filter, then $y \to x, x \to y \in < F \cup J >$. Hence $\bar{y} \to \bar{x}, \bar{x} \to \bar{y} \in < F \cup J > /J$, that is $\bar{x} \equiv_{\bar{F}} \bar{y}$. Then $\bar{y} \in U_{\bar{F}}[\bar{x}]$ and $\pi(U_F[x]) \subseteq U_{\bar{F}}[\bar{x}]$.

Conversely, let $\bar{y} \in U_{\bar{F}}[\bar{x}]$. Then $\bar{y} \to \bar{x}, \bar{x} \to \bar{y} \in \bar{F}$. There exist $a_1, a_2 \in < F \cup J >$ such that $\bar{y} \to \bar{x} = \bar{a_1}$ and $\bar{x} \to \bar{y} = \bar{a_2}$. By Theorem 2.7, we have

$$a_1 \to (y \to x) \in < F \cup J > \quad , \quad (y \to x) \to a_1 \in < F \cup J >,$$
$$a_2 \to (y \to x) \in < F \cup J > \quad , \quad (y \to x) \to a_2 \in < F \cup J >.$$

Since $< F \cup J >$ is a filter, then $y \to x, x \to y \in < F \cup J >$. By Theorem 2.6, there exist $b_1, b_2 \in F$ and $c_1, c_2 \in J$ such that $b_1 * c_1 \leq x \to y$ and $b_2 * c_2 \leq y \to x$. By Proposition 2.2 part (6),

$$b_1 \leq c_1 \to (x \to y) = (c_1 * x) \to y,$$
$$b_2 \leq c_2 \to (y \to x) = (c_2 * y) \to x,$$

Let $z = (b_1 * x) \vee (c_2 * y)$. By Proposition 2.2 part (4) and (8)

$$\begin{aligned} x \to z = x \to ((b_1 * x) \vee (c_2 * y)) &\geq (x \to (b_1 * x)) \vee (x \to (c_2 * y)) \\ &\geq x \to (b_1 * x) \geq b_1. \end{aligned}$$

That is $x \to z \in F$. By Proposition 2.2 part (1) and (9)

$$\begin{aligned} z \to x &= ((b_1 * x) \vee (c_2 * y)) \to x = ((b_1 * x) \to x) \wedge ((c_2 * y) \to x) \\ &= 1 \wedge ((c_2 * y) \to x = (c_2 * y) \to x) \geq b_2. \end{aligned}$$

Hence $z \to x \in F$. We have $x \equiv_F z$ and then $z \in U_F[x]$.
Now, we will show that $\bar{z} = \bar{y}$. By Proposition 2.2 part (4) and (8)

$$\begin{aligned} y \to z = y \to ((b_1 * x) \vee (c_2 * y)) &\geq (y \to (b_1 * x)) \vee (y \to (c_2 * y)) \\ &\geq (y \to (c_2 * y)) \geq c_2. \end{aligned}$$

Therefore $y \to z \in J$. By Proposition 2.2 part (1) and (9)

$$\begin{aligned} z \to y &= ((b_1 * x) \vee (c_2 * y)) \to y = ((b_1 * x) \to y) \wedge ((c_2 * y) \to y) \\ &= ((b_1 * x) \to y) \wedge 1 = (b_1 * x) \to y \geq c_1. \end{aligned}$$

Hence $z \to y \in J$. We have $z \equiv_J y$, that is $\bar{z} = \bar{y}$ in L/J. Then $\pi(z) = \bar{z} = \bar{y}$ in L/J. We get that $U_{\bar{F}}[\bar{x}] \subseteq \pi(U_F[x])$. □

Theorem 3.3 *$\tau = \tau_{\Lambda^*}$ in L/J.*

Proof. Let $O \in \tau$ and $\bar{x} \in O$ be arbitrary. Then $\pi^{-1}(O)$ is open set in τ_Λ and contain x. Hence there exists $F \in \Lambda$ such that $U_F[x] \subseteq \pi^{-1}(O)$. By Theorem 3.2, $U_{\bar{F}}[\bar{x}] \subseteq O$. Hence $O \in \tau_{\Lambda^*}$. Therefore $\tau \subseteq \tau_{\Lambda^*}$ and then $\tau = \tau_{\Lambda^*}$, by Theorem 3.2. □

