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BQ–semigroups of generalised transformations

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Abstract. If X and Y are non-empty sets, we let $T(X,Y)$ denote the set of all (total) transformations from X into Y. In addition, if $\theta \in T(Y,X)$, we define a 'sandwich operation' $*$ on $T(X,Y)$ by: $\alpha * \beta = \alpha\theta\beta$, for all $\alpha, \beta \in T(X,Y)$. Then $(T(X,Y),*)$ is a semigroup of so-called 'generalised transformations', which we denote by $T(X,Y,\theta)$. In 2003, Kemprasit and Namnak showed that $T(X,Y,\theta)$ belongs to the class BQ of all semigroups whose sets of bi-ideals and quasi-ideals coincide. Like Chinram in 2005, they also determined when certain subsemigroups $T(X,Y,\theta)$ belong to BQ. In this paper, we simplify and extend their work by showing how some of it can be derived from a simple result for abstract semigroups.

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1 Introduction

Suppose X, Y are non-empty sets and let $P(X,Y)$ denote the set of all *partial* transformations from X to Y: that is, all $\alpha : A \to B$ where $A \subseteq X$ and $B \subseteq Y$. For each $\theta \in P(Y,X)$, we define a *sandwich* operation on $P(X,Y)$ by:

$$\alpha * \beta = \alpha \circ \theta \circ \beta \quad \text{for all } \alpha, \beta \in P(X,Y).$$

Then $(P(X,Y),*)$ is a so-called *generalised transformation semigroup* which we denote by $P(X,Y,\theta)$. In particular, if $X = Y$ and $\theta = \mathrm{id}_X$, the identity mapping on X, then $P(X,X,\mathrm{id}_X)$ equals $(P(X),\circ)$: that is, the semigroup $P(X)$ consisting of all partial transformations from X to X under composition. Note that both $P(X,Y,\theta)$ and $P(X)$ always contain a zero element: namely, $\emptyset$.

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As in [11], a subsemigroup Q of a semigroup S is called a *quasi-ideal* of S if $SQ \cap QS \subseteq Q$, and we say a subsemigroup B of S is a *bi-ideal* of S if $BSB \subseteq B$. Note that every right and every left ideal of S is a quasi-ideal, and every quasi-ideal Q of a semigroup S is a bi-ideal of S since $QSQ \subseteq SQ \cap QS$. It is known that regular semigroups, right [left] simple semigroups and right [left] 0-simple semigroups are in the class BQ of all semigroups whose sets of bi-ideals and quasi-ideals coincide: see [3] vol 1, Exercise 2.7.18(d) (corrected in [3] vol 2, p 341) and [5] Propositions 2.6 and 2.8. We say a semigroup S is a *BQ–semigroup* if $S \in BQ$. Kapp appears to have been the first to use this terminology: see [5] Definition 2.3.

Some authors have determined when certain subsemigroups of $P(X, Y, \theta)$ belong to BQ. To explain this, for each $\alpha \in P(X, Y)$, we let $\operatorname{dom}\alpha$ denote the *domain* of α and $\operatorname{ran}\alpha$ the *range* of α. Now let

$$\begin{aligned} T(X,Y) &= \{\alpha \in P(X,Y) : \operatorname{dom}\alpha = X\}, \\ I(X,Y) &= \{\alpha \in P(X,Y) : \alpha \text{ is injective}\}, \\ S(X,Y) &= \{\alpha \in P(X,Y) : \alpha \text{ is surjective}\}, \\ M(X,Y) &= I(X,Y) \cap T(X,Y), \\ E(X,Y) &= S(X,Y) \cap T(X,Y). \end{aligned}$$

Clearly, $S(X, Y) \neq \emptyset$ if and only if $|X| \geq |Y|$, and $M(X, Y) \neq \emptyset$ if and only if $|X| \leq |Y|$. Provided no confusion will arise, we assume these conditions hold without further mention.

In addition, if $\theta \in T(Y, X)$, we let $T(X, Y, \theta)$ denote the semigroup $T(X, Y)$ under the sandwich operation determined by θ, and likewise for each of the above sets. Also, like before when $X = Y$ and $\theta = \mathrm{id}_X$, we write $T(X, X, \mathrm{id}_X)$ more simply as $T(X)$ under composition, and similarly for each of the above sets.

In [7] Theorem 3.1, the authors showed that each of $P(X, Y, \theta), T(X, Y, \theta)$ and $I(X, Y, \theta)$ belongs to BQ. On the other hand, in [7] Theorem 3.2, they showed that $M(X, Y, \theta) \in BQ$ if and only if $|X| = |Y| < \aleph_0$; and they obtained a similar result for $E(X, Y, \theta)$ in [7] Theorem 3.3. Likewise, in [2] Theorem 2.3, Chinram showed that for each cardinal $k > 0$ and each $\theta \in T(Y, X)$, we have:

$$T_k(X, Y, \theta) = \{\alpha \in T(X, Y) : |\operatorname{ran}\alpha| \leq k\} \in BQ.$$

Also, in [2] Theorem 3.4, he proved that, if $|X| = |Y| \geq q \geq \aleph_0$ and $\theta \in M(Y, X)$, and we let

$$MOE(X, Y, q) = \{\alpha \in M(X, Y) : |Y \setminus \operatorname{ran}\alpha| \geq q\},$$

then the sandwich semigroup $(MOE(X,Y,q),\theta)$ belongs to BQ if and only if $|X| = |Y| = q$.

In this paper, we derive the above work from a simple result for algebraic semigroups, and we extend such work to a wider class of generalised transformation semigroups. Similar results can be proved for linear transformations between vector spaces V and W (compare [9] and [10] for the case $V = W$), and we will consider such ideas in a subsequent paper.

2 Basic properties

In what follows, $Y = A \dot{\cup} B$ means Y is a *disjoint* union of sets A and B, and we write id_Y for the identity transformation on Y.

Given a non-empty subset X of S, the quasi-ideal and the bi-ideal generated by X will be denoted by $(X)_Q$ and $(X)_B$, respectively. If $X = \{x\}$ then we write $(x)_Q$ and $(x)_B$ instead of $(\{x\})_Q$ and $(\{x\})_B$. By [3] vol. 1, pp 84-85, Exercises 15 and 17, if X is a non-empty subset of a semigroup S, then

$$\begin{aligned} (X)_Q &= S^1X \cap XS^1 = (SX \cap XS) \cup X, \text{ and} \\ (X)_B &= (XS^1X) \cup X = XSX \cup X \cup X^2. \end{aligned}$$

Clearly, $S \in BQ$ if and only if $(X)_Q \subseteq (X)_B$ for each non-empty subset X of S (compare [11] Theorem 9.11). Hence, if there exists $x \in S$ such that $(x)_Q \not\subseteq (x)_B$, then $S \notin BQ$: some of our proofs depend on this simple observation.

In [9] Lemma 1, the authors proved that if S is a regular semigroup, then any right ideal R of S belongs to BQ, and this result was very useful in [12]. Here, we need to extend it in a natural way to a wider class of semigroups. First, if S is a subsemigroup of a semigroup T and $R \subseteq T$ is non-empty, we say R is *right-closed by S* if $RS \subseteq R$ (note that R need not be a subset of S: see Example 1 below); dually, $L \subseteq T$ is *left-closed by S* if $SL \subseteq L$. In addition, if $z \in T$, we define a *sandwich operation* $*$ on T by writing:

$$a * b = azb$$

and denote the *sandwich semigroup* $(T, *)$ by (T, z).

