PU. M. A. Vol. **21** (2010), No. 1, pp. 79 – 98

IC Quasi-semiadequate Semigroups Satisfying the Congruence Condition

Huiling Yao
Guangdong University of Finance
Guangzhou, Guangdong, 510520, China
e-mail: sunsome@163.com

and

Yizhi Chen*
Department of Mathematics
Huizhou University
Huizhou, Guangdong, 516007, China
email: yizhichen1980@126.com

and

Yonghua Li
Department of Mathematics
South China Normal University
Guangzhou, 510631, China
email: liyh@scnu.edu.cn

(Received: September 28, 2009)

Abstract. In this paper, we introduce a class of semigroups, IC quasi-semiadequate semigroups which are a generalization of IC quasi-adequate semigroups. To describe a construction of such quasi-semiadequate semigroups satisfying the congruence condition, we define a binary relation η on the semigroups and prove that S/η is a weakly ample semigroup if S is an IC quasi-semiadequate semigroup. We use Hall semigroups and weakly ample semigroups to establish a construction of IC quasi–semiadequate semigroups which satisfy the congruence condition.

Mathematics Subject Classification(2000). 20M10.

Keywords: IC quasi–semiadequate semigroup, weakly ample semigroup, congruence condition, the minimum semiadequate good congruence.

Foundation items: Natural Science Foundation of Guangdong Province(8151601501000002, 9151051501000066); Natural Science Foundation of Huizhou University ($C_2 10 \cdot 0216$).

*Corresponding author

1 Introduction

A way of generalizing regular semigroups is to consider generalized Green's relations on semigroups which were introduced by McAlister [14] and Pastijn [15]. They called two elements a and b of a semigroup S $\mathcal{L}^*-$related, denoted by $a\mathcal{L}^*b$, if

$$ax = ay \text{ if and only if } bx = by \text{ for all } x, y \in S^1.$$

The relation $\mathcal{R}^*$ is defined dually. The intersection $\mathcal{L}^* \cap \mathcal{R}^*$ is denoted by $\mathcal{H}^*$. A semigroup S is called an *abundant semigroups* if there exists an idempotent in every $\mathcal{L}^*-$class and every $\mathcal{R}^*-$class of S. An abundant semigroup is said to be *adequate* (*quasi–adequate*) if its idempotents are commutative (form a subsemigroup of S) ([3], [5]). To study quasi–adequate semigroups, in [2], El-Qallali and Fountain defined the idempotent–connected condition, in brevity the IC condition, and an equivalent relation δ on quasi–adequate semigroups. El-Qallali and Fountain [3] proved that if δ is a congruence on a quasi–adequate semigroup S then δ is the least adequate congruence and S/δ is a type A semigroup. A quasi–adequate semigroup S is called a type W semigroup [3] if it satisfies the IC condition and δ is a congruence on it. In [10] Guo showed that if a quasi–adequate semigroup satisfies IC condition then δ is a congruence on it. Hence, a quasi–adequate semigroup is type W if and only if it satisfies the IC condition. In [3] the authors use Hall semigroups and type A semigroups to establish a construction of IC quasi–adequate semigroups.

In [1], the relations $\widetilde{\mathcal{L}}, \widetilde{\mathcal{R}}, \widetilde{\mathcal{H}}$ on a semigroup S are defined by

$$\widetilde{\mathcal{L}} = \{(a,b) \in S \times S : (\forall e \in E(S))\ ae = a \Leftrightarrow be = b\},$$

$$\widetilde{\mathcal{R}} = \{(a,b) \in S \times S : (\forall e \in E(S))\ ea = a \Leftrightarrow eb = b\},$$

$$\widetilde{\mathcal{H}} = \widetilde{\mathcal{L}} \cap \widetilde{\mathcal{R}},$$

where $E(S)$ is the set of all idempotents of S. A semigroup S is said to be *semiabundant* if each $\widetilde{\mathcal{L}}-$class and each $\widetilde{\mathcal{R}}-$class contains at least one idempotent. We say a semiabundant semigroup to be *semiadequate* (quasi–semiadequate) if its set of idempotents is a semilattice (if its set of idempotents forms a band). It is clear that $\mathcal{L} \subseteq \mathcal{L}^* \subseteq \widetilde{\mathcal{L}}$ and $\mathcal{R} \subseteq \mathcal{R}^* \subseteq \widetilde{\mathcal{R}}$. When restricted to the regular elements of S, $\mathcal{L}$, $\mathcal{L}^*$ and $\widetilde{\mathcal{L}}$ ($\mathcal{R}$, $\mathcal{R}^*$ and $\widetilde{\mathcal{R}}$) coincide. In a semigroup S, both equivalent relations $\mathcal{L}$ ($\mathcal{R}$)and $\mathcal{L}^*$ ($\mathcal{R}^*$) are right (left) compatible but that may not be the case for $\widetilde{\mathcal{L}}$ ($\widetilde{\mathcal{R}}$) (see [8]). A semigroup is said to satisfy the *CR* (*CL*) *condition* if $\widetilde{\mathcal{L}}$ ($\widetilde{\mathcal{R}}$) is a right (left) congruence. Further, a semigroup is said to satisfy the *congruence condition* if it satisfies both the *CR* condition and the *CL* condition.

In this paper our aim is to characterize *IC quasi-semiadequate semigroups* which are a generalization of *IC* quasi-adequate semigroups. We use Hall semigroups and *weakly ample semigroups* ([8], [9]) to establish a construction of *IC* quasi-semiadequate semigroups.

