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Duo semigroups of generalised transformations of sets

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Abstract. By definition, every quasi-ideal of a semigroup S is a bi-ideal of S, and many people have studied algebraic semigroups and transformation semigroups which satisfy the converse condition: that is, when every bi-ideal of S is a quasi-ideal of S. Likewise, several authors have investigated algebraic properties of semigroups S which satisfy the 'duo' condition: namely, every one-sided ideal of S is a two-sided ideal of S. Here, we determine when so-called 'generalised transformation semigroups' which satisfy a simple transitivity condition are left (or right) duo. We also consider similar ideas involving the notion of 'almost left (or right) ideal': that is, subsets L (or R) of a semigroup S such that $xL \cap L \neq \emptyset$ (or $Rx \cap R \neq \emptyset$) for all $x \in S$.

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1 Introduction

As in [22], a subsemigroup Q of a semigroup S is called a *quasi-ideal* of S if $SQ \cap QS \subseteq Q$; and a subsemigroup B of S is a *bi-ideal* of S if $BSB \subseteq B$. Note that every quasi-ideal Q of a semigroup S is a bi-ideal of S since

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$QSQ \subseteq SQ \cap QS$. Many authors have studied semigroups S which satisfy the converse property, namely:

(BQ) every bi-ideal of S is a quasi-ideal of S.

For example, in [9] Kapp showed that all regular semigroups and all right simple semigroups satisfy (BQ). Also, in [20] Mielke studied Green's relations on semigroups which satisfy (BQ). And, more recently in [18] Lemma 2.1, the authors extended one of Kapp's results by proving that any right ideal of a regular semigroup satisfies (BQ).

In addition, some authors have determined when certain transformation semigroups satisfy (BQ). To illustrate the nature of their work, we need some notation.

Suppose X, Y are non-empty sets and let $P(X,Y)$ denote the set of all *partial* transformations from X to Y: that is, all $\alpha : A \to B$ where $A \subseteq X$ and $B \subseteq Y$. If $\alpha \in P(X,Y)$, we let $\operatorname{dom} \alpha$ denote the *domain* of α and $\operatorname{ran} \alpha$ the *range* of α. In addition, if $\lambda \in P(X,Y)$ and $\mu \in P(Y,Z)$, we often write $\lambda\mu$ for the *composition* $\lambda \circ \mu$ whose domain is $(\operatorname{ran} \lambda \cap \operatorname{dom} \mu)\lambda^{-1}$. Next, for each $\theta \in P(Y,X)$, we define a *sandwich operation* '$*$' on $P(X,Y)$ as follows. For each $\alpha, \beta \in P(X,Y)$, we write

$$x(\alpha * \beta) = x(\alpha\theta\beta) \quad \text{for all } x \in \operatorname{dom}(\alpha\theta\beta).$$

We say $(P(X,Y), *)$ is a *generalised transformation semigroup* and denote it by $P(X,Y,\theta)$. In particular, if $X = Y$ and $\theta = \mathrm{id}_X$, the identity mapping on X, then $P(X,X,\mathrm{id}_X)$ equals $(P(X), \circ)$: that is, the semigroup $P(X)$ consisting of all partial transformations from X to X under composition. Now let

$$\begin{aligned}
T(X,Y) &= \{\alpha \in P(X,Y) : \operatorname{dom} \alpha = X\},\\
I(X,Y) &= \{\alpha \in P(X,Y) : \alpha \text{ is injective}\},\\
S(X,Y) &= \{\alpha \in P(X,Y) : \alpha \text{ is surjective}\},\\
M(X,Y) &= I(X,Y) \cap T(X,Y),\\
E(X,Y) &= S(X,Y) \cap T(X,Y).
\end{aligned}$$

In addition, if $\theta \in T(Y,X)$, we let $T(X,Y,\theta)$ denote the semigroup $T(X,Y)$ under the sandwich operation determined by θ, and likewise for each of the above sets. Also, like before when $X = Y$ and $\theta = \mathrm{id}_X$, we write $T(X,X,\mathrm{id}_X)$ more simply as $T(X)$ under composition, and similarly for each of the above sets.

In [11] Theorem 3.1, the authors showed that each of $P(X,Y,\theta)$, $T(X,Y,\theta)$ and $I(X,Y,\theta)$ satisfy (BQ). On the other hand, in [11] Theorem 3.2, they showed that $M(X,Y,\theta)$ satisfies (BQ) if and only if $|X| = |Y| < \aleph_0$; and they obtained a similar result for $E(X,Y,\theta)$ in [11] Theorem 3.3. For other examples, see the references and results in [23].

In this paper, we determine when certain generalised transformation semigroups satisfy properties analogous to (BQ). For example, by definition, every two-sided ideal of a semigroup is one-sided, and several authors have studied semigroups with the converse property: namely, every one-sided ideal is two-sided (that is, so-called *duo semigroups*: see [4] and the references therein). In particular, following [1], we say S is *left duo* if it satisfies

(LD) every left ideal of S is a right ideal of S,

and we say S is *right duo* if it satisfies the dual (RD) of (LD). In Section 2, we consider (LD) and (RD) for some subsemigroups of $P(X,Y,\theta)$ which satisfy a simple transitivity condition. And in Section 3, we consider similar ideas which involve the notion of 'almost left (or right) ideal': that is, subsets L (or R) of a semigroup S such that $xL \cap L \neq \emptyset$ (or $Rx \cap R \neq \emptyset$) for all $x \in S$. Surprisingly, there is no duality between our results for one condition and those for the corresponding dual condition.

In passing, we briefly note that (left/right) duo semigroups arise naturally in other areas of semigroup theory. For example, Lajos [14] showed that a semigroup S is a semilattice of right groups if and only if S is regular and right duo. And in [13] he established a connection between them and quasi-ideals: namely, S is regular and right duo if and only if $R \cap Q = RQ$ for every right ideal R and quasi-ideal Q of S (the same condition holds for bi-deals B of S). Of course, duo semigroups also arise when studying semigroups in which every one-sided congruence is two-sided (see [2]). In fact, they are also important when describing certain semigroups S which are *permutable*: that is, $\rho \circ \sigma = \sigma \circ \rho$ for all congruences ρ, σ on S (see [3]). In addition, duo semigroups constitute one class of semigroups for which there is an affirmative answer to Hotzel's Problem: is every semigroup with maximal condition on right congruences finitely-generated? – see [12].