Corollary 3.4 *Let J be a filter of L. The* quotient topology *on L/J is the strongest topology on L/J for which the canonical epimorphism $\pi : L \longrightarrow L/J$ is continuous and open.*

Proof. It follows from Theorem 3.3. □

Theorem 3.5 *Let S be a topological subalgebra of L and J be a filter of L such that $J \subseteq S$. Then the quotient topology on S/J is stronger than the topology induced on the subalgebra S/J of L/J by the quotient topology on L/J.*

Proof. Let $\pi_S : S \to S/J$ be the canonical map and U be an open set in the subspace topology of S/J. Then $U = W \cap S/J$ where W is open in L/J. We obtain that $\pi_S^{-1}(U) = \pi^{-1}(W) \cap S$ is open in S. Hence U is open in the quotient topology of S/J. □

Example 3.6 Let L be a residuated lattice with the universe $\{0, a, b, c, 1\}$ such that $0 < c < a, b < 1$ and a, b are incomparable. The operations $*$ and $\rightarrow$ are given by the tables below:

$*$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	c	c	a
b	0	c	b	c	b
c	0	c	c	c	c
1	0	a	b	c	1

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	b	b	1
b	0	a	1	a	1
c	0	1	1	1	1
1	0	a	b	c	1

Define $F = \{b, 1\}$ and $\Lambda = \{F\}$. We have $U_F[0] = \{0\}$, $U_F[a] = U_F[c] = \{a, c\}$ and $U_F[b] = U_F[1] = \{b, 1\}$. Hence

$$\tau_\Lambda = \{\emptyset, L, \{0\}, \{a, c\}, \{b, 1\}, \{0, a, c\}, \{0, b, 1\}, \{b, a, c, 1\}\}.$$

Consider $S = \{0, c, 1\}$. Then S is a subalgebra of L and the subspace topology on S is the discrete topology.
Define $J = \{a, 1\}$. Then J is a filter of L. By Theorem 2.8, L/J is a residuated lattice with the universe $\{\bar{0}, \bar{b}, \bar{1}\}$ where $\bar{b} = \bar{c}$, $\bar{a} = \bar{1}$ and $\bar{0} < \bar{b} < \bar{1}$. The operations $*$ and $\rightarrow$ are given by the tables below:

$*$	$\bar{0}$	$\bar{b}$	$\bar{1}$
$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$
$\bar{b}$	$\bar{0}$	$\bar{b}$	$\bar{b}$
$\bar{1}$	$\bar{0}$	$\bar{b}$	$\bar{1}$

$\rightarrow$	$\bar{0}$	$\bar{b}$	$\bar{1}$
$\bar{0}$	$\bar{1}$	$\bar{1}$	$\bar{1}$
$\bar{b}$	$\bar{0}$	$\bar{1}$	$\bar{1}$
$\bar{1}$	$\bar{0}$	$\bar{b}$	$\bar{1}$

Since $< F \cup J >= \{a, b, c, 1\}$, then $\bar{F} =< F \cup J > /J = \{\bar{b}, \bar{1}\}$. Therefore $U_F[\bar{0}] = \{\bar{0}\}$ and $U_F[\bar{b}] = U_F[\bar{1}] = \{\bar{b}, \bar{1}\}$. Hence we have

$$\tau_{\Lambda^*} = \{\emptyset, L/J, \{\bar{0}\}, \{\bar{1}, \bar{b}\}\},$$

Since $S/J = L/J$, then the subspace topology on S/J induced by the quotient topology on L/J is τ_{Λ^*}, but the quotient topology on S/J is the discrete topology.

Notation. Let A, B be subsets of L. Then we define

$$A \star B = \{a \star b : a \in A, b \in B\} \quad , \quad \bar{A} = \{\bar{a} : a \in A\}$$

where $\star \in \{\wedge, \vee, *, \rightarrow\}$. If $A = \{a\}$, then we write $a \rightarrow B$ instead of $\{a\} \rightarrow B$.

DEFINITION 3.7 Let L be a residuated lattice and T be a topology defined on set L. Then we say that the pair (L, T) is a *topological residuated lattice* if the operations $\wedge, \vee, *$ and $\rightarrow$ be continuous with respect to T.