Section 2 contains the main technical results used in the paper. In Section 3, we describe the largest congruece contianed in $\widetilde{\mathcal{H}}$ on a quasi-semiadequate semigroup. In Section 4, we discuss the equivalence relation η on a quasi-semiadequate semigroup, which is analogous to the relation δ on a quasi-adequate semigroup. An example which is a quasi-semiadequate semigroup but the equivalence relation η is not a congruence on it is given in this section. In Section 5, IC quasi-semiadequate semigroup is introduced, and two examples are given, one is a quasi-semiadequate semigroup satisfying the congruence condition but not IC quasi-semiadequate, and the other one is an *IC* quasi-semiadequate semigroup satisfying the congruence condition. Moreover, it is showed that if S is an *IC* quasi-semiadequate semigroup satisfying the congruence condition then the equivalence relation η is a congruence on S. Finally, we obtain a construction of IC quasi-semiadequate semigroups satisfying the congruence condition.

We use the terminology and notion in [7], other undefined terms can be found in [8] and [9].

2 Preliminaries

Let S be a semiabundant semigroup. The $\widetilde{\mathcal{L}}-$class ($\widetilde{\mathcal{R}}-$class) containing the element a of the semigroup S is denoted by $\widetilde{L}_a$ ($\tilde{R}_a$) or by $\widetilde{L}_a(S)$ ($\widetilde{R}_a(S)$) in case of ambiguity. For $a \in S$, a^* denotes a typical idempotent in $\widetilde{L}_a(S) \cap E(S)$ and a^+ denotes a typical element in $\widetilde{R}_a(S) \cap E(S)$.

The following lemma is easy to check.

LEMMA 2.1 *Let a be an element of a semigroup S and e be an idempotent of S. Then the following conditions are equivalent :*

(1) $a\widetilde{\mathcal{L}}e$;

(2) $ae = a$ *and for all* $f \in E(S)$, $af = a$ *implies* $ef = e$.

We say that a semigroup homomorphism ϕ from S to T is a *good homomorphism* if for all elements a, b of S, $a\widetilde{\mathcal{L}}b$ implies that $a\phi\widetilde{\mathcal{L}}b\phi$ and $a\widetilde{\mathcal{R}}b$ implies that $a\phi\widetilde{\mathcal{R}}b\phi$. By Lemma 2.1 and its dual result, we have the following proposition.

PROPOSITION 2.2 *Let S be a semiabundant semigroup, T a semigroup and let $\phi : S \to T$ be a semigroup homomorphism. Then the following conditions are equivalent :*

(1) *ϕ is good;*

(2) *for each element a of S, there exist idempotents e in $\widetilde{L}_a$ and f in $\widetilde{R}_a$ such that $a\phi\widetilde{\mathcal{L}}(T)e\phi$ and $a\phi\widetilde{\mathcal{R}}(T)f\phi$.*

Clearly, a good homomorphic image of a semiabundant semigroup is also a semiabundant semigroup.

PROPOSITION 2.3 (*Proposition 1.2 of [12]*) *Let S be an arbitrary semigroup. Then the following conditions are equivalent:*

(1) *For all idempotents e and f of S the element ef is regular;*

(2) *$< E(S) >$ is a regular subsemigroup;*

(3) *$Reg(S)$ is a regular semigroup, where $Reg(S)$ is the set of regular elements of S.*

Any semigroup satisfying one of the conditions of the above proposition will be said to satisfy *the regularity condition* [12]. The following example is a semiabundant semigroup which does not satisfy the regularity condition.

Let $S = \{0, a, e, f\}$ with the following cayley table:

$\cdot$	0	e	f	a
0	0	0	0	0
e	0	e	a	a
f	0	a	f	0
a	0	0	a	0

which is a semiabundant semigroup but not abundant. Furthermore, since $E(S) = \{0, e, f\}$ and $< E(S) >= S$, S does not satisfy the regularity condition.

In a semigroup satisfying the regularity condition, Nambooripad [17] showed that the sandwich set of e and f, where e and f are idempotents, takes the following form :

$$S(e, f) = \{h \in E(S) : he = h = fh, ehf = ef\}$$

and $S(e, f) \neq \varnothing$. The proofs of Lemma 2.4 and Proposition 2.5 below are same as Proposition 1.3 and Theorem 1.6 in [12], respectively.

LEMMA 2.4 *Let S be a semiabundant semigroup satisfying the regularity condition. Then for $x, y \in S$ and $h \in S(x^*, y^+)$, $xy = (xh)(hy)$. Further, if S satisfies the congruence condition, then $xh\widetilde{\mathcal{L}}h$ and $hy\widetilde{\mathcal{R}}h$.*

Proof. Let S be a semiabundant semigroup satisfying the regularity condition and x, y in S. If $h \in S(x^*, y^+)$, then $x^*hy^+ = x^*y^+$ so that

$$xy = (xx^*)(y^+y) = x(x^*hy^+)y = xhy = (xh)(hy).$$

Further, if S satisfies the congruence condition, then $xh\widetilde{\mathcal{L}}x^*h$. Since $(x^*h)h = x^*h$ and $h(x^*h) = h$, we know that $x^*h\mathcal{L}h$ and so $xh\widetilde{\mathcal{L}}h$. Similarly, we can show that $hy\widetilde{\mathcal{R}}h$. □

PROPOSITION 2.5 *Let S be a semiabundant semigroup satisfying the regularity condition and $\alpha : S \to T$ a good homomorphism from S into a semigroup T. Let $a\alpha$ be an idempotent of T for some element a of S. Then there exists an idempotent h in S such that $h\alpha = a\alpha$.*