LEMMA 2.1 *Suppose S is a regular subsemigroup of a semigroup $(T,\cdot)$, and let $R \subseteq T$ be right-closed by S. If $zR \subseteq S$ for some $z \in T$, then (R, z) is a semigroup which belongs to BQ.*

Proof. If $b, c \in R$ then $zc \in S$ and so $b.zc \in R$. Hence (R, z) is a subsemigroup of (T, z). Now, let X be a non-empty subset of R. We know that $(X)_B \subseteq (X)_Q$ always. We assert that $(X)_Q \subseteq (X)_B$ in the semigroup (R, z). Let $a \in R * X \cap X * R$. Then, there exist $b, c \in R$ and $s, t \in X$ such that $a = bzs = tzc$. Since S is regular, $zs = zs.x.zs$ for some $x \in S$. Since R is right-closed by S, $cx \in R$. Therefore, $a = b.zs = bzs.xzs = tz.cx.zs \in X * R * X$. Hence, $R * X \cap X * R \subseteq X * R * X$ and so

$$\begin{aligned}(X)_Q = R^1 * X \cap X * R^1 &= (R * X \cap X * R) \cup X \\ &\subseteq (X * R * X) \cup X \cup (X * X) = (X)_B.\end{aligned}$$

Thus, $(X)_B = (X)_Q$ for every non-empty subset X of R and so $(R, z) \in BQ$. □

EXAMPLE 2.2 Clearly, $R = P(X, Y)$ is a subset of the semigroup $T = P(X \cup Y)$ under composition, and it is right-closed by $S = P(Y) \subseteq T$. Also, if $\theta \in P(Y, X) \subseteq T$, then $\theta R \subseteq S$. Therefore, since S is regular, the Lemma implies that $(R, \theta) \in BQ$. Similarly, if $R = I(X, Y)$ then $R \subseteq T$ and this R is right-closed by the regular semigroup $S = I(Y) \subseteq T$. Also, if $\theta \in I(Y, X) \subseteq T$, then $\theta R \subseteq S$ and we deduce that $(I(X, Y), \theta) \in BQ$. Likewise, if $R = T(X, Y)$ then $R \subseteq T$ and this R is right-closed by the regular semigroup $S = T(Y) \subseteq T$. In addition, if $\theta \in T(Y, X) \subseteq T$, then $\theta R \subseteq S$, so Lemma 1 implies that $(T(X, Y), \theta) \in BQ$. Finally, if $k > 0$ and

$$R = T_k(X, Y) = \{\alpha \in T(X, Y) : |X\alpha| \leq k\},$$

then $R \subseteq T$ and it is right-closed by $S = T(Y)$ (since $|X\alpha\beta| \leq |X\alpha|$ for all $\alpha \in R$ and $\beta \in S$). Moreover, if $\theta \in T(Y, X) \subseteq T$, then $\theta R \subseteq S$ and hence $(T_k(X, Y), \theta) \in BQ$. In other words, Lemma 2.1 provides an alternative proof of [7] Theorem 3.1 and [2] Theorem 2.3.

3 Transformations of a set

Throughout this section, we let $q \geq \aleph_0$, and suppose $|X| \geq q$ and $|Y| \geq q$.

If $\alpha \in T(X, Y)$, we say $r(\alpha) = |\operatorname{ran} \alpha|$ is the *rank* of α. Also, we write

$$\begin{aligned}D(\alpha) &= Y \setminus \operatorname{ran} \alpha, & d(\alpha) &= |D(\alpha)|, \\ C(\alpha) &= \bigcup\{y\alpha^{-1} : |y\alpha^{-1}| \geq 2\}, & c(\alpha) &= |C(\alpha)|.\end{aligned}$$

and we refer to these cardinals as the *defect* and the *collapse* of α, respectively.

We extend the convention introduced in [3] vol 2, p 241: namely, if $\alpha \in P(X,Y)$ is non-empty then we write

$$\alpha = \begin{pmatrix} A_i \\ y_i \end{pmatrix}$$

and take as understood that the subscript i belongs to some (unmentioned) index set I, that the abbreviation $\{y_i\}$ denotes $\{y_i : i \in I\}$, and that $X\alpha = \operatorname{ran}\alpha = \{y_i\}$, $y_i\alpha^{-1} = A_i$ and $\operatorname{dom}\alpha = \bigcup\{A_i : i \in I\}$.

We extend the notation in [12] by defining the following sets:

$$\begin{aligned}
AM(X,Y,q) &= \{\alpha \in T(X,Y) : c(\alpha) < q\},\\
AE(X,Y,q) &= \{\alpha \in T(X,Y) : d(\alpha) < q\},\\
OM(X,Y,q) &= \{\alpha \in T(X,Y) : c(\alpha) \geq q\},\\
OE(X,Y,q) &= \{\alpha \in T(X,Y) : d(\alpha) \geq q\},\\
MAE(X,Y,q) &= M(X,Y) \cap AE(X,Y,q),\\
MOE(X,Y,q) &= M(X,Y) \cap OE(X,Y,q),\\
EAM(X,Y,q) &= E(X,Y) \cap AM(X,Y,q),\\
EOM(X,Y,q) &= E(X,Y) \cap OM(X,Y,q).
\end{aligned}$$

A transformation $\alpha \in T(X)$ is said to be *almost one-to-one* if $c(\alpha)$ is finite, and α is an *almost onto* transformation of X if $d(\alpha)$ is finite. This terminology was used by Kemprasit in [6] where she showed that $(AM(X,X,\aleph_0),\circ)$, the semigroup of all almost one-to-one transformations of X, and $(AE(X,X,\aleph_0),\circ)$, the semigroup of all almost onto transformations of X, do not belong to BQ (here, the notation 'M' signifies 'mono', and 'E' denotes 'epi'). This explains the origin of the notation displayed above.

Clearly, if $\alpha \in AM(X,Y,q)$ then $|X \setminus C(\alpha)| = |X|$ and, since α is injective on $X \setminus C(\alpha)$, we deduce that $|X| \leq |Y|$. Conversely, if $|X| \leq |Y|$ then $M(X,Y)$ is a non-empty subset of $AM(X,Y,q)$. Thus, we have:

$$AM(X,Y,q) \neq \emptyset \quad \text{if and only if} \quad |X| \leq |Y|. \tag{1}$$

Similarly, if $\alpha \in AE(X,Y,q)$ then $|X\alpha| = |Y \setminus D(\alpha)| = |Y|$, and it follows that $|X| \geq |Y|$. Conversely, if $|Y| \leq |X|$ then $E(X,Y)$ is a non-empty subset of $AE(X,Y,q)$. Hence, we have:

$$AE(X,Y,q) \neq \emptyset \quad \text{if and only if} \quad |X| \geq |Y|. \tag{2}$$

On the other hand, both $OM(X,Y,q)$ and $OE(X,Y,q)$ contain each constant map in $T(X,Y)$, so these sets are always non-empty. In fact, since

$C(\alpha) \subseteq C(\alpha\theta\beta)$ for all $\alpha, \beta \in T(X, Y)$ and $\theta \in T(Y, X)$, $OM(X, Y, q)$ is a right ideal of $T(X, Y, \theta)$. Likewise, since $X\beta\theta\alpha \subseteq X\alpha$ for all $\alpha, \beta \in T(X, Y)$ and $\theta \in T(Y, X)$, we know $OE(X, Y, q)$ is a left ideal of $T(X, Y, \theta)$.

To show that $AM(X, Y, q)$ and $AE(X, Y, q)$ are subsemigroups of $T(X, Y, \theta)$ for specific θ, we need the following result. In [8] Lemma 2.1, the authors showed that if $\alpha, \beta \in AM(X, X, q)$ then $c(\alpha\beta) \leq c(\alpha) + c(\beta)$. However, we need a more general result as follows.

Lemma 3.1 *If $\lambda : A \to B$ and $\mu : B \to C$, then*
(a) $d(\mu) \leq d(\lambda\mu) \leq d(\lambda) + d(\mu)$, and
(b) $c(\lambda) \leq c(\lambda\mu) \leq c(\lambda) + c(\mu)$.