Finally, we note that the notion of a generalised transformation semigroup has a long and interesting history. To briefly illustrate this, we let $M(m,n,F)$ denote the set of all $m \times n$ matrices over a field F, and fix an $n \times m$ matrix P over F. Then $M(m,n,F)$ is a semigroup under the *sandwich operation*: $A * B = A \cdot P \cdot B$, where '$\cdot$' denotes usual matrix multiplication. Moreover, $(M(m,n,F), +, *)$ is a ring, provided '$+$' is usual matrix addition. Something like this semigroup and this ring play a crucial role in the study of Rees matrix semigroups and Munn matrix algebras (see [5] vol. 1, pp. 88 and 162). Magill [15] was the first to consider an analogous 'sandwich operation' for transformations between two sets. His aim was to create an algebraic structure for the set $C(X,Y)$ of all continuous maps from one topological space X into another space Y; and then, to decide when that algebraic structure determines the topological structures of X and Y, and vice versa. To do this, he considered certain subsemigroups S of $P(X,Y,\theta)$ when $\operatorname{dom}\theta = Y$ and described all homomorphisms and isomorphisms between such semigroups. And, in [16], the authors described the regular elements in any subsemigroup S of $P(X,Y,\theta)$, where $\operatorname{dom}\theta \subseteq Y$, and they characterised when those elements are related under Green's relations on S. In the last three decades, various authors have studied other questions about $P(X,Y,\theta)$ and its subsemigroups, as well as semigroups and rings defined in a similar way in other algebraic contexts (for a survey of this work, see [24]).

2 Left (right) ideals

To avoid trivialities, our basic assumption is that $\theta \neq \emptyset$. In addition, $Y = A \dot{\cup} B$ means Y is a *disjoint* union of sets A and B, and we write id_Y for the identity transformation on Y.

We extend the convention introduced in [5] vol. 2, p. 241: namely, if $\alpha \in P(X,Y)$ is non-empty then we write

$$\alpha = \begin{pmatrix} A_i \\ y_i \end{pmatrix}$$

and take as understood that the subscript i belongs to some (unmentioned) index set I, that the abbreviation $\{y_i\}$ denotes $\{y_i : i \in I\}$, and that $X\alpha = \operatorname{ran}\alpha = \{y_i\}$, $y_i\alpha^{-1} = A_i$ and $\operatorname{dom}\alpha = \bigcup\{A_i : i \in I\}$. In particular, A_y denotes the constant map with domain $A \subseteq X$ and range $\{y\}$ in Y, and we write this as x_y when $A = \{x\}$.

First we consider the 'left duo' property:

(LD) every left ideal of S is a right ideal of S.

It is easy to see that a semigroup S is left duo if and only if, for all $a, b \in S$, there exists $y \in S^1$ such that $ab = ya$ (and right duo if and only if $ab = bx$ for some $x \in S^1$).

THEOREM 2.1 *Suppose S is a subsemigroup of $P(X,Y,\theta)$ where $\theta \neq \emptyset$ and S contains all injective constants. Then S is left duo if and only if $|Y| = 1$.*

Proof. Suppose S is left duo. Since $\theta \neq \emptyset$, there exists $y \in \operatorname{dom}\theta$. Let $y\theta = x$. For each $b \in Y$, $x_b \in S$ and so, by assumption, there exists some $\mu \in S^1$ such that $x_y\theta x_b = \mu\theta x_y$. Since the product on the left is non-zero, we deduce that $b = y$ and hence $|Y| = 1$.

Conversely, suppose $Y = \{y\}$. Then, every non-empty element of S has the form A_y for some $A \subseteq X$. Also, $\theta = y_x$ for some $x \in X$. Let $A_y, B_y \in S$. If $x \in B$, then $A_y\theta B_y = A_y$. If $x \notin B$, then $\emptyset = A_y\theta B_y \in S$ and so $A_y\theta B_y = \emptyset\theta A_y$. Therefore, for every $\alpha, \beta \in S$, there exists $\mu \in S^1$ such that $\alpha\theta\beta = \mu\theta\alpha$. By the remark above, S is left duo. □

There is no obvious duality between the above result and the next, simply because the proof of Theorem 2.1 involves constants of the form A_y where $A \subseteq X$ and $Y = \{y\}$, whereas the converse of Theorem 2.2 obliges us to use constants like x_b where $X = \{x\}$ and $b \in Y$. Nonetheless, a proof of Theorem 2 can be given by closely following the one for Theorem 1, so we omit the details.

THEOREM 2.2 *Suppose S is a subsemigroup of $P(X, Y, \theta)$ where $\theta \neq \emptyset$ and S contains all injective constants. Then S is right duo if and only if $|X| = 1$.*

A similar approach using total constants leads to the following result.

THEOREM 2.3 *Suppose S is a subsemigroup of $T(X, Y, \theta)$ where S contains all total constants. Then S is left duo if and only if $|Y| = 1$.*

Next we consider (RD) for total transformations. Clearly, if $Y = \{y\}$ then $T(X, Y) = \{X_y\}$ and $\theta = y_a$ for some $a \in X$. Then $X_y y_a X_y = X_y y_a \mu$, where $\mu = X_y$, so $T(X, Y, \theta)$ is right duo when $|Y| = 1$. For a more general result, we say a subsemigroup S of $T(X, Y, \theta)$ is $(2,1)$-*transitive* if, for any distinct $x_0, x_1, x_2 \in X$ and $y_1, y_2 \in Y$, there exists $\alpha \in S$ such that $x_0\alpha = x_1\alpha = y_1$ and $x_2\alpha = y_2$. It is easy to see that, if $|X| \geq 3$ and $|Y| \geq 2$, then $T(X, Y, \theta)$ and many of its subsemigroups are $(2,1)$-transitive.

THEOREM 2.4 *Suppose $|X| \geq 3$ and $|Y| \geq 2$, and let S be a $(2,1)$-transitive subsemigroup of $T(X, Y, \theta)$. Then S is right duo if and only if θ is a constant.*

Proof. First, suppose there exist distinct $c_0, c_1 \in Y\theta \subseteq X$ and choose (distinct) $b_0, b_1 \in Y$ such that $b_i\theta = c_i$ for $i = 0, 1$. Let $d_1 \neq d_2$ in Y. Since

$|X| \geq 3$, we can choose $c_2 \in X \setminus \{c_0, c_1\}$. By $(2,1)$-transitivity, there exists $\alpha \in S$ such that $c_0\alpha = c_2\alpha = d_1$ and $c_1\alpha = d_2$, and we write

$$S(c_0, c_2) = \{\beta \in S : c_0\beta = c_2\beta\}.$$

Clearly, $\alpha \in S(c_0, c_2)$ and this is a right ideal of S. Therefore, if S is right duo, then $S(c_0, c_2)$ is also a left ideal. However, this is false since there exists $\lambda \in S$ such that $c_0\lambda = c_1\lambda = b_0$ and $c_2\lambda = b_1$, and thus

$$c_0\lambda\theta\alpha = b_0\theta\alpha = c_0\alpha = d_1 \neq d_2 = c_1\alpha = b_1\theta\alpha = c_2\lambda\theta\alpha.$$

Hence, $\lambda\theta\alpha \notin S(c_0, c_2)$. Therefore, if S is right duo then $|Y\theta| = 1$. Conversely, if $\alpha, \beta \in S$ and $\theta = Y_x$ for some $x \in X$, then $\alpha Y_x \beta = \beta Y_x \mu$ where $\mu = \beta \in S$, so S is right duo by the remark before Theorem 2.1. □

EXAMPLE 2.5 There are many subsemigroups of $P(X, Y, \theta)$ and $T(X, Y, \theta)$ which satisfy the conditions of Theorems 2.1-2.4. In particular, Theorems 2.1 and 2.2 determine when $I(X, Y, \theta)$ is left (or right) duo. Also, if $2 \leq r \leq |Y|$ and

$$P_r(X, Y) = \{\alpha \in P(X, Y) : |\operatorname{ran}\alpha| < r\}$$

then $P_r(X, Y, \theta)$ is a semigroup for any $\theta \in P(Y, X)$, and Theorem 2.1 implies that it is not left duo. Likewise, if $T_r(X, Y) = T(X, Y) \cap P_r(X, Y)$ where $|X| \geq 3$, $|Y| \geq 2$ and $r \geq 3$, then $T_r(X, Y)$ is a semigroup for any $\theta \in T(X, Y)$, and it is right duo only when θ is constant.