Proof. Let S be a semiabundant semigroup satisfying the regularity condition. Then, for each $a \in S$, there exist e and f in $E(S)$ such that $f\widetilde{\mathcal{R}}a\widetilde{\mathcal{L}}e$. The mapping is good so $f\alpha\widetilde{\mathcal{R}}a\alpha\widetilde{\mathcal{L}}e\alpha$. If $a\alpha$ is an idempotent in T, then $f\alpha\mathcal{R}a\alpha\mathcal{L}e\alpha$ and so $a\alpha f\alpha = f\alpha$ and $e\alpha a\alpha = e\alpha$. By the regularity condition and Lemma 2.4, for $h \in S(e, f)$ we have

$$\begin{aligned} a\alpha &= (aa)\alpha = (aha)\alpha = (afhea)\alpha = (af)\alpha h\alpha(ea)\alpha \\ &= f\alpha h\alpha e\alpha = (fhe)\alpha = h\alpha. \end{aligned}$$

□

We say that a congruence ρ on a semigroup S is a *good congruence* when the natural homomorphism $S \to S/\rho$ is good. It is clear that a surjective homomorphism is good if and only if its kernel is a good congruence. By Lemma 2.1, Proposition 2.2 and 2.5, it is easy to prove Corollary 2.6.

COROLLARY 2.6 *Let ρ be a congruence on a semiabundant semigroup S. Then the following conditions are equivalent:*

(1) *ρ is good;*

(2) *for any element $a \in S$ there exists idempotents e, f with e in $\tilde{L}_a$ and f in $\tilde{R}_a$ such that for all $g \in E(S)$,*

(*a*) *$(ag, a) \in \rho$ implies $(eg, e) \in \rho$;*

(*b*) *$(ga, a) \in \rho$ implies $(ge, e) \in \rho$.*

PROPOSITION 2.7 *Let S be a quasi-semiadequate semigroup and ρ is a good congruence on S. Then S/ρ is quasi-semiadequate.*

Proof. Since ρ is good, S/ρ is semiabundant, by Proposition 2.5, the set of idempotents in S/ρ is $\{e\rho : e \in E(S)\}$, which is clearly a subsemigroup of S/ρ. □

COROLLARY 2.8 *Let $\phi : S \to T$ be a good homomorphism from a quasi-semiadequate semigroup S into a semigroup T. Then $S\phi$ is quasi-semiadequate, and if f is an idempotent in $S\phi$, then there exists an idempotent e in S such that $e\phi = f$.*

3 The largest congruence contianed in $\widetilde{\mathcal{H}}$

In this section we introduce the largest congruence contained in $\widetilde{\mathcal{H}}$ on a quasi-semiadequate semigroup.

Let S be a quasi-semiadequate semigroup. We define binary relations μ_L and μ_R on S as follows :

$$\mu_L = \{(a,b) \in S : (\forall e \in E(S)) \quad ea\widetilde{\mathcal{L}}eb\};$$

$$\mu_R = \{(a,b) \in S : (\forall e \in E(S)) \quad ae\widetilde{\mathcal{R}}be\}.$$

Let $\mu = \mu_L \cap \mu_R$. Then we have the following proposition.

PROPOSITION 3.1 *Let S be a quasi-semiadequate semigroup satisfying the condition (CR). Then μ_L is the largest congruence on S contained in $\widetilde{\mathcal{L}}$.*

Proof. Clearly, μ_L is an equivalent relation on S. Since S satisfies the condition (CR), it is clear that μ_L is right compatible. Let $a\mu_L b$ for $a, b \in S$. Since there exists f in $E(S)$ such that $f\widetilde{\mathcal{L}}ec$ for $e \in E(S)$ and $c \in S$, we have $(fa, eca), (fb, ecb) \in \widetilde{\mathcal{L}}$. By $fa\widetilde{\mathcal{L}}fb$ we have $eca\widetilde{\mathcal{L}}ecb$ and so $ca\mu_L ca$. It means that μ_L is left compatible.

Now, we show that $\mu_L \subseteq \widetilde{\mathcal{L}}$. Assume that $a\mu_L b$ for $a, b \in S$ and $f\widetilde{\mathcal{R}}a\widetilde{\mathcal{L}}e, b\widetilde{\mathcal{L}}g$ for $f, e, g \in E(S)$. Since $fa\widetilde{\mathcal{L}}fb$ and $f\widetilde{\mathcal{R}}a$ implies that $a = fa$, we have $e\widetilde{\mathcal{L}}fb$. Further, $bg = b$ implies that $fbg = fb$. By the definition of $\widetilde{\mathcal{L}}$, $eg = e$. A similar argument shows that $ge = g$ so that $e\mathcal{L}g$ and hence $a\widetilde{\mathcal{L}}b$.

Finally, let α be a congruence contained in $\widetilde{\mathcal{L}}$. For $a, b \in S$, if $a\alpha b$, then for all $e \in E(S)$ we have $ea\alpha eb$, i.e., $ea\widetilde{\mathcal{L}}eb$. Hence, $a\mu_L b$ which shows that μ_L is the largest congruence on S contained in $\widetilde{\mathcal{L}}$. □

From Proposition 3.1 and its dual, we have

COROLLARY 3.2 *Let S be a quasi–semiadequate satisfying the congruence condition. Then μ is the largest congruence on S contained in $\widetilde{\mathcal{H}}$.*

4 The minimum semiadequate good congruence

A congruence ρ on a semigroup S is called a semiadequate congruence if S/ρ is a semiadequate semigroup. In this section, we shall demonstrate the existence of a minimum semiadequate good congruence γ on a quasi-semiadequate semigroup satisfying the congruence condition. We firstly introduce an equivalence relation η on a quasi-semiadequate semigroup S satisfying the congruence condition and then prove that η is a good congruence and $\eta = \gamma$ if it is a congruence. Finally, we obtain that if S is an IC quasi-semiadequate semigroup satisfying the congruence condition, then $\eta = \gamma$.