Proof. For part (a), we observe that $A\lambda\mu \subseteq B\mu$, and so $D(\lambda\mu) \supseteq D(\mu)$. Also, we have:

$$C \setminus A\lambda\mu = (C \setminus B\mu) \cup (B\mu \setminus A\lambda\mu) \subseteq (C \setminus B\mu) \cup (B \setminus A\lambda)\mu,$$

and hence $d(\lambda\mu) \leq d(\mu) + d(\lambda)$. For part (b), it is easy to see that $C(\lambda) \subseteq C(\lambda\mu)$, and so $c(\lambda) \leq c(\lambda\mu)$. We assert that $C(\lambda\mu) \subseteq C(\lambda) \cup C(\mu)\lambda^{-1}$. For, if $x \in C(\lambda\mu)$, then $x(\lambda\mu) = y(\lambda\mu)$ for some $y \in A$ with $y \neq x$. If $x\lambda = y\lambda$, then $x \in C(\lambda)$. If $x\lambda \neq y\lambda$, then $x\lambda \in C(\mu)$ and the assertion follows. Let $C(\mu) \cap \operatorname{ran} \lambda = \{b_j\}$. Then either $|b_j\lambda^{-1}| \geq 2$ or $|b_j\lambda^{-1}| = 1$. Clearly, if the former occurs then $b_j\lambda^{-1} \subseteq C(\lambda)$; and, since each $b_j \in C(\mu)$, the number of b_j's such that $|b_j\lambda^{-1}| = 1$ is at most $c(\mu)$. In other words,

$$C(\lambda) \cup C(\mu)\lambda^{-1} = C(\lambda) \cup \{b_j\lambda^{-1} : |b_j\lambda^{-1}| = 1\},$$

and it follows that $c(\lambda\mu) \leq c(\lambda) + c(\mu)$. □

Since $q \geq \aleph_0$, half of the next two results follows directly from the last one.

Lemma 3.2 *Suppose $|Y| \geq |X| \geq q \geq \aleph_0$ and $\theta \in T(Y, X)$. Then $(AM(X, Y, q), \theta)$ is a subsemigroup of $T(X, Y, \theta)$ if and only if $\theta \in AM(Y, X, q)$. Consequently, when this occurs, $|X| = |Y|$.*

Proof. For simplicity, write $AM(X, Y, q) = AM$ and recall that, by (1), $AM \neq \emptyset$ if and only if $|Y| \geq |X|$. Suppose $\theta \notin AM(Y, X, q)$ and write

$$\theta = \begin{pmatrix} C_i & y_j \\ x_i & x_j \end{pmatrix},$$

where $|C_i| \geq 2$ for each i and $c(\theta) = r \geq q$. If $r < |X|$ then $|J| = |Y|$ (since $|Y| \geq |X| \geq \aleph_0$) and we can write $X = \{x_m\} \dot\cup \{x_n\}$ where $|M| = r$ and

$|N| = |X|$. Let $\lambda_1 : \{x_m\} \to C(\theta)$ be a bijection and $\lambda_2 : \{x_n\} \to \{y_j\}$ be an injection. Then $\alpha = \lambda_1 \dot\cup \lambda_2 : X \to Y$ is an injection, so $\alpha \in AM$. Moreover, since $|C_i\lambda_1^{-1}| \geq 2$ for each i, we see that $C(\alpha\theta) = \{x_m\}$ and so $c(\alpha\theta) \geq q$. Consequently, for each $\beta \in AM$, $c(\alpha\theta\beta) \geq q$ and thus $\alpha\theta\beta \notin AM$. Hence, in this case, $(AM(X,Y,q),\theta)$ is not a subsemigroup of $T(X,Y,\theta)$.

Instead, suppose $r \geq |X|$. We assert that there is a partition $\{A_k\}$ of X where $K \subseteq I$ and $2 \leq |A_k| \leq |C_k|$ for each k. To see this, choose $B \subseteq C(\theta)$ with $|B| = |X|$ and write

$$K = \{i \in I : B \cap C_i \neq \emptyset\} \quad \text{and} \quad B_k = (B \cap C_k) \cup [z_k],$$

where z_k is an element of $C_k \setminus B$ (if this set is non-empty: note that $B \cap C_k$ may contain only one element, whereas $|C_k| \geq 2$ for each k) and $[z_k]$ denotes the set containing the single element z_k. Now, since the B_k are pairwise disjoint and $B = \bigcup(B \cap C_k)$, we have

$$|B| \leq |\bigcup B_k| \leq |B| + |K|,$$

where $|K| \leq |B|$ (since $\{B \cap C_k\}$ is a partition of B). Since B is infinite, it follows that $|B| = |\bigcup B_k|$, and clearly $2 \leq |B_k| \leq |C_k|$ for each k. Although some B_k may not be a subset of B, bijections $\bigcup B_k \to B \to X$ can now be used to show that X can be partitioned as asserted. Finally, if $\pi_k : A_k \to C_k$ is an injection for each k, then $\alpha = \dot\bigcup \pi_k : X \to Y$ is an injection, so $\alpha \in AM$. But $C(\alpha\theta) = \bigcup A_k$, and so the result follows like before. □

LEMMA 3.3 *Suppose* $|X| \geq |Y| \geq q \geq \aleph_0$ *and* $\theta \in T(Y,X)$. *Then* $(AE(X,Y,q),\theta)$ *is a subsemigroup of* $T(X,Y,\theta)$ *if and only if* $\theta \in AE(Y,X,q)$. *Consequently, when this occurs,* $|X| = |Y|$.

Proof. For simplicity, write $AE(X,Y,q) = AE$ and recall that, by (2), $AE \neq \emptyset$ if and only if $|X| \geq |Y|$. Suppose $\theta \notin AE(Y,X,q)$ and write $Y\theta = \{x_i\}$. Let $D(\theta) = X \setminus Y\theta = \{x_j\}$, so $|J| \geq q$. If $|I| < |Y|$, then $|J| \geq |Y|$ (since $|X| \geq |Y| \geq \aleph_0$) and we can write $Y = \{y_i\} \dot\cup \{y_k\}$ where $|K| = |Y|$. Let $\mu_1 : \{x_i\} \to \{y_i\}$ be a bijection and $\mu_2 : \{x_j\} \to \{y_k\}$ be a surjection. Then $\beta = \mu_1 \dot\cup \mu_2 : X \to Y$ is a surjection, so $\beta \in AE$. However, for each $\alpha \in AE$, $X\alpha\theta\beta \subseteq Y\theta\beta = \{y_i\}$ and so $D(\alpha\theta\beta) \supseteq \{y_k\}$. Hence $\alpha\theta\beta \notin AE$ and so, in this case, $(AE(X,Y,q),\theta)$ is not a subsemigroup of $T(X,Y,\theta)$. Instead, suppose $|I| \geq |Y|$ and write $Y = \{y_m\} \dot\cup \{y_n\}$ where $|M| = |Y|$ and $|N| = q$ (possible since $|Y| \geq q \geq \aleph_0$). Now let $\mu_1 : \{x_i\} \to \{y_m\}$ and

$\mu_2 : \{x_j\} \to \{y_n\}$ be surjections. Then $\beta = \mu_1 \dot{\cup} \mu_2 : X \to Y$ is a surjection, and $D(\alpha\theta\beta) \supseteq \{y_n\}$ for each $\alpha \in AE$, so the result follows like before. $\square$

Clearly, $M(X, Y) \neq \emptyset$ if and only if $|X| \leq |Y|$, so this fact and (2) imply that

$$MAE(X, Y, q) \neq \emptyset \quad \text{if and only if} \quad |X| = |Y|. \tag{3}$$