Clearly, $M(X, Y) \neq \emptyset$ if and only if $|X| \leq |Y|$. Hence, $M(X, Y, \theta)$ is a semigroup for a given $\theta \in M(Y, X)$ precisely when $|X| = |Y|$. We assert that if $|X| = |Y| < \aleph_0$, then $M(X, Y, \theta)$ is a group, so it is the only left (in fact, two-sided) ideal of itself and hence, in this case, $M(X, Y, \theta)$ is left duo. For, if $|X| = |Y| < \aleph_0$, then each $\alpha \in M(X, Y)$ is bijective, and the same is true for $\theta \in M(Y, X)$. Therefore, $\theta^{-1} \in M(X, Y)$ and $\alpha\theta\theta^{-1} = \theta^{-1}\theta\alpha = \alpha$, so θ^{-1} is an identity for $M(X, Y, \theta)$. Also, for each $\alpha \in M(X, Y)$, $\alpha\theta(\alpha\theta)^{-1}\theta^{-1} = \theta^{-1}$ and $(\alpha\theta)^{-1}\theta^{-1} \in M(X, Y)$, hence each element of $M(X, Y)$ has an inverse in $M(X, Y)$, and so the assertion holds. In fact, in this case, there is an isomorphism $(M(X, Y), \theta) \to (G(X), \circ)$, $\alpha \to \alpha\theta$, where $G(X)$ denotes the symmetric group on X.

We now consider the case when $|X| = |Y| \geq \aleph_0$. As usual, we say a subsemigroup S of $T(X, Y, \theta)$ is *transitive* if, for each $x \in X$ and $y \in Y$, $x\beta = y$ for some $\beta \in S$.

THEOREM 2.6 *If* $|X| = |Y| \geq \aleph_0$ *and* S *is a transitive subsemigroup of* $M(X, Y, \theta)$ *which contains a non-surjective element, then* S *is not left duo.*

Proof. Assume $\gamma \in S$ is non-surjective and suppose S is left duo. Fix $a \in X$, suppose $a\gamma = b, b\theta = c$, and choose $\gamma' \in S$ such that $c\gamma' = y \notin X\gamma$ (possible by transitivity). Then, since S is left duo, $\gamma\theta\gamma' = \mu\theta\gamma$ for some $\mu \in S^1$. But, this is a contradiction since the range of the product on the left contains y, whereas $X\mu\theta\gamma \subseteq X\gamma$ and $y \notin X\gamma$. Thus, S is not left duo. □

As in [23], if $\alpha \in T(X, Y)$, we write $d(\alpha) = |Y \setminus X\alpha|$ and call this the *defect* of α.

EXAMPLE 2.7 Let $|X| = |Y| \geq \aleph_0$, suppose r is a cardinal such that $|Y| \geq r \geq 0$ and write

$$M_r(X, Y) = \{\beta \in M(X, Y) : d(\beta) \geq r\}.$$

Then $M_r(X, Y, \theta)$ is a subsemigroup of $M(X, Y, \theta)$ for any $\theta \in M(Y, X)$ (since $X\alpha\theta\beta \subseteq X\beta$ implies $d(\alpha\theta\beta) \geq d(\beta)$). Clearly, $M_0(X, Y) = M(X, Y)$ and, for each $r \geq 0$, $M_r(X, Y)$ contains a non-surjective element. Furthermore, $M_r(X, Y, \theta)$ is transitive. For example, if $X = \{x\} \dot\cup \{x_i\}$ and $Y = \{y\} \dot\cup \{y_i\} \dot\cup \{y_j\}$, where $|J| \geq r$ (possible since $|X| = |Y| \geq \aleph_0$) then

$$\alpha = \begin{pmatrix} x & x_i \\ y & y_i \end{pmatrix} \in M_r(X, Y).$$

Consequently, by Theorem 2.6 and the remark before it, we deduce that $M(X, Y, \theta)$ is left duo if and only if $|X| = |Y| < \aleph_0$.

Unlike most of the results in this paper, there seems to be no reasonable condition under which a subsemigroup S of $M(X, Y, \theta)$ is right duo when $|X| = |Y| \geq \aleph_0$. We believe that most natural examples of such S are right duo, as we now show for $M_r(X, Y, \theta)$. However, in Remark 1 below, we observe that, if $X = Y$ and $\theta = \mathrm{id}_X$, then there is a subsemigroup of $G(X)$, the symmetric group on an infinite set X, which is not right duo.

Suppose $|X| = |Y| \geq \aleph_0$, and let $\theta \in M(Y, X)$ and $\alpha, \beta \in M_r(X, Y)$ where $r \geq 0$. Then, since α, θ, β are total and injective, we have:

$$\begin{aligned} Y \setminus X\alpha\theta\beta &= Y \setminus X\beta \,\dot\cup\, (X \setminus X\alpha\theta)\beta \\ &= Y \setminus X\beta \,\dot\cup\, [(X \setminus Y\theta) \,\dot\cup\, (Y \setminus X\alpha)\theta]\beta. \end{aligned}$$

Therefore, $d(\alpha\theta\beta) = d(\beta) + d(\theta) + d(\alpha) = d(\beta\theta) + d(\alpha)$. Moreover, if $Y \setminus X\alpha\theta\beta = \{b_j\}$ and $X \setminus X\beta\theta = \{a_k\}$, we can write $J = K \dot\cup L$, where $|L| = d(\alpha)$. Thus, we can write

$$\alpha\theta\beta = \begin{pmatrix} x_i \\ b_i \end{pmatrix}, \quad \beta\theta = \begin{pmatrix} x_i \\ a_i \end{pmatrix}, \quad \mu = \begin{pmatrix} a_i & a_k \\ b_i & b_k \end{pmatrix}$$

where $\{b_j\} = \{b_k\} \dot\cup \{b_\ell\}$ and $d(\mu) = d(\alpha) \geq r$, so $\mu \in M_r(X, Y)$ and clearly $\alpha\theta\beta = \beta\theta\mu$. In other words, $M_r(X, Y, \theta)$ is right duo for any $\theta \in M(Y, X)$.