It is easy to prove that the intersection of good congruences on a semi-abundant semigroup satisfying the regularity condition is a good congruence. Since the universal relation on a quasi-semiadequate semigroup S is a semiadequate good congruence, we can define γ to be the intersection of all semiadequate good congruences on S.

PROPOSITION 4.1 *γ is the minimum semiadequate good congruence on a quasi–semiadequate semigroup S.*

Proof. In view of Proposition 2.5, $E(S/\gamma) = \{e\gamma | e \in E(S)\}$. If ρ is a semiaequate congruence on S, then for any $e, f \in E(S)$ we have $ef\rho fe$ and so $ef\gamma fe$. By Proposition 2.7, $E(S/\gamma)$ is a semilattice. Hence γ is the minimum semiadequate good congruence on S. □

For the rest of this section, S is a quasi-semiadequate semigroup satisfying the congruence condition with a band of idempotents B. We denote the $\mathcal{J}-$class in B containing an element of e in B by $E(e)$. The $\mathcal{J}-$relation on B is the minimum semilattice congruence [7].

We now define the relation η on S by the rule

$$a\eta b \text{ if and only if } E(a^+)aE(a^*) = E(b^+)bE(b^*) \text{ for some } a^+, a^*, b^+, b^*.$$

The definition is independent of the choices of a^+, a^*, b^+, b^*. Obviously, η is an equivalent relation. We now give some properties of η.

LEMMA 4.2 *If $a, b \in S$ and $b = eaf$ where $e \in E(a^+)$, $f \in E(a^*)$, then $e\widetilde{\mathcal{R}}b$ and $f\widetilde{\mathcal{L}}b$.*

Proof. Notice that $eb = b$. If $gb = b$ for $g \in B$, then $geaf = eaf$. Since $f \in E(a^*)$, we have $gea = geaa^*fa^* = eaa^*fa^* = ea$. Further, since

$e \in E(a^+)$ and $\widetilde{\mathcal{R}}$ is a left congruence, $e\widetilde{\mathcal{R}}ea^+\widetilde{\mathcal{R}}ea$ and so $ge = e$. Thus $e\widetilde{\mathcal{R}}b$.

Similarly, we can prove that $f\widetilde{\mathcal{L}}b$. □

Since the proofs of following Corollary 4.3, 4.4, 4.8 and Lemma 4.5 and Proposition 4.10, 4.7, 4.11 are same as Corollary 2.3, 2.4, 2.8 and Lemma 2.5 and Proposition 2.7, 2.9, 2.10 in [3] respectively, we omit their proofs. Readers can refer to [3].

COROLLARY 4.3 *If $a, b \in S$ and $b = eaf$ where $e \in E(a^+)$, $f \in E(a^*)$, then*
(1) $E(a^+) = E(b^+)$;
(2) $E(a^*) = E(b^*)$;
(3) $E(a^+)aE(a^*) = E(b^+)bE(b^*)$.

COROLLARY 4.4 *Let $a, b \in S$. Then*
(1) $a\eta = E(a^+)aE(a^*)$;
(2) $a\eta b$ *if and only if* $b = eaf$ *for some* $e \in E(a^+), f \in E(a^*)$.

LEMMA 4.5 *η is contained in any semiadequate congruence ρ on S.*

PROPOSITION 4.6 *If η is a congruence on S, then $\eta = \gamma$, where γ is the minimum semiadequate good congruence on S.*

Proof. Suppose that η is a congruence on S. We first prove that η is a good congruence. Let $a\widetilde{\mathcal{L}}e$ for $a \in S$ and $e \in B$. If $(af, a) \in \eta$ for $f \in B$, then there exist elements p in $E((af)^+)$ and q in $E((af)^*)$ such that $a = pafq$. Thus $aq = a$ and so $eq = e$. By $af\widetilde{\mathcal{L}}ef$,

$$E(ef) \subseteq E(e) = E(eq) \subseteq E(q) = E((af)^*) = E(ef),$$

i.e., $E(ef) = E(e)$. Hence $(ef, e) \in \eta$. From this and its dual, η is a good congruence on S.

By Proposition 2.7 and Lemma 4.5 it suffices to show that the idempotents of S/η commute. From Proposition 2.5, the set of idempotents of S/η is $\{e\eta : e \in B\}$ since $\eta \cap (B \times B) = \mathcal{J}(B)$. Therefore, $\eta = \gamma$. □

In view of Proposition 4.6 we now give a criterion for η to be a congruence.

PROPOSITION 4.7 *η is a congruence on S if and only if, for all $a, b \in S$, $aE(a^*)E(b^+)b \subseteq E((ab)^+)abE((ab)^*)$.*

COROLLARY 4.8 *If B is normal, then η is a congruence on S.*

However, η is not necessarily a congruence.

Example 4.9 [16] Let $S = \{e, f, g, h, z, a, b, c\}$ with the following Cayley table

$\cdot$	e	f	g	h	z	a	b	c
e	e	f	g	z	z	a	b	c
f	f	f	g	z	z	b	b	c
g	g	f	g	z	z	c	b	c
h	z	z	z	h	z	z	z	z
z	z	z	z	z	z	z	z	z
a	z	z	z	a	z	z	z	z
b	z	z	z	b	z	z	z	z
c	z	z	z	c	z	z	z	z

.