To see that $MOE(X, Y, q) \neq \emptyset$ when $|X| \leq |Y|$, write $X = \{x_i\}$ and $Y = \{y_i\} \dot{\cup} \{y_j\}$ where $|J| = |Y| \geq q$ (possible since $\aleph_0 \leq q \leq |X| \leq |Y|$). Then

$$\alpha = \begin{pmatrix} x_i \\ y_i \end{pmatrix} \in MOE(X, Y, q).$$

Clearly, $E(X, Y) \neq \emptyset$ if and only if $|X| \geq |Y|$, so this fact and (1) imply that

$$EAM(X, Y, q) \neq \emptyset \quad \text{if and only if} \quad |X| = |Y|. \tag{4}$$

To see that $EOM(X, Y, q) \neq \emptyset$ when $|X| \geq |Y|$, write $Y = \{y_i\}$ and $X = \{x_i\} \dot{\cup} \{x_j\}$ where $|J| = |X| \geq q$ (possible since $\aleph_0 \leq q \leq |Y| \leq |X|$). If $0 \in I$ and $K = I \setminus \{0\}$, then

$$\alpha = \begin{pmatrix} \{x_0\} \cup \{x_j\} & x_k \\ y_0 & y_k \end{pmatrix} \in EOM(X, Y, q).$$

Thus, under suitable assumptions about the cardinals of X and Y, each of these four intersections are semigroups under sandwich operations induced on them by appropriate choices of θ. For example, if $|X| = |Y| \geq q \geq \aleph_0$, then $(EAM(X, Y, q), \theta)$ is a semigroup if and only if $\theta \in EAM(Y, X, q)$ (by Lemma 3.2). On the other hand, Lemma 3.1(b) implies that $c(\alpha\theta\beta) \geq c(\alpha)$ for each $\alpha, \beta \in T(X, Y)$ and $\theta \in T(Y, X)$. Hence $(EOM(X, Y, q), \theta)$ is a semigroup for each $\theta \in E(Y, X)$.

As in [12], we now determine when some of the semigroups listed above belong to BQ.

Clearly, if $R = OM(X, Y, q)$ then $R \subseteq T = P(X \cup Y)$ and it is right-closed by the regular semigroup $S = T(Y) \subseteq T$ (since $c(\alpha) \leq c(\alpha\beta)$ for all $\alpha \in R$ and $\beta \in S$). Also, if $\theta \in T(Y, X) \subseteq T$, then $\theta R \subseteq S$, so Lemma 2.1 implies that $(R, \theta) \in BQ$.

Similarly, if $L = OE(X, Y, q)$ then $L \subseteq T = P(X \cup Y)$ and L is left-closed by the regular semigroup $S = T(X) \subseteq T$ (since $d(\alpha) \leq d(\beta\alpha)$ for all $\alpha \in L$ and $\beta \in S$). Also, if $\theta \in T(Y, X) \subseteq T$, then $L\theta \subseteq S$ and so the dual of Lemma 2.1 implies that $(L, \theta) \in BQ$.

In view of the last two paragraphs, we have proved the following result.

THEOREM 3.4 *Suppose* $q \geq \aleph_0$, $|X| \geq q$ *and* $|Y| \geq q$. *If* $\theta \in T(Y,X)$ *then* $(OM(X,Y,q),\theta)$ *and* $(OE(X,Y,q),\theta)$ *belong to* BQ.

Two more semigroups which always belong to BQ are the *generalised Baer-Levi semigroups* and their duals (compare [3] section 8.1 and [1], respectively). We define them as follows. If $|X| = |Y| = p \geq q \geq \aleph_0$ then

$$\begin{aligned} BL(X,Y,q) &= \{\alpha \in T(X,Y) : c(\alpha) = 0 \text{ and } d(\alpha) = q\}, \\ DBL(X,Y,q) &= \{\alpha \in T(X,Y) : d(\alpha) = 0 \text{ and } |y\alpha^{-1}| = q \text{ for each } y \in Y\}. \end{aligned}$$

Clearly these are non-empty sets. From the next result, it follows that $(BL(X,Y,q),\theta)$ belongs to BQ when $\theta \in M(Y,X)$ and $d(\theta) \leq q$ (compare [2] Lemma 3.3).

THEOREM 3.5 *Suppose* $|X| = |Y| = p \geq q \geq \aleph_0$ *and* $\theta \in M(Y,X)$. *Then* $(BL(X,Y,q),\theta)$ *is a semigroup if and only if* $d(\theta) \leq q$. *Moreover, when this occurs,* $(BL(X,Y,q),\theta)$ *is right simple.*

Proof. For simplicity, write $BL(X,Y,q) = BL$. Suppose (BL,θ) is a semigroup and let $\alpha,\beta \in BL$. Then, since α,θ and β are injective on their domains,

$$\begin{aligned} Y \setminus X\alpha\theta\beta &= Y \setminus X\beta) \,\dot\cup\, (X \setminus X\alpha\theta)\beta, \\ &= (Y \setminus X\beta) \,\dot\cup\, [(X \setminus Y\theta) \,\dot\cup\, (Y \setminus X\alpha)\theta]\beta, \end{aligned}$$

and so $d(\theta) = |X \setminus Y\theta| \leq d(\alpha\theta\beta) = q$.

Conversely, suppose $d(\theta) > q$ and write $Y\theta = \{x_i\}$, so $|I| = p$ (since θ is injective). Let $D(\theta) = X \setminus Y\theta = \{x_j\}$, where $|J| = r > q$. Write $Y = \{y_i\} \,\dot\cup\, \{y_j\} \,\dot\cup\, \{y_k\}$, where $|K| = q$ (possible since $p \geq r > q \geq \aleph_0$) and define $\beta \in M(X,Y)$ by

$$\beta = \begin{pmatrix} x_i & x_j \\ y_i & y_j \end{pmatrix}.$$

Then $D(\beta) = \{y_k\}$, so $\beta \in BL$. But, for each $\alpha \in BL$, $X\alpha\theta\beta \subseteq Y\theta\beta = \{y_i\}$, and hence $D(\alpha\theta\beta) \supseteq Y \setminus Y\theta\beta = \{y_j\} \,\dot\cup\, \{y_k\}$, which has cardinal $r > q$. That is, (BL,θ) is not closed.

Finally, suppose $d(\theta) \leq q$ and write $\alpha,\beta \in BL$ and $\theta \in M(Y,X)$ as

$$\alpha = \begin{pmatrix} x_i \\ a_i \end{pmatrix}, \quad \theta = \begin{pmatrix} a_i & a_j \\ u_i & u_j \end{pmatrix}, \quad \beta = \begin{pmatrix} x_i \\ b_i \end{pmatrix},$$

where $D(\alpha) = \{a_j\}, D(\theta) = \{u_k\}$ and $D(\beta) = \{b_j\} = \{c_j\} \,\dot\cup\, \{d_j\} \,\dot\cup\, \{e_k\}$ (possible since $|J| = q \geq \aleph_0$ and $|K| = d(\theta) \leq q$). Now define $\gamma \in M(X,Y)$

by

$$\gamma = \begin{pmatrix} u_i & u_j & u_k \\ b_i & c_j & e_k \end{pmatrix}.$$

Then $D(\gamma) = \{d_j\}$, so $\gamma \in BL$ and clearly $\beta = \alpha\theta\gamma$. $\square$

We now prove a similar result for $DBL(X, Y, q)$ and we conclude that $(DBL(X, Y, q), \theta) \in BQ$ when $\theta \in E(Y, X)$ and $|x\theta^{-1}| \leq q$ for each $x \in X$.