Now we consider (LD) for $E(X, Y, \theta)$ where $|X| = |Y| \geq \aleph_0$. For this we need the following result from [23] Lemma 7.

Lemma 2.8 *Suppose $|X| = |Y| \geq \aleph_0$ and let $\alpha, \gamma \in E(X, Y)$. In addition, suppose $\theta \in T(Y, X)$ and $\theta\alpha \in E(Y)$. Then $\gamma = \lambda\theta\alpha$ for some $\lambda \in E(X, Y)$ if and only if $|y\gamma^{-1}| \geq |y(\theta\alpha)^{-1}|$ for each $y \in Y$.*

Suppose $|X| = |Y| \geq \aleph_0$, and let $y \in Y, \alpha, \beta \in E(X, Y)$ and $\theta \in E(Y, X)$. Clearly, for some $x' \in X$, we have:

$$|y(\alpha\theta\beta)^{-1}| \geq |x'\theta^{-1}\alpha^{-1}|.$$

Therefore, if $|x\theta^{-1}| = |Y|$ for each $x \in X$ then, for each $y \in Y$,

$$|y(\alpha\theta\beta)^{-1}| \geq |Y| = |(y\alpha^{-1})\theta^{-1}| = |y(\theta\alpha)^{-1}|,$$

and, by the above Lemma, we deduce that $\alpha\theta\beta = \lambda\theta\alpha$ for some $\lambda \in E(X, Y)$. In other words, $E(X, Y, \theta)$ is left duo for such θ.

We say a subsemigroup S of $E(X, Y, \theta)$ is *multi-transitive* if, whenever $X = A_1 \dot\cup A_2 \dot\cup A_3$ and $Y = \{y\} \dot\cup B_2 \dot\cup B_3$, where $|A_2| = |B_2|$ and $|A_3| = |X| = |Y| = |B_3|$, then there exists $\gamma \in S$ such that $y\gamma^{-1} = A_1$ and $B_2\gamma^{-1} = A_2$.

Theorem 2.9 *Suppose $|X| = |Y| \geq \aleph_0$. If $\theta \in E(Y, X)$ and $|x\theta^{-1}| < |Y|$ for some $x \in X$, then no multi-transitive subsemigroup S of $E(X, Y, \theta)$ is left duo.*

Proof. Suppose $|x\theta^{-1}| < |Y|$ and write $B_2 = x\theta^{-1}$. Since $|B_2| < |Y| = |X|$, we can write $X = A_1 \dot\cup A_2 \dot\cup A_3$ and $Y = \{y\} \dot\cup B_2 \dot\cup B_3$, where $|A_2| = |B_2|$, $|A_1| = |A_3| = |X|$ and $|B_3| = |Y|$. Consequently, we can choose $\gamma \in S$ such that $y\gamma^{-1} = A_1$ and $B_2\gamma^{-1} = A_2$. Also, by multi-transitivity, there exists $\omega \in S$ such that $y\omega^{-1} = \{x\}$. Now $y(\gamma\theta\omega)^{-1} = x\theta^{-1}\gamma^{-1} = B_2\gamma^{-1} =$

A_2 and this has cardinal less than $|Y|$. On the other hand, $|y(\theta\gamma)^{-1}| = |A_1\theta^{-1}| = |Y|$. That is, if $\beta = \gamma\theta\omega \in S$, then $|y\beta^{-1}| < |y(\theta\gamma)^{-1}|$ and so $\beta \notin E(X,Y)\theta\gamma$ by Lemma 1. Hence $\beta \notin S\theta\gamma$ also. Moreover, $\beta \neq \gamma$ since $A_2\beta = B_2\theta\omega = x\omega = y$, whereas $A_1\gamma = y$ and $A_1 \cap A_2 = \emptyset$. In other words, the principal left ideal $S\theta\gamma \cup \{\gamma\}$ is not a right ideal of S, and therefore S is not left duo. □

As in [23] Section 3, if $\alpha \in T(X,Y)$, we write $c(\alpha) = |C(\alpha)|$, where

$$C(\alpha) = \bigcup\{y\alpha^{-1} : |y\alpha^{-1}| \geq 2\}.$$

This cardinal number is called the *collapse* of α, and it is often used to define and study important examples of transformation semigroups for the case when $X = Y$. For example, the notion was introduced in [7] where it was used to characterise products of idempotents in $T(X)$ for infinite X. During the next three decades, the notion was explicitly involved in various problems related to the semigroup of balanced transformations of an infinite set (for example, see [8] and [17]). Then in [10] it was used to define the semigroup of all 'almost injective' transformations of an infinite set which was then shown to not satisfy (BQ). More recently, the authors of [19] extended Kemprasit's result to the semigroup $AM(X,q) = \{\alpha \in T(X) : c(\alpha) < q\}$, where $|X| \geq q \geq \aleph_0$. They also determined Green's relations and ideals for $AM(X,q)$ and studied other algebraic properties of this semigroup.

Example 2.10 With the above in mind, let $\lambda : A \to B$ and $\mu : B \to C$ be mappings between sets A, B and C, and suppose $a\lambda\mu = a'\lambda\mu$ for some $a \neq a'$ in A. Then either $a\lambda = a'\lambda$ or $a\lambda \neq a'\lambda$: that is, either $a \in C(\lambda)$ or $a\lambda \in C(\mu)$, and hence $C(\lambda\mu) \subseteq C(\lambda) \cup C(\mu)\lambda^{-1}$. Therefore, since $\lambda^{-1}\lambda \subseteq \mathrm{id}_X$ always, we deduce that

$$C(\lambda\mu)\lambda\mu \subseteq [C(\lambda)\lambda]\mu \cup C(\mu)\mu. \quad (1)$$

Now let $|X| = |Y| > r \geq \aleph_0$ and write

$$C_r(X,Y) = \{\beta \in E(X,Y) : |C(\beta)\beta| < r\}.$$

By (1), if $\theta \in C_r(Y,X)$ and $\alpha \in C_r(X,Y)$, then $|C(\alpha\theta)\alpha\theta| < r$ and it follows that $C_r(X,Y)$ is a subsemigroup of $E(X,Y,\theta)$. Moreover, if $X = A_1 \dot\cup A_2 \dot\cup A_3$ and $Y = \{y\} \dot\cup B_2 \dot\cup B_3$, where $|A_2| = |B_2|$ and $|A_3| = |X| =$

$|Y| = |B_3|$, then there are bijections $\pi_2 : A_2 \to B_2$ and $\pi_3 : A_3 \to B_3$. Consequently,

$$\gamma = \begin{pmatrix} A_1 \\ y \end{pmatrix} \cup \pi_2 \cup \pi_3 \in C_r(X,Y),$$

since $C(\gamma) \subseteq A_1$ and $|C(\gamma)\gamma| \leq 1 < r$ (note that $|A_1|$ may equal 1). In other words, if $\theta \in C_r(Y,X)$, then each $C_r(X,Y)$ is a multi-transitive subsemigroup of $E(X,Y,\theta)$.