By Proposition 4.7 it is not difficult to prove that S is a quasi-semiadequate semigroup but the equivalence relation η is not a congruence on S.

Proposition 4.10 $\widetilde{\mathcal{H}} \cap \eta = \iota$.

Proposition 4.11 *If T is a full subsemigroup of S, then $\eta_T = \eta_S \cap (T \times T)$. Consequently, if η_S is a congruence, then so is η_T.*

5 The IC quasi–semiadequate semigroups

In this section we are going to establish a structure of IC quasi-semiadequate semigroups satisfying the congruence condition.

In [2], the authors introduced the concept of IC abundant semigroup. An abundant semigroup S is called *idempotent-connected*, for short, IC, if for each element a in S and for some $a^+ \in \mathcal{R}^* \cap E(S)$, $a^* \in \mathcal{L}^* \cap E(S)$, there exists a bijection $\alpha :< a^+ > \to < a^* >$ satisfying $xa = a(x\alpha)$ for all $x \in < a^+ >$. Such a bijection α is a isomorphism [2].

Remark: Let S be an abundant semigroup, $a \in S$, $a^+ \in \mathcal{R}^* \cap E(S)$ and $a^* \in \mathcal{L}^* \cap E(S)$. Assume that α is a mapping from $< a^+ >$ to $< a^* >$ satisfying $xa = a(x\alpha)$ for all $x \in S$. If $x\alpha = y\alpha$ for $x, y \in < a^+ >$, then $xa = a(x\alpha) = a(y\alpha) = ya$. By the definition of Green's $*-$relations, we deduce that $xa^+ = ya^+$, that is, $x = y$. This means that α is an injection. Further, for $x, y \in < a^+ >$, $xya = a((xy)\alpha) = a(x\alpha)(y\alpha)$. It means that $a^*(xy\alpha) = a^*(x\alpha)(y\alpha)$. Since $(xy\alpha), (x\alpha)(y\alpha) \in < a^* >$, we have $(xy\alpha) = (x\alpha)(y\alpha)$, that is, α is a homomorphism. Hence, we obtain an equivalent definition of idempotent–connected abundant semigroups, that is, an abundant semigroup S is idempotent–connected if for each element a

in S and for some $a^+ \in \mathcal{R}^* \cap E(S), a^* \in \mathcal{L}^* \cap E(S)$, there exists a subjection $\alpha :< a^+ > \to < a^* >$ satisfying $xa = a(x\alpha)$ for all $x \in < a^+ >$.

The following example is a quasi-semiadequate semigroup S in which there exist $a \in S, a^+ \in \mathcal{R}^* \cap E(S)$ and $a^* \in \mathcal{L}^* \cap E(S)$ and an epimorphism α from $a^+E(S)a^+ \to a^*E(S)a^*$ satisfying $xa = a(x\alpha)$ for all $x \in a^+E(S)a^+$, but α is not injective.

EXAMPLE 5.1 Let $S = \{e, f, g, h, z, a, b, \}$ with the following Cayley table

$\cdot$	a	b	h	e	f	g	z
a	z	z	z	a	b	b	z
b	z	z	z	b	b	b	z
h	a	b	h	z	z	z	z
e	z	z	z	e	f	g	z
f	z	z	z	f	f	f	z
g	z	z	z	g	g	g	z
z	z	z	z	z	z	z	z

.

The $\widetilde{\mathcal{R}}-$classes of S are $\{\{a, b, h\}, \{e\}, \{f\}, \{g\}, \{z\}\}$ and the $\widetilde{\mathcal{L}}-$classes of S are $\{\{a, e\}, \{b, f, g\}, \{h\}, \{z\}\}$. Then clearly S is a quasi-semiadequate semigroup satisfying the congruence condition. For $a \in S$, $e \in \widetilde{L}_a$ and $h \in \widetilde{R}_a$, $eE(S)e = \{e, f, g, z\}$ and $hE(S)h = \{h, z\}$. Define a mapping $\alpha : eE(S)e \to hE(S)h$ by $e\alpha = f\alpha = g\alpha = h$ and $z\alpha = z$. It is easy to check that α is an epimorphism and satisfies that for any $x \in eE(S)e$, $xa = a(x\alpha)$.

Now, we introduce the concept of idempotent–connected semiabundant semigroups. A semiabundant semigroup S is called *idempotent–connected*, for short, IC, if for each element a in S and for some $a^+ \in \widetilde{\mathcal{R}} \cap E(S), a^* \in \widetilde{\mathcal{L}} \cap E(S)$, there exists an isomorphism $\alpha :< a^+ > \to < a^* >$ satisfying $xa = a(x\alpha)$ for all $x \in < a^+ >$. The phrase "for some" in this definition may be replaced by "for all". It is not difficult to verify these results by the way in Section 3 of [3].

A semiaedquate semigroup S that satisfies the congruence condition and in which for all $a \in S$ and $e \in E(S)$

$$ea = a(ea)^* \text{ and } ae = (ae)^+a$$

is said to be weakly ample ([8], [9]). Accoroding to Lemma 2.9 and 5.1 in [8] we have the following lemma.

LEMMA 5.2 *A semiadequate semigroup S satisfying the congruence condition is weakly ample if and only if it is IC.*

To prove Theorem 5.4 we need the following lemma.