THEOREM 3.6 *Suppose $|X| = |Y| = p \geq q \geq \aleph_0$ and $\theta \in E(Y, X)$. Then $(DBL(X, Y, q), \theta)$ is a semigroup if and only if $|x\theta^{-1}| \leq q$ for each $x \in X$. Moreover, when this occurs, $(DBL(X, Y, q), \theta)$ is left simple.*

Proof. For simplicity, write $DBL(X, Y, q) = DBL$. Suppose (DBL, θ) is a semigroup and let $\alpha, \beta \in DBL$. If $x \in X$ then $x \in y\beta^{-1}$ for some $y \in Y$ and

$$y\beta^{-1}\theta^{-1}\alpha^{-1} = \bigcup\{z\alpha^{-1} : z \in y\beta^{-1}\theta^{-1}\} \supseteq \bigcup\{z\alpha^{-1} : z \in x\theta^{-1}\}.$$

From this and the supposition, we deduce that $|x\theta^{-1}| \leq q$ for each $x \in X$.

Conversely, suppose $Y\theta = X = \{x_i\} \,\dot\cup\, \{x_0\}$, where $C_0 = x_0\theta^{-1} = \{y_j\}$ has cardinal greater than q. Note that $x_i\theta^{-1} \neq \emptyset$ for each i, so $|Y| \geq |Y \setminus C_0| \geq |I| = |X| = |Y|$ and thus we can write $Y = \{y_i\} \,\dot\cup\, \{y_j\}$, where $\{y_i\} = Y \setminus C_0$. Since $|X| = |Y| \geq q \geq \aleph_0$, there is a partition $\{A_i\} \,\dot\cup\, \{A_j\}$ of X where $|A_i| = |A_j| = q$ for each i and j. Clearly

$$\alpha = \begin{pmatrix} A_i & A_j \\ y_i & y_j \end{pmatrix} \in DBL$$

and, for each $\beta \in DBL$, $(\bigcup A_j)\alpha\theta\beta = x_0\beta$. Since $|\bigcup A_j| > q$, this implies that $\alpha\theta\beta \notin DBL$. That is, (DBL, θ) is not closed.

Finally, suppose $|x\theta^{-1}| \leq q$ for each $x \in X$ and write $\alpha, \beta \in DBL$ and $\theta \in E(Y, X)$ as

$$\alpha = \begin{pmatrix} A_i \\ y_i \end{pmatrix}, \quad \theta = \begin{pmatrix} C_{ij} \\ x_{ij} \end{pmatrix}, \quad \beta = \begin{pmatrix} B_i \\ y_i \end{pmatrix},$$

where, for each i, $B_i = \{x_{ij} : j \in J_i\}$ for an appropriate index set J_i (possible since $\{B_i\}$ is a partition of $X = Y\theta$). For each i, let $\lambda_i : A_i \to D_i = \bigcup\{C_{ij} : j \in J_i\}$ be a surjection such that $|y\lambda_i^{-1}| = q$ for each $y \in D_i$ (possible since $|A_i| = |B_i| = q$ and $1 \leq |C_{ij}| \leq q$ for each i and j, and consequently $|D_i| = |J_i|.\sum\{|C_{ij}| : j \in J_i\} = q$). Then $\lambda = \dot\cup\, \lambda_i \in DBL$ and $\alpha = \lambda\theta\beta$ as required. $\square$

4 Injective transformations

In [7] Theorem 3.2, the authors prove that if $|X| = |Y| \geq \aleph_0$ and $\theta_1 \in M(Y, X)$, then $M(X, Y, \theta_1) \notin BQ$. They do this by reducing the given problem to showing that if X is infinite and $\theta_2 \in M(X)$ then $M(X, X, \theta_2) \notin BQ$. On the other hand, in [2] Theorem 3.4, Chinram proves that if $|X| = |Y| \geq q \geq \aleph_0$ and $\theta \in M(Y, X)$, then $(MOE(X, Y, q), \theta)$ belongs to BQ if and only if $|X| = |Y| = q$. In essence, he does this by choosing a particular $\alpha \in M(X, Y)$ and showing that $\beta\theta\alpha = \alpha\theta\gamma$ for some $\beta, \gamma \in M(X, Y)$ where $\beta \neq \alpha\theta\mu$ for any $\mu \in M(X, Y)$.

To simplify and extend the work just mentioned, we prove the following Lemmas. The first can be used to characterise Green's $\mathcal{R}$–relation on the semigroup $(M(X, Y), \theta)$ when $\theta \in M(Y, X)$ (compare [4] Lemma 3(b) for arbitrary partial transformations). The second extends [12] Lemma 4.

Lemma 4.1 *Suppose $|X| = |Y| \geq q \geq \aleph_0$ and let $\alpha, \beta \in M(X, Y)$. In addition, suppose $\theta \in T(Y, X)$ and $\alpha\theta \in M(X)$. Then $\beta = \alpha\theta\mu$ for some $\mu \in M(X, Y)$ if and only if $d(\beta) \geq d(\alpha\theta)$.*

Proof. If $\beta = \alpha\theta\mu$ for some $\mu \in M(X, Y)$ then, since μ is injective, we have:

$$Y \setminus X\beta = (Y \setminus X\mu) \,\dot\cup\, (X \setminus X\alpha\theta)\mu,$$

and it follows that $d(\beta) \geq d(\alpha\theta)$. Conversely, suppose this condition holds. If $X = \{x_i\}$, we can write

$$\alpha\theta = \begin{pmatrix} x_i \\ a_i \end{pmatrix}, \quad \beta = \begin{pmatrix} x_i \\ y_i \end{pmatrix}.$$

If $\{a_j\} = X \setminus X\alpha\theta = D(\alpha\theta)$ then, by supposition, there exists $\{y_j\} \subseteq D(\beta)$. Then $\beta = \alpha\theta\mu$, where $\mu \in M(X, Y)$ is defined by

$$\mu = \begin{pmatrix} a_i & a_j \\ y_i & y_j \end{pmatrix}.$$

□

Lemma 4.2 *Suppose $|X| = |Y| = p \geq \aleph_0$ and let q be any (finite or infinite) cardinal such that $p \geq q$. If $\alpha \in M(X, Y)$ and $\theta \in M(Y, X)$, then there exist $\beta, \gamma \in M(X, Y)$ such that $\beta\theta\alpha = \alpha\theta\gamma$ and $d(\beta) = d(\gamma) = q$.*

Proof. Write $X = \{x_i\}$ and $x_i\alpha = a_i$ for each i, and let $Y \setminus X\alpha = \{a_j\}$ and $X \setminus Y\theta = \{x_\ell\}$ (note that J and/or L may be empty). Consequently, since θ is injective, we have:

$$X = (X \setminus Y\theta) \,\dot\cup\, (Y \setminus X\alpha)\theta \,\dot\cup\, (X\alpha\theta) = \{x_\ell\} \,\dot\cup\, \{a_j\theta\} \,\dot\cup\, \{a_i\theta\}.$$

Now $Y\theta = \{x_i\} \setminus \{x_\ell\}$, which has cardinal p since $|Y| = p$ and θ is injective. Moreover, since α is injective, $(\{x_i\} \setminus \{x_\ell\})\alpha = \{a_i\} \setminus \{a_\ell\}$, hence this set also has cardinal p. Choose $\{a_k\} \subsetneq \{a_i\} \setminus \{a_\ell\}$ so that $|K| = q$ and $|(\{a_i\} \setminus \{a_\ell\}) \setminus \{a_k\}| = p$ (possible since $p \geq \aleph_0$ and $p \geq q$). Write $\{a_i'\} = \{a_i\} \setminus (\{a_k\} \cup \{a_\ell\})$ and let $x_i' = a_i'\alpha^{-1}$ for each i. Clearly, $\{x_i'\} = \{x_i\} \setminus (\{x_k\} \cup \{x_\ell\}) \subseteq Y\theta$. Therefore, for each i, there exists $y_i \in Y$ such that $x_i' = y_i\theta$. Likewise, since $\{x_k\} \subseteq \{x_i\} \setminus \{x_\ell\} = Y\theta$, there exists $y_k \in Y$ such that $y_k\theta = x_k$. That is,

$$Y\theta = \{x_i\} \setminus (\{x_k\} \cup \{x_\ell\}) \,\dot\cup\, \{x_k\} = (\{y_i\} \,\dot\cup\, \{y_k\})\theta$$

and so $Y = \{y_i\} \,\dot\cup\, \{y_k\}$ (since θ is injective). Now define $\beta, \gamma \in M(X, Y)$ by

$$\beta = \begin{pmatrix} x_i \\ y_i \end{pmatrix}, \quad \gamma = \begin{pmatrix} a_i\theta & a_j\theta & x_\ell \\ a_i' & a_j & a_\ell \end{pmatrix}.$$