As usual, we say a subsemigroup S of $P(X,Y,\theta)$ is 2-*transitive* if, for each $x \neq x'$ in X and each $y \neq y'$ in Y, there exists $\alpha \in S$ such that $x\alpha = y$ and $x'\alpha = y'$. Clearly 2-transitive implies transitive.

THEOREM 2.11 *Suppose $|X| = |Y| \geq 2$, and let S be a 2-transitive subsemigroup of $E(X,Y,\theta)$. Then S is right duo if and only if*

(a) θ and each element of S is injective, and

(b) for each $\alpha, \beta \in S$, $(\beta\theta)^{-1}\alpha\theta\beta \in S$.

Proof. Suppose S is right duo and, for contradiction, assume that there exists $\alpha \in S$ which is not injective, and write $c_0\alpha = c_1\alpha = d_1$ where $c_0 \neq c_1$ in X. Then, as in the proof of Theorem 2.4, $S(c_0, c_1)$ is a right ideal of S and we assert that it is not a left ideal of S. To show this, first observe that there exist $b_0 \neq b_1$ in Y such that $b_i\theta = c_i$ for $i = 0, 1$ (since θ is surjective). Now choose $d_2 \neq d_1$ in Y (possible since $|Y| \geq 2$) and $c_2 \in X$ such that $c_2\alpha = d_2$ (possible since α is surjective). Then $c_2 \notin d_1\alpha^{-1}$ and there exists $b_2 \in Y \setminus \{c_0, c_1\}\theta^{-1}$ such that $b_2\theta = c_2$. By 2-transitivity, there exists $\lambda \in S$ such that $c_0\lambda = b_0$ and $c_1\lambda = b_2$. Then

$$c_0\lambda\theta\alpha = b_0\theta\alpha = c_0\alpha = d_1 \neq d_2 = c_2\alpha = b_2\theta\alpha = c_1\lambda\theta\alpha,$$

hence $\lambda\theta\alpha \notin S(c_0, c_1)$ and the assertion follows. That is, if S is right duo, then each element of S is injective.

Now let $b_0, b_1 \in Y$ be such that $b_0\theta = b_1\theta$ and choose $\alpha \in S$. Since α is surjective, there exist $a_0, a_1 \in X$ such that $a_i\alpha = b_i$ for $i = 0, 1$. Then $a_0\alpha\theta = a_1\alpha\theta$, and this implies $a_0\alpha\theta\alpha = a_1\alpha\theta\alpha$. But $\alpha\theta\alpha \in S$, hence it is injective (as shown above) and so $a_0 = a_1$. Thus, $b_0 = a_0\alpha = a_1\alpha = b_1$, and therefore θ is injective. In other words, we have now shown that if S is right duo then (a) holds. Moreover, in this case, for each $\alpha, \beta \in S$, there exists $\mu \in S$ such that $\alpha\theta\beta = \beta\theta\mu$ and so $(\beta\theta)^{-1}\alpha\theta\beta = \mu \in S$ as required for (b).

Conversely, if $\alpha, \beta \in S$ and S satisfies (a) and (b), then $\mu = (\beta\theta)^{-1}\alpha\theta\beta$ is a well-defined surjective element of S, and clearly $\alpha\theta\beta = \beta\theta\mu$, so S is right duo. □

Remark 1. Suppose $|X| = |Y| \geq 2$ and let $G(X, Y, \theta)$ denote the group of bijections from X to Y. It follows from Theorem 2.11 that, if S is a subsemigroup of $G(X, Y, \theta)$ which contains some α and β such that $(\beta\theta)^{-1}\alpha\theta\beta \notin S$, then S is not right duo. In other words, such an S is a subsemigroup of $M(X, Y, \theta)$ which is not right duo. In particular, if $X = Y$ and $\theta = \mathrm{id}_X$, then there exists a subsemigroup S of $G(X)$, where X is infinite, for which $\beta^{-1}\alpha\beta \notin S$ for some $\alpha, \beta \in S$. For example, suppose $|X| = q \geq \aleph_0$ and write

$$X = A \,\dot\cup\, C_2 \,\dot\cup\, C_3 = B_1 \,\dot\cup\, B_2 \,\dot\cup\, B_3,$$

where $A = B_1 \,\dot\cup\, B_2$ and each set in this display has cardinal q. Clearly, if $S(A) = \{\alpha \in G(X) : A\alpha \subseteq A\}$, then $S(A)$ is a subsemigroup of $G(X)$. In particular, $\alpha, \beta \in S(A)$ where

$$\alpha = \begin{pmatrix} A & C_2 & C_3 \\ A & C_3 & C_2 \end{pmatrix}, \quad \beta = \begin{pmatrix} A & C_2 & C_3 \\ B_1 & B_2 & B_3 \end{pmatrix},$$

and this display means there is a bijection between each set in the top row and the set immediately below it. Now $\beta^{-1} \notin S(A)$ (since $A\beta^{-1} = A \cup C_2 \not\subseteq A$) and

$$\beta^{-1}\alpha\beta = \begin{pmatrix} B_1 & B_2 & B_3 \\ B_1 & B_3 & B_2 \end{pmatrix} \notin S(A),$$

since $B_2 \subseteq A$ but $B_2\beta^{-1}\alpha\beta = B_3 \not\subseteq A$.

3 Almost left (right) ideals

As in [21], we say a subset L of a semigroup S is an *almost left ideal* (more briefly, 'AL-ideal') if $xL \cap L \neq \emptyset$ for all $x \in S$ (the notion of an *almost right ideal* or 'AR-ideal' is defined dually). Clearly, left ideals are AL-ideals, but not conversely. For example, if $b \neq d$ in Y and $\theta \in T(Y, X)$, then

$$L_b = \{\beta \in T(X, Y) : b \in \operatorname{ran}\beta\}$$

is not a left ideal of $T(X, Y, \theta)$ (since, if $d\theta = c$ and $c\beta = d$ for some $\beta \in L_b$, then $X_d\theta\beta = X_d \notin L_b$). But L_b is an almost left ideal since $X_b \in L_b$ and, for each $\alpha \in T(X, Y)$, $\alpha\theta X_b \in L_b$: that is, $\alpha\theta L_b \cap L_b \neq \emptyset$.

Now we consider the following property for transformation semigroups S.

(AL) every almost left ideal of S is a left ideal of S.

We say $S \subseteq P(X,Y)$ *covers* Y if, for each $y \in Y$, there is a constant $\alpha \in S$ with range $\{y\}$.