LEMMA 5.3 *Let B be a band. Then, for $e, f, u, v \in B$,*
(1) *$e = eufv$ and $f = fveu$ imply $E(e) = E(f)$,*
(2) *$e = euv$ implies $eu, ev \in E(e)$.*

Proof. Let $Y = B/\mathcal{J}$. For $\alpha \in Y$, we denote the inverse image of α under the natural homomorphism from B onto $B/\mathcal{J}$ by J_α. For $e, f, u, v \in B$, there exist $\alpha, \beta, \gamma, \delta$ in Y such that $E(e) = J_\alpha$, $E(f) = J_\beta$, $E(u) = J_\gamma$, $E(v) = J_\delta$.

(1) We only prove that $\alpha = \beta$. If $e = eufv$ and $f = fuev$, then $\alpha = \alpha\beta\gamma\delta = \beta$.

(2) If $e = euv$, then $\alpha\gamma\delta = \alpha$. It means that $\alpha\gamma = \alpha$ and $\alpha\delta = \alpha$. Hence, $J_\alpha J_\gamma = J_\alpha$ and $J_\alpha J_\delta = J_\alpha$, that is, $eu, ev \in E(e)$. □

THEOREM 5.4 *If S is an IC quasi-semiadequate semigroup satisfying the congruence condition, then $\eta = \gamma$.*

Proof. For $a, b, c \in S$, if $a\eta b$, then $a = ebf$ for some $e \in E(b^+)$ and $f \in E(b^*)$. Moreover, $b = b^+ab^*$. Since S is IC, there exists $h \in < b^* >$ and $g \in < a^* >$ such that $eb = bh$ and $b^+a = ag$. Notice that $ca = cebf = cb(hf) = (cb)^+cb((cb)^*(hf))$. By Corollary 4.4 (2) we only prove that $(cb)^*(hf) \in E((cb)^*)$. Since $ca = cb(hf) = cb(cb)^*(hf)$ and $cb = cb^+ab^* = ca(gb^*) = ca(ca)^*(gb^*)$, we have

$$\begin{aligned} & ca = ca(gb^*)(cb)^*(hf) \text{ and } cb = cb(hf)(ca)^*(gb^*) \\ \Longrightarrow \quad & (ca)^* = (ca)^*(gb^*)(cb)^*(hf) \text{ and } (cb)^* = (cb)^*(hf)(ca)^*(gb^*) \\ \Longrightarrow \quad & E((ca)^*) = E((cb)^*) \quad \text{(by Lemma 5.3 (1))} \end{aligned}$$

and

$$\begin{aligned} & cb = cb(hf)(gb^*) \\ \Longrightarrow \quad & (cb)^* = (cb)^*(hf)(gb^*) \\ \Longrightarrow \quad & (cb)^*(hf) \in E((cb)^*) \quad \text{(by Lemma 5.3 (2))}. \end{aligned}$$

In view of Corollary 4.4 (2), $ca\eta cb$, that is, η is left compatible.

Dually, we can verify that η is right compatible. According to Proposition 4.6, $\eta = \gamma$. □

The example below is exactly an *IC* quasi-semiadequate semigroup satisfying the congruence condition.

EXAMPLE 5.5 ([13]) Let $a = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. Let $a_n = 3^n a$ and $S = \{e, f, g, h, u, v, a, a_n\}$, where $n = 1, 2, 3, \ldots$. The Caley table of S is as follows:

$\cdot$	a	a_n	e	f	g	h	u	v
a	a	a_n	e	f	g	h	u	v
a_m	a_m	a_{m+n}	e	f	g	h	u	v
e	e	e	e	f	g	h	u	v
f	f	e	e	f	g	h	u	v
g	g	g	g	h	u	v	e	f
h	h	g	g	h	u	v	e	f
u	u	u	u	v	e	f	g	h
v	v	u	u	v	e	f	g	h

The $\widetilde{\mathcal{L}}-$classes on S are the sets $\{e, g, u\}$,$\{f, h, v\}$, $\{a, a_n\}$ $(n \in N)$ and the $\widetilde{\mathcal{R}}-$classes on S are the sets $\{e, f, g, h, u, v\}$, $\{a, a_n\}$ $(n \in N)$. It is routine to verify that S is an IC semiabundant semigroup satisfying the congruence condition. But S is not abundant since for any $n \in N$, a_n is not $\mathcal{R}^*-$related to any idempotents in S.

We recall [7] that on a band B, the relation $\mathcal{U}$ on B is defined by the rule

$$\mathcal{U} = \{(e, f) \in B \times B : eBe \cong fBf\}$$

and that $W_{e,f}$ denotes the set of all isomorphisms from eBe onto fBf. Notice that the Green's $\sim$–relations coincide with Green's relations on B. If $\alpha \in W_{e,f}$, then we can define $\alpha_l \in \mathcal{J}(B/\mathcal{L})$ and $\alpha_r \in \mathcal{J}(B/\mathcal{R})$ by

$$L_x\alpha_l = L_{x\alpha}; \quad R_x\alpha_r = R_{x\alpha} \ \ (x \in eBe).$$

For $e \in B$, the mappings $\rho_e \in \mathcal{J}(B/\mathcal{L})$ and $\lambda_e \in \mathcal{J}(B/\mathcal{R})$ are defined by

$$L_x\rho_e = L_{exe}; \quad R_x\lambda_e = R_{exe} \ \ (x \in B).$$

The Hall semigroup

$$W_B = \{(\rho_e\alpha_l, \lambda_f\alpha_r^{-1}) : \alpha \in W_{e,f}, (e, f) \in \mathcal{U}\}$$

is an orthodox subsemigroup of $\mathcal{J}(B/\mathcal{L}) \times \mathcal{J}^*(B/\mathcal{R})$ with the band of idempotents $B^* = \{(\rho_e, \lambda_e) : e \in B\}$ isomorphic to B. We refer the reader to ([7], Chapter VI) for further details.