Then $\beta\theta\alpha = \alpha\theta\gamma$ since

$$\begin{aligned} \beta\theta\alpha &: x_i \mapsto y_i \mapsto x_i' \mapsto a_i', \\ \alpha\theta\gamma &: x_i \mapsto a_i \mapsto a_i\theta \mapsto a_i'. \end{aligned}$$

Moreover, from the above remarks, we see that $D(\beta) = \{y_k\}$ and $D(\gamma) = \{a_k\}$, so $d(\beta) = d(\gamma) = q$, as required. □

THEOREM 4.3 *Suppose* $|X| = |Y| \geq q \geq \aleph_0$. *If* $\theta \in MAE(Y, X, q)$, *then the sandwich semigroup* $(MAE(X, Y, q), \theta)$ *does not belong to* BQ.

Proof. For simplicity, write $MAE(X, Y, q) = MAE$. Choose $\alpha \in MAE$ with defect 2 and suppose $(\alpha)_Q \subseteq (\alpha)_B$ in the sandwich semigroup (MAE, θ). Since $\alpha \in M(X, Y)$ and $\theta \in M(Y, X)$, Lemma 4.2 implies that there exist $\beta, \gamma \in M(X, Y)$ such that $\beta\theta\alpha = \alpha\theta\gamma$ and $d(\beta) = d(\gamma) = 1$. That is, $\beta, \gamma \in MAE$ and so $\beta\theta\alpha \in (\alpha)_Q$. Hence, by supposition, $\beta\theta\alpha \in \{\alpha, \alpha\theta\alpha\}$ or $\beta\theta\alpha = \alpha\theta\mu\theta\alpha$ for some $\mu \in MAE$. Now, since injective transformations are right cancellative, if $\beta\theta.\alpha = \alpha = \mathrm{id}_X.\alpha$ then $\beta\theta = \mathrm{id}_X$, hence θ is surjective and so it is a bijection; thus, $\beta = \theta^{-1}$ and so $d(\beta) = 0$, a contradiction. On the other hand, if $\beta\theta\alpha = \alpha\theta\alpha$ then $\beta = \alpha$, another contradiction. Therefore, $\beta\theta\alpha = \alpha\theta\mu\theta\alpha$, and so $\beta = \alpha\theta\mu$. Then Lemma 4.1 implies that $d(\beta) \geq d(\alpha\theta)$ and, since θ is injective, $d(\alpha\theta) \geq d(\alpha)$, so we again contradict our choice of α and β. Consequently, $(\alpha)_Q \not\subseteq (\alpha)_B$ and the result follows. □

If $|X| = |Y| = p \geq \aleph_0$ and $\alpha \in M(X,Y)$ then $d(\alpha) \leq |Y|$. Thus, $M(X,Y) = MAE(X,Y,q)$ where $q = p'$, the *successor* of p. Consequently, [7] Theorem 3.2 does not follow directly from the above result. However, it is easy to see that the above proof with $MAE(X,Y,q)$ replaced throughout by $M(X,Y)$ produces the following result: namely, [7] Theorem 3.2.

THEOREM 4.4 *Suppose $|X| = |Y| \geq \aleph_0$. If $\theta \in M(Y,X)$, then the sandwich semigroup $(M(X,Y),\theta)$ does not belong to BQ.*

The next result provides a simpler proof of [2] Theorem 3.4.

THEOREM 4.5 *Suppose $|X| = |Y| \geq q \geq \aleph_0$. If $\theta \in M(Y,X)$, then the sandwich semigroup $(MOE(X,Y,q),\theta)$ belongs to BQ if and only if $|X| = q$.*

Proof. For simplicity, write $MOE(X,Y,q) = MOE$. If $|X| = q$, then certainly $d(\theta) \leq q$ and thus $MOE = BL(X,Y,q)$, which is right simple by Theorem 3.5, and hence it belongs to BQ.

Conversely, suppose $|X| = p > q$. Choose $\alpha \in MOE$ with defect p and suppose $(\alpha)_Q \subseteq (\alpha)_B$ in the sandwich semigroup (MOE,θ). Since $\alpha \in M(X,Y)$, Lemma 4.2 implies that there exist $\beta,\gamma \in M(X,Y)$ such that $\beta\theta\alpha = \alpha\theta\gamma$ and $d(\beta) = d(\gamma) = q$. That is, $\beta,\gamma \in MOE$ and so $\beta\theta\alpha \in (\alpha)_Q$. As in the proof of Theorem 4.3, $\beta\theta\alpha \notin \{\alpha,\alpha\theta\alpha\}$, so $\beta\theta\alpha = \alpha\theta\mu\theta\alpha$ for some $\mu \in MOE$. Like before, this leads to a contradiction, so $(\alpha)_Q \not\subseteq (\alpha)_B$ and the result follows. □

5 Surjective transformations

The next result is comparable with [12] Lemma 3.

LEMMA 5.1 *Suppose $|X| = |Y| \geq q \geq \aleph_0$ and let $\alpha,\gamma \in E(X,Y)$. In addition, suppose $\theta \in T(Y,X)$ and $\theta\alpha \in E(Y)$. Then $\gamma = \lambda\theta\alpha$ for some $\lambda \in E(X,Y)$ if and only if $|y\gamma^{-1}| \geq |y(\theta\alpha)^{-1}|$ for each $y \in Y$. Consequently, when this occurs, $c(\gamma) \geq c(\theta\alpha)$.*

Proof. Suppose $\gamma = \lambda\theta\alpha$ for some $\lambda \in E(X,Y)$. Then, since λ is surjective, we have:

$$y\gamma^{-1} = y(\theta\alpha)^{-1}\lambda^{-1} = \bigcup\{z\lambda^{-1} : z \in y(\theta\alpha)^{-1}\},$$

and it follows that $|y\gamma^{-1}| \geq |y(\theta\alpha)^{-1}|$ for each $y \in Y$. Conversely, suppose this condition holds. If $Y = \{y_i\}$, we can write

$$\theta\alpha = \begin{pmatrix} A_i \\ y_i \end{pmatrix}, \quad \gamma = \begin{pmatrix} B_i \\ y_i \end{pmatrix}.$$

Since $|B_i| \geq |A_i|$ for each i, there are surjective maps $\lambda_i : B_i \to A_i$ for each i. Then $\lambda = \dot{\cup}\, \lambda_i : X \to Y$ is surjective and clearly $\gamma = \lambda\theta\alpha$. □

EXAMPLE 5.2 Unfortunately, the condition: $c(\gamma) \geq c(\theta\alpha)$ does not imply that $\gamma = \lambda\theta\alpha$ for some $\lambda \in E(X,Y)$. For example, let $X = \{a_0\} \dot{\cup} \{a_i\} \dot{\cup} \{b_i\}$ and $Y = \{y_0\} \dot{\cup} \{y_i\}$, and define $\gamma, \alpha \in T(X,Y)$ by

$$\gamma = \begin{pmatrix} \{a_0\} \cup \{a_i\} & b_i \\ y_0 & y_i \end{pmatrix}, \quad \alpha = \begin{pmatrix} a_0 & \{a_i, b_i\} \\ y_0 & y_i \end{pmatrix}.$$

Clearly, $\gamma, \alpha \in E(X,Y)$ and $c(\gamma) = c(\alpha)$. Also, if $\theta : Y \to X$ is any bijection then $\theta\alpha \in E(Y)$ and $c(\alpha) = c(\theta\alpha)$. But $|y_i\gamma^{-1}| \ngeq |y_i(\theta\alpha)^{-1}|$ for each i.