THEOREM 3.1 *Suppose $|X| \geq 2$, $|Y| \geq 2$ and $\theta \in P(Y,X)$ is non-empty. If S is a 2-transitive subsemigroup of $P(X,Y,\theta)$ which covers Y, then S does not satisfy (AL).*

Proof. Suppose $b\theta = c$ and $A_b \in S$. Choose $d \in Y \setminus \{b\}$ and $a \in X \setminus \{c\}$, and let $\beta \in S$ satisfy $a\beta = d$ and $c\beta = b$. Now, let

$$S_d = \{\gamma \in S : d \in \operatorname{ran} \gamma\},$$

and $S_d^0 = S_d \cup \{\emptyset\}$. Since S covers Y, there exists a constant $B_d \in S_d$; and, for all $\alpha \in S$, $\alpha\theta B_d$ belongs to S_d (if $\emptyset \notin S$) or to S_d^0 (if $\alpha\theta B_d = \emptyset \in S$). That is, if $\emptyset \notin S$, then S_d is an AL-ideal of S; and, if $\emptyset \in S$, then S_d^0 is an AL-ideal of S. However, $\beta \in S_d$, whereas $A_b\theta\beta = A_b \notin S_d$, and so S_d (or S_d^0) is not a left ideal of S. Hence, S does not satisfy (AL). □

Theorems 2.1 and 2.3 are similar, but the second cannot be deduced from the first without placing more restrictive conditions on the given semigroup. However, in the present context, our next result follows directly from Theorem 3.1.

THEOREM 3.2 *Suppose $|X| \geq 2$ and $|Y| \geq 2$. If S is a 2-transitive subsemigroup of $T(X,Y,\theta)$ which contains all total constants, then S does not satisfy (AL).*

Now we consider (AR) for subsemigroups of $P(X,Y,\theta)$ and of $T(X,Y,\theta)$, with the latter being harder to handle.

THEOREM 3.3 *Suppose $|X| \geq 2$, $|Y| \geq 2$ and $\theta \in P(Y,X)$ is non-empty. If S is a 2-transitive subsemigroup of $P(X,Y,\theta)$ which contains all injective constants, then S does not satisfy (AR).*

Proof. First note that if $b\theta = a$ and $x \neq a$ in X, then $a_b\theta x_b = \emptyset \in S$. Now fix $a \in X$ and write

$$T_a = \{\gamma \in S : a \in \operatorname{dom} \gamma\} \cup \{\emptyset\}.$$

Then, for all $\alpha \in S$, $\emptyset \circ \theta\alpha = \emptyset \in T_a$. That is, T_a is an AR-ideal of S, but it is not a right ideal of S if $|Y\theta| \geq 2$. For example, suppose $b\theta = a$ and $y\theta = x$ where $a \neq x$ in $Y\theta$. By 2-transitivity, there exists $\gamma \in S$ such that $a\gamma = b$ and $x\gamma = y$. Then $\gamma \in T_a$ and $x_y \in S$, so $\gamma\theta x_y = C_y$ where $a \notin C = x(\gamma\theta)^{-1}$. In other words, $\gamma\theta x_y \neq \emptyset$ and $a \notin \operatorname{dom}(\gamma\theta x_y)$, so $\gamma\theta x_y \notin T_a$.

Suppose instead that $\theta = B_a$ for some $B \subseteq Y$ and $a \in X$. Choose $b \in B$ and let $R = \{a_b, \emptyset\}$. If $\alpha \in S$, then $\emptyset \circ B_a\alpha = \emptyset \in R$, so R is an AR-ideal of S. However, if $b \neq d$ in Y, then $a_d \in S$ and $a_b \circ B_a \circ a_d \notin R$, so R is not a right ideal of S. That is, for each non-empty $\theta \in P(Y, X)$, S does not satisfy (AR). □

For the next result, we assume that $|X| \geq 4$, $|Y| \geq 3$ and write

$$\begin{aligned} K_1 &= \{\alpha \in T(X,Y) : |X\alpha| = 1\}, \\ K_2 &= \{\alpha \in T(X,Y) : |X\alpha| = 2 \text{ and } b\alpha^{-1} = \{a\} \text{ for some } b \in Y,\ a \in X\}, \\ K_3 &= \{\alpha \in T(X,Y) : |X\alpha| = 3 \text{ and } b\alpha^{-1} = \{a\} \text{ for some } b \in Y,\ a \in X\}. \end{aligned}$$

Suppose $X = Y$ and $\theta = \mathrm{id}_X$. If $a_i \in A_i$ for $i = 1, 2$, $a_3 \in A_3 = A_1 \setminus \{a_1\}$ (if this is non-empty) and $A_0 = X \setminus \{a_2\}$, then

$$\begin{pmatrix} A_1 & A_2 \\ x_1 & x_2 \end{pmatrix} = \begin{pmatrix} a_1 & A_2 & A_3 \\ a_1 & a_2 & a_3 \end{pmatrix} \circ \begin{pmatrix} A_0 & a_2 \\ x_1 & x_2 \end{pmatrix}.$$

Hence, in this case, the semigroup generated by $K_1 \cup K_2 \cup K_3$ contains $T_3(X)$, the ideal of $T(X)$ consisting of all elements with rank less than 3. Although this containment may not occur if $\theta \neq \mathrm{id}_X$, it motivates the supposition in the next result. We extend some notation for $T(X)$ in [5] vol. 1, p. 51, and let π_β denote the partition induced on X by a given $\beta \in T(X, Y)$.

THEOREM 3.4 *Suppose $|X| \geq 4$, $|Y| \geq 3$, and let S be a subsemigroup of $T(X, Y, \theta)$ which contains $K_1 \cup K_2 \cup K_3$. Then S satisfies (AR) if and only if θ is constant.*

Proof. Fix $a \in X$ and let

$$P_a = K_1 \cup K_2 \cup \{\beta \in S : \{a\} \in X/\pi_\beta\}.$$

Suppose $\alpha \in S$. If there exist distinct $y_1, y_2, y_3 \in Y\theta\alpha$, choose $b_i \in y_i(\theta\alpha)^{-1}$ for each $i = 1, 2, 3$ and note that

$$\beta = \begin{pmatrix} a & A_2 & A_3 \\ b_1 & b_2 & b_3 \end{pmatrix} \in P_a, \quad \beta\theta\alpha = \begin{pmatrix} a & A_2 & A_3 \\ y_1 & y_2 & y_3 \end{pmatrix} \in P_a.$$

Likewise, if $|Y\theta\alpha| = 2$, then there exists $\beta \in K_2 \subseteq P_a$ such that $\beta\theta\alpha \in P_a$. And if $|Y\theta\alpha| = 1$, then $\beta\theta\alpha \in K_1 \subseteq P_a$ for each $\beta \in P_a$. In other words, for each $\alpha \in S$, there exists $\beta \in P_a$ such that $\beta\theta\alpha \in P_a$, and hence P_a is an AR-ideal of S for each $a \in X$. Now we show that P_a is not a right ideal of S if θ is non-constant.