Now, let S be an IC quasi-semiadequate semigroup satisfying the congruence condition and B be the set of idempotents of S. For $a \in S$, we define $\rho_a \in \mathcal{J}(B/\mathcal{L})$ and $\lambda_a \in \mathcal{J}(B/\mathcal{R})$ by

$$L_x\rho_a = L_{(xa)^*}; \quad R_x\lambda_a = R_{(ax)^+} \ \ (x \in B).$$

Clearly, ρ_a, λ_a are well–defined since S satisfies the congruence condition. Moreover, for any elements $a, b \in S$ and $x \in B$, we have

$$L_x\rho_a\rho_b = L_{(xa)^*}\rho_b = L_{((xa)^*b)^*} = L_{(xab)^*} = L_x\rho_{ab}.$$

Thus $\rho_{ab} = \rho_a\rho_b$. Similarly, we have $\lambda_{ab} = \lambda_b\lambda_a$. Furthermore, we define

$$\xi : S \to \mathcal{J}(B/\mathcal{L}) \times \mathcal{J}^*(B/\mathcal{R}), a \mapsto (\rho_a, \lambda_a).$$

Then ξ is a homomorphism. We now show that $ker\xi = \mu$. If $a\mu_L b$ for $a, b \in S$, then $xa\mu_L xb$ for any $x \in B$ so that $\widetilde{L}_{xa} = \widetilde{L}_{xb}$, and so $\widetilde{L}_{(xa)^*} = \widetilde{L}_{(xb)^*}$. Thus, $\rho_a = \rho_b$. From this and its dual we have $\mu \subseteq ker\xi$. On the other hand, if $\rho_a = \rho_b$, then for all element $x \in B$ we have $\widetilde{L}_{(xa)^*} = \widetilde{L}_{(xb)^*}$ and so $ea\widetilde{\mathcal{L}}eb$ for all $e \in E$, that is, $a\mu_L b$. It and its dual show that $ker\xi = \mu$.

Theorem 5.6 *Let S be an IC quasi-semiadequate semigroup satisfying the congruence condition and B be the set of idempotents of S. The mapping ξ : $S \to \mathcal{J}(B/\mathcal{L}) \times \mathcal{J}^*(B/\mathcal{R})$ defined by $a\xi = (\rho_a, \lambda_a)$ is a good homomorphism with kernel μ and $S\xi$ is a full subsemigroup of W_B.*

Proof. From the discussion above we only prove that $S\xi$ is a full subsemigroup of W_B. Let $a \in S, e \in \widetilde{R}_a \cap E(S), f \in \widetilde{L}_a \cap E(S)$ and $\alpha : eE(S)e \to fE(S)f$ be an connecting isomorphism. Notice that, for any $x \in E(S)$, $L_{xe} = L_{exe}$. Then,

$$\begin{aligned}
\widetilde{L}_x\rho_a &= \widetilde{L}_{(xa)^*} = \widetilde{L}_{(xea)^*} = \widetilde{L}_{xe}\rho_a \\
&= \widetilde{L}_{exe}\rho_a = \widetilde{L}_{(exea)^*} = \widetilde{L}_{(a(exe)\alpha)^*} \\
&= \widetilde{L}_{a(exe)\alpha} = \widetilde{L}_{a^*(exe)\alpha} \\
&= \widetilde{L}_{(exe)\alpha} \quad \text{(since } (exe)\alpha \in fE(S)f) \\
&= \widetilde{L}_x\rho_e\alpha_l,
\end{aligned}$$

that is, $\rho_a = \rho_e\alpha_l$. Symmetrically, we can prove that $\lambda_a = \lambda_f\alpha_r^{-1}$. Therefore, $a\xi = (\rho_e\alpha_l, \lambda_f\alpha_r^{-1})$. It means $S\xi \subseteq W_B$. Since $E(W_B) = \{(\rho_e, \lambda_e) \in W_B : e \in B\} \subseteq E(S\xi)$, we obtain that $S\xi$ is a full subsemigroup of W_B. □

We now consider good semiadequate homomorphic images of *IC* quasi-semiadequate semigroups satisfying the congruence condition.

LEMMA 5.7 *Let S be an IC quasi-semiadequae semigroup satisfying the congruence condition and a be an element of S. Let $e \in \widetilde{R}_a \cap E(S)$, $f \in \widetilde{L}_a \cap E(S)$ and let $\alpha : eE(S)e \to fE(S)f$ be the connecting isomorphism. Then, for $x \in eE(S)e$, $xa\widetilde{\mathcal{L}}x\alpha$.*

Proof. It follows from $xa = a(x\alpha)\widetilde{\mathcal{L}}f(x\alpha) = x\alpha$. □

PROPOSITION 5.8 *Let S be an IC quasi-semiadequate semigroup satisfying the congruence condition and $\theta : S \to T$ be a good homomorphism from S onto a semiadequate semigroup T. Then T is an IC semiadequate semigroup. Therefore, if T satisfies the congruence condition, then T is weakly ample.*