Next we quote [12] Lemma 5. This is a result for the special case when $X = Y$ and $\theta = \mathrm{id}_X$: that is, for the semigroup $(E(X), \circ)$ of all mappings from X onto X under composition.

LEMMA 5.3 *Suppose $|X| = p \geq \aleph_0$ and let q be any (finite or infinite) cardinal such that $p \geq q \geq 2$. If $\alpha \in E(X)$ and $|X \setminus C(\alpha)| = p$, then there exist $\beta, \gamma \in E(X)$ such that $\beta\alpha = \alpha\gamma$ and $c(\beta) = c(\gamma) = q$.*

We wish to determine when $(EAM(X,Y,q), \theta)$ belongs to BQ and the same for the semigroup $(EOM(X,Y,q), \theta)$. However, Lemma 3.2 implies that θ must be surjective and have collapse less than q in the first case, whereas it can be any element of $E(Y,X)$ in the second case. Thus, the myriad of possibilities for θ dissuades us from attempting to prove a general result like Lemma 5.3 for $E(X,Y)$. Instead, we adopt the approach used in the proof of [7] Theorem 3.3. To do this, first we extend [7] Lemma 2.1(2).

LEMMA 5.4 *Suppose $|X| = |Y| \geq q \geq \aleph_0$ and let $\varphi : X \to Y$ be a bijection. If $\theta \in T(Y,X)$ then*

$$\Phi : (T(X,Y), \theta) \to (T(X,X), \varphi\theta), \ \alpha \to \alpha\varphi^{-1},$$

is an isomorphism. Moreover, for each $\alpha \in T(X,Y)$,

$$c(\alpha) = c(\alpha\varphi^{-1}), \ d(\alpha) = d(\alpha\varphi^{-1}) \quad \text{and} \quad c(\theta) = c(\varphi\theta), \ d(\theta) = d(\varphi\theta).$$

Consequently, if $\theta \in EAM(X,Y,q)$ then Φ induces an isomorphism from $(EAM(X,Y,q), \theta)$ onto $(EAM(X,X,q), \varphi\theta)$; and, for each $\theta \in E(Y,X)$, Φ also induces an isomorphism from $(EOM(X,Y,q), \theta)$ onto $(EOM(X,X,q), \varphi\theta)$.

Proof. It suffices to note that $x_1\alpha = x_2\alpha$ if and only if $x_1\alpha\varphi^{-1} = x_2\alpha\varphi^{-1}$, and $Y \setminus X\alpha = (X \setminus X\alpha\varphi^{-1})\varphi$. Also, $y_1\theta = y_2\theta$ if and only if $x_1(\varphi\theta) = x_2(\varphi\theta)$, where $y_i = x_i\varphi$ for some unique $x_i \in X$: that is, $C(\theta) = C(\varphi\theta)\varphi$. In addition, $X \setminus Y\theta = X \setminus X(\varphi\theta)$. □

We also need [7] Lemma 2.3. We give a simple proof of this result which mimics that of [12] Theorem 3.

Lemma 5.5 *If* $|X| = p \geq \aleph_0$, *then* $(E(X), \circ) \notin BQ$.

Proof. Choose $\alpha \in E(X)$ with collapse 3 and suppose $(\alpha)_Q \subseteq (\alpha)_B$. Then, since $|X \setminus C(\alpha)| = p$, Lemma 5.3 implies that there exist $\beta, \gamma \in E(X)$ such that $\beta\alpha = \alpha\gamma$ and $c(\beta) = c(\gamma) = 2$. Then $\alpha\gamma \in (\alpha)_Q$ and so, by supposition, $\alpha\gamma \in \{\alpha, \alpha^2\}$ or $\alpha\gamma = \alpha\lambda\alpha$ for some $\lambda \in E(X)$. However, since surjective transformations are left cancellative and $\gamma \notin \{\mathrm{id}_X, \alpha\}$, we deduce that $\gamma = \lambda\alpha$. From Lemma 5.1 (with $X = Y$ and $\theta = \mathrm{id}_X$), we deduce that $c(\gamma) \geq c(\alpha)$, a contradiction. So, $(\alpha)_Q \not\subseteq (\alpha)_B$ and the result follows. □

We now prove a result similar to [12] Theorem 3. Here, we let $G(X) = M(X) \cap E(X)$.

Theorem 5.6 *Suppose* $|X| = |Y| = p \geq q \geq \aleph_0$. *If* $\theta \in EAM(Y, X, q)$, *then the sandwich semigroup* $(EAM(X, Y, q), \theta)$ *does not belong to* BQ.

Proof. Since the BQ–property is preserved by isomorphisms, it suffices by Lemma 5.4 to consider the semigroup $(EAM(X, X, q), \theta)$ where $\theta \in EAM(X, X, q)$. For simplicity, write $EAM(X, X, q) = EAM$. If $\theta \in G(X) \subset EAM$ then

$$(EAM, \theta) \to (EAM, \circ),\ \alpha \to \theta\alpha,$$

is an isomorphism, and we showed in [12] Theorem 3 that $(EAM, \circ)$ does not belong to BQ.

Thus, we may assume that $\theta \notin G(X)$. If $(EAM, \theta) \in BQ$ then, in particular, $(\theta)_Q \subseteq (\theta)_B$ in the semigroup (EAM, θ). Then, since $\mathrm{id}_X \in EAM$, we know $\theta.\theta.\,\mathrm{id}_X = \mathrm{id}_X.\theta.\theta \in (\theta)_B$ and so $\theta^2 \in \{\theta, \theta^3\}$ or $\theta^2 = \theta^2\lambda\theta^2$ for some $\lambda \in EAM$. If $\theta^2 \in \{\theta, \theta^3\}$ then $\theta = \mathrm{id}_X$ (since surjective transformations are left cancellative) which contradicts our assumption. Hence $\theta^2 = \theta^2\lambda\theta^2$ for some $\lambda \in EAM$, so $\theta^2\lambda = \mathrm{id}_X$ (since id_X is the only idempotent in $E(X)$) and hence θ is injective, another contradiction. So, $(EAM, \theta) \notin BQ$ as required. □

To prove a general result for $(EOM(X, Y, q), \theta)$ that is similar to [12] Theorem 4 appears to be far more complicated. Instead, we consider the following special case.