Remark. The continuity of operation $\star$ where $\star \in \{\wedge, \vee, *, \rightarrow\}$ is equivalent to having the following property satisfied:
Let O be an open set and $x, y \in L$ such that $x \star y \in O$. Then there exist open sets O_1 and O_2 such that $x \in O_1$ and $y \in O_2$ and $O_1 \star O_2 \subseteq O$.

THEOREM 3.8 *Let J be a filter of L. Then $(L/J, \tau)$ is a topological residuated lattice.*

Proof. Since (L, τ_Λ) is a topological residuated lattice and π is continuous, then each operation of L/J is continuous in the quotient topology. □

THEOREM 3.9 *Let J be a filter of L. Then $(L/J, \tau)$ is Hausdorff if and only if J is closed.*

Proof. Suppose that $(L/J, \tau)$ is Hausdorff. We will show that J^c is open. Let $x \in J^c$. Then $x \notin J$. We have $\bar{x} \neq \bar{1}$ in L/J. Then there exist open neighborhoods U, V of $\bar{x}, \bar{1}$ respectively in $(L/J, \tau)$ such that $U \cap V = \emptyset$. So $\pi^{-1}(U), \pi^{-1}(V)$ are open neighborhoods of $x, 1$ respectively. Let $z \in J$. Then $\pi(z) = \bar{z} = \bar{1} \in V$. Hence $z \in \pi^{-1}(V)$, that is $J \subseteq \pi^{-1}(V)$. We will show that $\pi^{-1}(U) \subseteq J^c$. Let $z \in \pi^{-1}(U)$. Then $\bar{z} \in U$. We get that $\bar{z} \notin V$. Thus $z \notin \pi^{-1}(V)$. We have $z \notin J$ and then $z \in J^c$. Therefore J is closed.
Conversely, suppose that J is closed and $\bar{x} \neq \bar{y}$ in L/J. Then $\bar{x} \rightarrow \bar{y} \neq \bar{1}$ or $\bar{y} \rightarrow \bar{x} \neq \bar{1}$. We assume that $\bar{x} \rightarrow \bar{y} \neq \bar{1}$. Hence $x \rightarrow y \notin J$. There exists an open neighborhood U of $x \rightarrow y$ such that $U \cap J = \emptyset$. We have $\pi(U)$ is open neighborhood of $\bar{x} \rightarrow \bar{y}$. We will show that $\bar{1} \notin \pi(U)$. Let $\bar{1} \in \pi(U)$. Then there exists $a \in U$ such that $\bar{a} = \bar{1}$. Thus $a = 1 \rightarrow a \in J$, that is $U \cap J \neq \emptyset$ which is a contradiction. By Remark, there exist open neighborhoods V, W of x, y respectively in L such that $V \rightarrow W \subseteq U$. Hence

$$\pi(V) \rightarrow \pi(W) = \bar{V} \rightarrow \bar{W} = \overline{V \rightarrow W} = \pi(V \rightarrow W) \subseteq \pi(U),$$

where $\pi(V), \pi(W)$ are open neighborhoods of $\bar{x}, \bar{y}$ respectively. We will show that $\pi(V) \cap \pi(W) = \emptyset$ and than $(L/J, \tau)$ is Hausdorff. Let $\bar{z} \in \pi(V) \cap \pi(W)$. Then there exist $a \in V$ and $b \in W$ such that $\bar{a} = \bar{z} = \bar{b}$. We get that $\bar{1} = \bar{z} \rightarrow \bar{z} \in \pi(V) \rightarrow \pi(W) \subseteq \pi(U)$. Hence $\bar{1} \in \pi(U)$ which is a contradiction. □

THEOREM 3.10 *Let J be a filter of L. Then $(L/J,\tau)$ is discrete if and only if J is open.*

Proof. If $(L/J,\tau_{\Lambda^*})$ is discrete, then it is clear that J is open.
Conversely, suppose that J is open and $\bar{x}\in L/J$ be arbitrary. Then $\pi(J)=\bar{J}=\{\bar{1}\}$ is open in $(L/J,\tau)$. By Theorem 3.2, $U_{\bar{J}}[\bar{x}]$ is open in the quotient topology. We have

$$\begin{aligned}U_{\bar{J}}[\bar{x}] &= \{\bar{y}\in L/J : \bar{x}\equiv_{\bar{F}}\bar{y}\}\\ &= \{\bar{y}\in L/J : \bar{x}\to\bar{y}\in\bar{J}, \bar{y}\to\bar{x}\in\bar{J}\}\\ &= \{\bar{y}\in L/J : \bar{x}\to\bar{y}=\bar{1}, \bar{y}\to\bar{x}=\bar{1}\}\\ &= \{\bar{y}\in L/J : \bar{x}=\bar{y}\}=\{\bar{x}\}.\end{aligned}$$

Hence $\{\bar{x}\}$ is open in the quotient topology and then $(L/J,\tau)$ is discrete. □

In [4], we prove that (L,τ_Λ) is locally compact. So, we have the following theorem.