First suppose $Y\theta$ contains distinct x_1, x_2, x_3, and choose $b_i \in x_i\theta^{-1}$ for $i = 1, 2, 3$. In this case, if $X = \{x_3\} \dot\cup A_0 = \{a\} \dot\cup A_2 \dot\cup A_3$ with $|A_3| \geq 2$ (possible since $|X| \geq 4$), then

$$\beta = \begin{pmatrix} a & A_2 & A_3 \\ b_1 & b_2 & b_3 \end{pmatrix} \in P_a, \quad \alpha = \begin{pmatrix} A_0 & x_3 \\ b_1 & b_3 \end{pmatrix} \in S,$$

$$\beta\theta\alpha = \begin{pmatrix} \{a\} \cup A_2 & A_3 \\ b_1 & b_3 \end{pmatrix} \notin P_a.$$

On the other hand, suppose $Y\theta = \{x_1, x_2\}$ and $b_i \in x_i\theta^{-1}$ for $i = 1, 2$ and $b_3 \in x_1\theta^{-1}$ (possible, and without loss of generality, since $|Y| \geq 3$ but $|Y\theta| = 2$). In this case, if $X = \{x_1\} \dot\cup A_0 = \{a\} \dot\cup A_2 \dot\cup A_3$ with $|A_2| \geq 2$, then

$$\beta = \begin{pmatrix} a & A_2 & A_3 \\ b_1 & b_2 & b_3 \end{pmatrix} \in P_a, \quad \alpha = \begin{pmatrix} x_1 & A_0 \\ b_1 & b_2 \end{pmatrix} \in S,$$

$$\beta\theta\alpha = \begin{pmatrix} \{a\} \cup A_3 & A_2 \\ b_1 & b_2 \end{pmatrix} \notin P_a.$$

That is, if S satisfies (AR), then θ must be constant.

Conversely, suppose $\theta = Y_x$ for some $x \in X$ and let R be an AR-ideal of S. Then, for each $X_b \in K_1 \subseteq S$, there exists $\beta \in R$ such that $X_b = \beta Y_x X_b \in R$. Thus $K_1 \subseteq R$. Consequently, for each $\alpha \in S$ and $\beta \in R$, $\beta Y_x \alpha = X_{x\alpha} \in K_1 \subseteq R$, and so R is a right ideal of S. □

Remark 2. As in Example 2.5, it is easy to see that many subsemigroups of $P(X, Y, \theta)$ and $T(X, Y, \theta)$ satisfy the conditions of Theorems 3.1-3.4. In particular, Theorems 3.1 and 3.2 provide simple conditions under which $I(X, Y, \theta)$ does not satisfy (AL) or (AR).

Although every group satisfies (LD) trivially, the same is not true for (AL). For example, as noted in [6] p. 237, if G is a group and $H = G \setminus \{a\}$ where $a \in G$ and $|G| \geq 3$ then, for all $x \in G$, $xH \cap H = G \setminus \{xa, a\} \neq \emptyset$ (clearly $xa, a \notin xH \cap H$ for all $x \in G$; and, if $x \in G$ and $g \in G \setminus \{xa, a\}$, then $g = x(x^{-1}g)$ where $x^{-1}g \neq a$, so $g \in xH \cap H$). Hence H is an AL-ideal of G. Moreover, if $a \neq 1$, then $1 \in H$ and $aH \nsubseteq H$, so H is not a left ideal of G, and thus G does not satisfy (AL). In particular, as shown before Theorem 2.6, if $3 \leq |X| = |Y| < \aleph_0$, then $M(X, Y, \theta)$ is a group with order at least 3, and so it does not satisfy (AL).

We say a subsemigroup S of $T(X, Y, \theta)$ is *strictly transitive* if, for each $x \in X$ and distinct $y, y' \in Y$, there exists $\alpha \in S$ such that $x\alpha = y$ and $y' \notin X\alpha$. For example, as defined in Example 2, each $M_r(X, Y, \theta)$ is strictly transitive. Indeed, if $|X| = |Y| \geq \aleph_0$ and $|Y| \geq r \geq 0$, write $X = \{x\} \dot\cup \{x_i\}$ and $Y = \{y, y'\} \dot\cup \{y_i\} \dot\cup \{y_j\}$ where $y \neq y'$ and $|J| = r$ (possible by the supposition). Then $M_r(X, Y, \theta)$ is strictly transitive since

$$\alpha = \begin{pmatrix} x & x_i \\ y & y_i \end{pmatrix} \in M_r(X, Y).$$

THEOREM 3.5 *Suppose $|X| = |Y| \geq \aleph_0$. If S is a strictly transitive subsemigroup of $M(X, Y, \theta)$, then S does not satisfy (AL).*

Proof. Fix $b \in Y$ and let

$$S_b = \{\beta \in S : b \in X\beta\}.$$

If $\alpha \in S$, choose $a \in X\alpha\theta$ and let $\gamma \in S$ be such that $a\gamma = b$ (possible by transitivity). Then, $\gamma \in S_b$. Moreover, $b = a\gamma \in X\alpha\theta\gamma$ and so $\alpha\theta\gamma \in S_b$. Therefore, S_b is an AL-ideal of S. Consequently, if S satisfies (AL), then S_b is a left ideal of S. However, this is false, as we now show. Let $d \in Y \setminus \{b\}$ and write $b\theta = c$. Choose $\beta \in S$ such that $c\beta = b$ (by transitivity). Also, by strict transitivity, there exists $\alpha \in S$ such that $c\alpha = d$ and $b \notin X\alpha$. Then $\beta \in S_b$ but $\alpha\theta\beta \notin S_b$, contrary to S_b being a left ideal of S. For, if $c\beta = b = x\alpha\theta\beta$ for some $x \in X$, then $x\alpha\theta = c$ (since β is injective) and so $x\alpha = b$, contradicting the choice of α. That is, $\alpha\theta\beta \notin S_b$. □

Next we say a subsemigroup S of $T(X, Y, \theta)$ is 2-*strictly transitive* if for all $x \in X$ and distinct $y_1, y_2, y_3 \in Y$, there exists $\alpha \in S$ such that $x\alpha = y_1$ and $y_i \notin X\alpha$ for $i = 2, 3$. Clearly, '2-strictly transitive' implies 'strictly transitive' as well as 'transitive'.

THEOREM 3.6 *Suppose $|X| = |Y| \geq \aleph_0$. If S is a 2-strictly transitive subsemigroup of $M(X, Y, \theta)$ then S does not satisfy (AR).*

Proof. Fix $b \in Y$ and let

$$R_b = \{\beta \in S : b \notin X\beta\}.$$

Let $y, y' \in Y \setminus \{b\}$ and $a \in X$. Since S is 2-strictly transitive, there exists $\gamma \in S$ such that $a\gamma = y$ and $y', b \notin X\gamma$. Clearly, $\gamma \in R_b$ and so $R_b \neq \emptyset$.