THEOREM 5.7 *Suppose $|X| = |Y| = p \geq q \geq \aleph_0$. If $\theta \in E(Y, X)$ satisfies $q < c(\theta) < p$, then the sandwich semigroup $(EOM(X, Y, q), \theta)$ does not belong to BQ.*

Proof. By Lemma 5.4, we may restrict our attention to $(EOM(X, X, q), \omega)$ where $\omega = \varphi\theta \in E(X)$ satisfies $q < c(\omega) < p$. For simplicity, write $EOM(X, X, q) = EOM$. By Lemma 2(b), $c(\omega^2) \leq c(\omega) + c(\omega) < p$ and so $|X \setminus C(\omega^2)| = p$. Hence, by Lemma 5.3, there exist $\beta, \gamma \in E(X)$ such that $\beta\omega^2 = \omega^2\gamma$ and $c(\beta) = c(\gamma) = q$. Therefore $\omega, \beta, \gamma \in EOM$. If $(EOM, \omega) \in BQ$ then, in particular, $(\omega)_Q \subseteq (\omega)_B$ in the semigroup (EOM, ω). This implies

$$\beta\omega^2 = \omega^2\gamma \in (EOM * \omega) \cap (\omega * EOM) \subseteq (\omega)_B,$$

and so $\omega^2\gamma \in \{\omega, \omega^3\}$ or $\omega^2\gamma = \omega^2\lambda\omega^2$ for some $\lambda \in EOM$. If $\omega^2\gamma = \omega$ then $\omega\gamma = \mathrm{id}_X$ and so ω is injective, contradicting our assumption that $c(\omega) > q$. If $\omega^2\gamma = \omega^3$ then $\gamma = \omega$ and so $c(\gamma) = c(\omega)$, contradicting our choice of γ. And if $\omega^2\gamma = \omega^2\lambda\omega^2$ for some $\lambda \in EOM$, then $\gamma = \lambda\omega^2$ where $\gamma, \lambda, \omega \in E(X)$. Then Lemma 5.1 implies that $q = c(\gamma) \geq c(\omega^2) \geq c(\omega) > q$, another contradiction. Thus, we have shown that $(\omega)_Q \not\subseteq (\omega)_B$, and hence (EOM, ω) does not belong to BQ under the stated condition on ω. □

Consequently, if $\theta \in E(Y, X)$ and $(EOM(X, Y, q), \theta)$ belongs to BQ, then either $c(\theta) \leq q$ or $c(\theta) = p$. In what follows, we consider special cases of each of these possibilities.

Suppose $\theta \in E(Y, X)$ satisfies $c(\theta) = p = |Y \setminus C(\theta)|$ and write

$$\theta = \begin{pmatrix} A_i & a_j & a_k \\ x_i & x_j & x_k \end{pmatrix},$$

where $|A_i| \geq 2$ for each i, and $|J| = |K| = p$ (note that possibly $|I| < p$). Choose any bijection $\pi : \{x_i\} \cup \{x_j\} \to (\bigcup A_i) \cup \{a_j\}$, and define a bijection $\varphi : X \to Y$ by

$$\varphi = \pi \cup \begin{pmatrix} x_k \\ a_k \end{pmatrix}.$$

Clearly, $\varphi\theta\varphi\theta$ is injective on $\{x_k\}$ and so $|X \setminus C(\varphi\theta\varphi\theta)| \geq |K| = p$. Therefore, if $\omega = \varphi\theta$ for this particular φ, then $c(\omega) = p$ and $|X \setminus C(\omega^2)| = p$.

In view of the above, we state the following result whose proof is similar to that for Theorem 5.7 (we simply note that, at the last step, $p > q = c(\gamma) \geq c(\omega^2) \geq c(\omega) = p$).

THEOREM 5.8 *Suppose $|X| = |Y| = p \geq q \geq \aleph_0$. Let $\theta \in E(Y, X)$ satisfy $c(\theta) = p = |X \setminus C(\theta)|$. If $p > q$ then the semigroup $(EOM(X, Y, q), \theta)$ does not belong to BQ.*

Next, we take a particular example of $\theta \in E(Y,X)$ with $c(\theta) = q$ and show that, if $(EOM(X,Y,q),\theta)$ belongs to BQ, then $p = q$. We start with:

$$\theta = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 & \dots & Y' \\ b_{11} & B_1 \setminus \{b_{11}\} & B_2 & B_3 & \dots & X' \end{pmatrix}, \tag{5}$$

where $b_{11} \in B_1$, $\{X', B_1, B_2, \dots\}$ and $\{Y', A_1, A_2, \dots\}$ are partitions of X and Y respectively, each A_i and B_i has cardinal $q \geq \aleph_0$, and $|X'| = |Y'| = p$. Also, $Y' \to X'$, $A_2 \to B_1 \setminus \{b_{11}\}$ and $A_{i+1} \to B_i$, for $i = 2, 3, \dots$, are arbitrary bijections. Then $C(\theta) = A_1$ and so $\theta \in EOM(X,Y,q)$. Now let $\{C_j\}$ be a partition of X' with $|C_j| = 2$ for each j, and write $Y' = \{d_j\}$. Choose $a_{11} \in A_1$ and define $\alpha \in E(X,Y)$ by

$$\alpha = \begin{pmatrix} b_{11} & B_1 \setminus \{b_{11}\} & B_2 & B_3 & \dots & C_j \\ a_{11} & A_1 \setminus \{a_{11}\} & A_2 & A_3 & \dots & d_j \end{pmatrix}, \tag{6}$$

where α is injective on $X \setminus X'$. Clearly, $c(\alpha) = p$ since $|J| = p$. Next define $\beta, \gamma \in E(X,Y)$ by

$$\begin{aligned} \beta &= \begin{pmatrix} B_1 & B_2 & B_3 & B_4 & \dots & C_j \\ a_{11} & A_1 \setminus \{a_{11}\} & A_2 & A_3 & \dots & C_j\theta^{-1} \end{pmatrix}, \\ \gamma &= \begin{pmatrix} B_1 & B_2 & B_3 & B_4 & \dots & d_j\theta \\ a_{11} & A_1 \setminus \{a_{11}\} & A_2 & A_3 & \dots & d_j \end{pmatrix}. \end{aligned}$$

Since $\theta|Y' : Y' \to X'$ is a bijection, $|C_j| = |C_j\theta^{-1}|$ for each j and $\{d_j\theta\} = Y'\theta = X'$. In other words, our notation means that β and γ are injective on $X \setminus B_1$ and so $c(\beta) = c(\gamma) = |B_1| = q$. By calculation, we find:

$$\beta\theta\alpha = \begin{pmatrix} B_1 \cup B_2 & B_3 & B_4 & B_5 & \dots & C_j \\ a_{11} & A_1 \setminus \{a_{11}\} & A_2 & A_3 & \dots & d_j \end{pmatrix} = \alpha\theta\gamma.$$

THEOREM 5.9 *Suppose $|X| = |Y| = p \geq q \geq \aleph_0$. If $\theta \in EOM(Y,X,q)$ is defined as in (5) and the sandwich semigroup $(EOM(X,Y,q),\theta)$ belongs to BQ, then $p = q$.*

Proof. For simplicity, write $EOM(X,Y,q) = EOM$. Suppose $p > q$ and let $\alpha, \beta, \gamma \in EOM$ be defined as in (6)-(7) with $\beta\theta\alpha = \alpha\theta\gamma$. Suppose $(\alpha)_Q \subseteq (\alpha)_B$ in the semigroup (EOM,θ). Then $\alpha\theta\gamma \in (\alpha)_Q$ and so $\alpha\theta\gamma \in \{\alpha, \alpha\theta\alpha\}$ or $\alpha\theta\gamma = \alpha\theta\lambda\theta\alpha$ for some $\lambda \in EOM$. If $\alpha.\theta\gamma = \alpha.\mathrm{id}_Y$ then $\theta\gamma = \mathrm{id}_Y$ and hence θ is injective, contradicting our choice of θ. Likewise, if $\alpha\theta.\gamma = \alpha\theta.\alpha$ then $\gamma = \alpha$, another contradiction. Therefore, $\alpha\theta.\gamma = \alpha\theta.\lambda\theta\alpha$ and so $\gamma = \lambda\theta\alpha$. Hence, $q = c(\gamma) \geq c(\theta\alpha)$ by Lemma 5.1. However, $|C_j\theta^{-1}| = 2$ for each j and thus $c(\theta\alpha) = p$. Therefore we have a contradiction. Consequently $(\alpha)_Q \not\subseteq (\alpha)_B$ and the result follows. ☐

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