THEOREM 3.11 *$(L/J,\tau)$ is locally compact.*

Proof. Let $\bar{x}\in L/J$. Since L is locally compact space by uniform topology induced by Λ, there exists an open neighborhood O of x and a compact set K such that $x\in O\subseteq K$. Since π is open and continuous map, we have that $\pi(O)$ is open and $\pi(K)$ is compact such that $\bar{x}\in\pi(O)\subseteq\pi(K)$. Thus L/J is locally compact at $\bar{x}$ and hence is locally compact. □

THEOREM 3.12 *([6]) Every locally compact, Hausdorff space is regular.*

COROLLARY 3.13 *If J is closed, then $(L/J,\tau)$ is regular.*

Proof. It follows from Theorem 3.9, Theorem 3.11. □

THEOREM 3.14 *Let $x\in L$ and $F_0=\bigcap\limits_{F\in\Lambda}F$. Then $U_{F_0}[x]$ is a connected subset of (L,τ_Λ).*

Proof. Let $U_{F_0}[x]=A\cup B$ where $A\cap B=\emptyset$ and A,B are non-empty open subset of $U_{F_0}[x]$ and hence L. We have $x\in U_{F_0}[x]$. Hence $x\in A$ or $x\in B$. We assume that $x\in A$. Since A is open in L, there exists $F\in\Lambda$ such that $U_F[x]\subseteq A$. We get that $U_{F_0}[x]\subseteq U_F[x]\subseteq A$ which is a contradiction. Hence $U_{F_0}[x]$ is a connected subset of (L,τ_Λ). □

COROLLARY 3.15 *(L,τ_Λ) is locally connected.*

Proof. Let $x\in L$ and O be a neighborhood of x. Then there exists $F\in\Lambda$ such that $U_F[x]\subseteq O$. We get that $U_{F_0}[x]\subseteq U_F[x]\subseteq O$ where $U_{F_0}[x]$ is a connected subset of (L,τ_Λ) by Theorem 3.14. Hence L is locally connected subset of (L,τ_Λ) at each of its point. □

PROPOSITION 3.16 *([5]) Every quotient space of a locally connected space is locally connected.*

COROLLARY 3.17 $(L/J, \tau)$ *is locally connected.*

LEMMA 3.18 *Let L be a residuated lattice such that $x * (x \to y) = x \wedge y$ and J be a filter of L. Then $\bar{x} = \bar{y}$ if and only if $x * a = y * b$ for some $a, b \in J$.*

Proof. Let $\bar{x} = \bar{y}$. Then $x \to y \in J$ and $y \to x \in J$. Hence there exists $a \in J$ such that $x \to y = a$. By Proposition 2.2 part (1), $a \to (x \to y) = 1$. We obtain that $(a * x) \to y = 1$, that is $a * x \leq y$. Hence $y * (y \to x * a) = x * a$. We will show that $y \to a * x \in J$. Since $a * y * (y \to x) \leq x * a$, then $a * (y \to x) \leq y \to (x * a)$. We have $a, y \to x \in J$. Hence $y \to a * x \in J$. Put $y \to a * x = b$. Then $x * a = y * b$ for some $a, b \in J$. Conversely, let $x * a = y * b$ for some $a, b \in J$. Since $x * a \leq x$, then $1 \leq (x * a) \to x = (y * b) \to x$. By Proposition 2.2 part (6), $b \to (y \to x) = 1$. Hence $b \leq y \to x$, that is $y \to x \in J$. Similarly, we can show that $x \to y \in J$. Hence $\bar{x} = \bar{y}$. □

DEFINITION 3.19 *([6]) A* component *of a space is a maximally connected subset, that is, a connected subset which is not properly contained in any connected subset of that space.*