Now let $\alpha \in S$. If $b \notin X\alpha$, then $b \notin X\beta\theta\alpha$ for each $\beta \in R_b$ (since $X\beta\theta\alpha \subseteq X\alpha$), and so $\beta\theta\alpha \in R_b$. If $b \in X\alpha$, then let $a\alpha = b$. If $a \in Y\theta$, then $a = d\theta$ for some $d \in Y$. Choose $\beta \in S$ such that $b, d \notin X\beta$, and suppose $b = x\beta\theta\alpha$ for some $x \in X$. Then, $d\theta\alpha = x\beta\theta\alpha$ and, since α and θ are injective, it follows that $d = x\beta$, a contradiction. Thus, $b \notin X\beta\theta\alpha$ and so $\beta\theta\alpha \in R_b$. On the other hand, if $a \notin Y\theta$, then $b \notin X\beta\theta\alpha$ for each $\beta \in R_b$. For, if $a\alpha = b = x\beta\theta\alpha$ for some $x \in X$, then $a = x\beta\theta \in Y\theta$, another contradiction. Therefore, for each $\alpha \in S$, there exists some $\beta \in R_b$ such that $\beta\theta\alpha \in R_b$, and so R_b is an (AR)-ideal of S.

We assert that R_b is not a right ideal of S. To see this, let $d \in Y \setminus \{b\}$ and $c = d\theta$. By transitivity, there exists $\alpha \in S$ such that $c\alpha = b$. Choose $\gamma \in S$ such that $c\gamma = d$ and $b \notin X\gamma$ (possible by strict transitivity). Clearly, $\gamma \in R_b$ but $c(\gamma\theta\alpha) = d\theta\alpha = c\alpha = b$, and so $\gamma\theta\alpha \notin R_b$. Therefore, R_b is not a right ideal of S, and hence S does not satisfy (AR). □

We say a subsemigroup S of $T(X, Y, \theta)$ is *doubly transitive* if, for all $a \neq a'$ in X and $b \in Y$, there exists $\alpha \in S$ such that $a\alpha = b = a'\alpha$. For example, if $|X| = |Y| \geq \aleph_0$, write $X = \{a, a'\} \dot\cup \{x_i\}$ and $Y = \{b\} \dot\cup \{y_i\}$ where $a \neq a'$ (possible by the supposition). Then $E(X, Y, \theta)$ is doubly transitive since

$$\alpha = \begin{pmatrix} \{a, a'\} & x_i \\ b & y_i \end{pmatrix} \in E(X, Y).$$

THEOREM 3.7 *Suppose $|X| = |Y| \geq \aleph_0$. If S is a doubly transitive subsemigroup of $E(X, Y, \theta)$, then S does not satisfy (AL).*

Proof. Fix $a \in X$. Choose b in $a\theta^{-1}$ and let

$$S_{(a,b)} = \{\beta \in S : a\beta = b\}.$$

For each $\alpha \in S$, if $a(\alpha\theta) = x \in X$, then there exists $\gamma \in S$ such that $x\gamma = b = a\gamma$ (by transitivity if $x = a$, and by double transitivity if $x \neq a$). That is, $\gamma \in S_{(a,b)}$ and $\alpha\theta\gamma \in S_{(a,b)}$, and so $S_{(a,b)}$ is an AL-ideal of S. But $S_{(a,b)}$ is not a left ideal of S.

To see this, let $\beta \in S_{(a,b)}$ and choose $a' \in X \setminus b\beta^{-1}$ (possible since β is surjective). Since θ is surjective, there exists $b' \in Y$ such that $b'\theta = a'$. By transitivity, there exists $\alpha \in S$ such that $a\alpha = b'$. In this case, $\beta \in S_{(a,b)}$ but $\alpha\theta\beta \notin S_{(a,b)}$. For, $a(\alpha\theta\beta) = b'\theta\beta = a'\beta \neq b$. □

EXAMPLE 3.8 Note that, if $|X| = |Y| \geq \aleph_0$, then there are proper subsemigroups of $E(X, Y, \theta)$ which are doubly transitive. For example, suppose $r \geq 2$ and let

$$E_r(X, Y) = \{\beta \in E(X, Y) : |y\beta^{-1}| \geq r \text{ for all } y \in Y\}.$$

If $\beta \in E_r(X, Y)$, $\alpha \in E(X, Y)$ and $y \in Y$, then $|y\beta^{-1}(\theta^{-1}\alpha^{-1})| \geq r$ (since α and θ are surjective). That is, $\alpha\theta\beta \in E_r(X, Y)$ and thus $E_r(X, Y)$ is a left ideal of $E(X, Y)$. It is also doubly transitive. For example, if $a \neq a'$ in X and $b \in Y$, we can write $Y = \{b\} \dot{\cup} \{y_i\}$ and $X = A_0 \dot{\cup} A_i$, where $a, a' \in A_0$ and $|A_0| = |A_i| = r$ for each i, and observe that

$$\alpha = \begin{pmatrix} A_0 & A_i \\ b & y_i \end{pmatrix} \in E_r(X, Y).$$

The supposition in the next result is comparable with that in Theorem 2.6.

THEOREM 3.9 *If $|X| = |Y| \geq \aleph_0$ and S is a 2-transitive subsemigroup of $E(X, Y, \theta)$ which contains a non-injective element, then S does not satisfy (AR).*

Proof. Suppose $\varepsilon \in S$ is non-injective and choose distinct $a_0, a_1 \in X$ such that $a_0\varepsilon = a_1\varepsilon$. Since θ is surjective, there exist $b_0 \neq b_1$ in Y such that $b_i\theta = a_i$ for $i = 0, 1$. By 2-transitivity, there exists $\beta \in S$ such that $a_i\beta = b_i$ for $i = 0, 1$. Now let

$$D(a_0, a_1) = \{\gamma \in S : a_0\gamma \neq a_1\gamma\}.$$

Clearly, $\beta \in D(a_0, a_1)$. But

$$a_0\beta\theta\varepsilon = b_0\theta\varepsilon = a_0\varepsilon = a_1\varepsilon = b_1\theta\varepsilon = a_1\beta\theta\varepsilon,$$

so $\beta\theta\varepsilon \notin D(a_0, a_1)$ and this is not a right ideal of S. Next we show that $D(a_0, a_1)$ is an AR-ideal.

For each $\alpha \in S$, there exist (distinct) c_0, c_1 in X such that $c_0\alpha = b_0$ and $c_1\alpha = b_1$ (since α is surjective). Let $d_0\theta = c_0$ and $d_1\theta = c_1$ (also, since θ is surjective). By 2-transitivity, there exists $\lambda \in S$ such that $a_0\lambda = d_0$ and $a_1\lambda = d_1$. Clearly, $\lambda \in D(a_0, a_1)$ and

$$a_0\lambda\theta\alpha = d_0\theta\alpha = c_0\alpha = b_0 \neq b_1 = c_1\alpha = d_1\theta\alpha = a_1\lambda\theta\alpha,$$

so $\lambda\theta\alpha \in D(a_0, a_1)$. Therefore, $D(a_0, a_1)$ is an AR-ideal of S, and we conclude that S does not satisfy (AR). $\square$

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