LEMMA 3.20 *Let L be a residuated lattice such that $x * (x \to y) = x \wedge y$ and J be a filter of L. If J is a connected component of 1 and C is a closed subset of L/J such that $\pi^{-1}(C)$ is disconnected, then C is disconnected.*

Proof. Let $\pi^{-1}(C) = A \cup B$ where $A \cap B = \emptyset$ and A, B are non-empty closed subset of $\pi^{-1}(C)$ and hence L. We will show that $A = \pi^{-1}(\pi(A))$. Clearly $A \subseteq \pi^{-1}(\pi(A))$. Now, let $y \in \pi^{-1}(\pi(A))$. Then $\bar{y} = \bar{x}$ for some $x \in A$. By Lemma 3.18, $x * a = y * b$ for some $a, b \in J$. We have $x * a \in x * J \subseteq J_x$ where J_x is the connected component of L containing x and $y * b \in y * J \subseteq J_y$ where J_y is the connected component of L containing y. Therefore $J_x \cap J_y \neq \emptyset$ and hence $J_x \cup J_y$ is connected, that is $J_x = J_y$. We have $y \in J_x \subseteq A$. Thus $A = \pi^{-1}(\pi(A))$. Similarly, we can show that $B = \pi^{-1}(\pi(B))$. Thus $\pi^{-1}(\pi(A) \cap \pi(B)) = A \cap B = \emptyset$. Hence $\pi(A) \cap \pi(B) = \emptyset$. We have $\pi(L - A) = L/J - \pi(A)$. Since $L - A$ is open and π is an open map, then $\pi(A)$ is closed. Similarly, $\pi(B)$ is closed. Now,

$$\pi^{-1}(C) = A \cup B = \pi^{-1}(\pi(A)) \cup \pi^{-1}(\pi(B)) = \pi^{-1}(\pi(A) \cup \pi(B)).$$

Therefore $C = \pi(A) \cup \pi(B)$ where $\pi(A) \cap \pi(B) = \emptyset$ and $\pi(A), \pi(B)$ are non-empty closed subsets of C and hence C is disconnected. □

LEMMA 3.21 *Let J be a filter of L and C be a connected subset of $(L/J, \tau)$. Then $\bar{x} \to C$ and $C \to \bar{x}$ are connected subsets of $(L/J, \tau)$.*

Proof. Define $L_{\bar{x}}, R_{\bar{x}} : L/J \to L/J$ by $L_{\bar{x}}(\bar{y}) = \bar{x} \to \bar{y}$ and $R_{\bar{x}}(\bar{y}) = \bar{y} \to \bar{x}$. Since L/J is a topological residuated lattice, the $L_{\bar{x}}, R_{\bar{x}}$ are continuous. Therefore $L_{\bar{x}}(C) = \bar{x} \to C$ and $R_{\bar{x}} = C \to \bar{x}$ are connected subsets of $(L/J, \tau)$. □

A space is said to be totally disconnected, if all components are singleton sets.[6]

THEOREM 3.22 *Let L be a residuated lattice such that $x * (x \to y) = x \wedge y$ and J be a filter of L. If J is a connected component of 1, then L/J is totally disconnected.*

Proof. Let C be the connected component of $\bar{1}$ in L/J. We Will show that $C = \{\bar{1}\}$. Suppose that $\bar{x} \in C$ such that $\bar{1} \neq \bar{x}$. Then J properly contained in $\pi^{-1}(C)$ and hence $\pi^{-1}(C)$ is disconnected. Since C is closed, by Lemma 3.20, we have K is disconnected which is a contradiction. Now, let $C_{\bar{x}}$ be the connected component of $\bar{x}$ in L/J. By Lemma 3.21, $\bar{x} \to C_{\bar{x}}$ is connected and contain $\bar{1}$. Hence $\bar{x} \to C_{\bar{x}} \subseteq C = \{\bar{1}\}$. So $\bar{x} \leq \bar{y}$ for all $\bar{y} \in C_{\bar{x}}$. Similarly, $C_{\bar{x}} \to \bar{x} \subseteq C = \{\bar{1}\}$ and hence $\bar{y} \leq \bar{x}$ for all $\bar{y} \in C_{\bar{x}}$. Thus $C_{\bar{x}} = \{\bar{x}\}$. □

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