Proof. By Lemma 5.2 we only prove that T is IC. Let $a \in T$. Since θ is surjective, there exists s in S such that $s\theta = a$. Then, for $g, k \in E(S)$ with $g\widetilde{\mathcal{R}}s\widetilde{\mathcal{L}}k$, $g\theta = a^+$ and $k\theta = a^*$. We now prove that $< g > \theta =< g\theta >=< a^+ >$. For any $x \in< g > \theta$, there exists $h \in< g >$ such that $x = h\theta$. Then $x = h\theta \leq g\theta$ since $h \leq g$, i.e., $x \in< g\theta >$ and $< g > \theta \subseteq< g\theta >$. Conversely, for any $x \in< g\theta >$, $x \leq g\theta \in< g > \theta$ implies that $< g\theta >\subseteq< g > \theta$. Therefore, $< g\theta >=< g > \theta$. Similarly, we can show that $< k > \theta =< k\theta >=< a^* >$. Now we define $\beta :< a^+ >=< g > \theta \to< a^* >=< k > \theta$ by $h\theta \mapsto (h\alpha)\theta$ for any $h \in< g >$. Clearly, $(h\alpha)\theta \in< a^* >=< k > \theta$. For $h, h^{'} \in< g >$, by Lemma 5.7 we have $hs\widetilde{\mathcal{L}}h\alpha, h^{'}s\widetilde{\mathcal{L}}h^{'}\alpha$. Notice that θ is good. If $h\theta = h^{'}\theta$, then

$$(h\alpha)\theta\widetilde{\mathcal{L}}(hs)\theta = (h\theta)(s\theta) = (h^{'}\theta)(s\theta) = (h^{'}s)\theta\widetilde{\mathcal{L}}(h^{'}\alpha)\theta.$$

From the fact that T is semiadequate, we have $(h\alpha)\theta = (h^{'}\alpha)\theta$. Hence, β is a mapping. On the other hand, if $(h\alpha)\theta = (h'\alpha)\theta$, then

$$\begin{aligned} h\theta &= (h\theta)(s^+\theta)\widetilde{\mathcal{R}}(h\theta)(s\theta) = (hs)\theta = (s(h\alpha))\theta \\ &= (s(h'\alpha))\theta = (h's)\theta = (h'\theta)(s\theta)\widetilde{\mathcal{R}}(h'\theta)(s^+\theta) = h'\theta. \end{aligned}$$

We deduce that β is injective. Now, for any $f\theta \in< k > \theta =< a^* >$, since $(f\alpha^{-1})\theta$ is in $< g > \theta$ such that $((f\alpha^{-1})\theta)\beta = f\theta$, β is a surjection.

For any $x \in< a^+ >$, there exists h in $< g >$ such that $x = h\theta \in< g > \theta =< a^* >$. Then $xa = (hs)\theta = (s(h\alpha))\theta = (s\theta)(h\alpha)\theta = a(h\theta)\beta = a(x\beta)$. Hence, T is IC. Further, if T satisfies the congruence condition, then T is weakly ample. □

Unfortunately, we have no idea whether the good homomorphic image of semiabundant semigroup satisfying the congruence condition necessarily satisfies the congruence condition. So if we remove the congruence condition on T in the above proposition, we can not guard that T is weakly ample.

Let S be a quasi-semiadequate semigroup with a band B of idempotents. We call a semigroup homomorphism $\theta : S \to T$ a $\sim$–homomorphism if for all elements a, b of S, $a\widetilde{\mathcal{L}}b$ if and only if $a\theta\widetilde{\mathcal{L}}b\theta$, and $a\widetilde{\mathcal{R}}b$ if and only if $a\theta\widetilde{\mathcal{R}}b\theta$. Clearly, $\sim$–homomorphisms are good homomorphisms. In fact, $\sim$–homomorphisms are just the idempotent–separating good homomorphisms as the following lemma shows:

LEMMA 5.9 *Let S be a quasi-semiadequate semigroup and $\theta : S \to T$ be a semigroup homomorphism. Then the following conditions are equivalent:*
(1) *θ is a $\sim$–homomorphism;*
(2) *θ is an idempotent–separating good homomorphism.*

The proof is the same as one of Lemma 2.1 of [4].

We say that a congruence ρ on a semigroup S is a $\sim$–congruence when the natural homomorphism $S \to S/\rho$ is a $\sim$–homomorphism. Clearly, μ is a $\sim$–homomorphism if it is good.

PROPOSITION 5.10 *Let $\phi : S \to T$ be a $\sim$–homomorphism from a quasi-semiadequate semigroup S onto a semigroup T. Then S satisfies the congruence condition if and only if T satisfies the congruence condition.*

Proof. We need only to prove that S satisfies (CR) if and only if T satisfies (CR). Symmetrically we can prove the dual case.

Now, suppose that S satisfies (CR). Let $a, b, c \in T$ with $a\widetilde{\mathcal{L}}(T)b$. Since ϕ is a $\sim$–homomorphism, there exist a', b', c' in S such that $a'\phi = a, b'\phi = b, c'\phi = c$ and $a'\widetilde{\mathcal{L}}(S)b'$. Hence, $a'c'\widetilde{\mathcal{L}}(S)b'c'$ and so $ac = (a'c')\phi\widetilde{\mathcal{L}}(T)(b'c')\phi = bc$. That is, T satisfies (CR).

Conversely, suppose that T satisfies (CR). Let $s, t, r \in S$ with $s\widetilde{\mathcal{L}}(S)t$. Then $s\phi\widetilde{\mathcal{L}}(T)t\phi$ so that $sr\phi = s\phi r\phi\widetilde{\mathcal{L}}(T)t\phi r\phi = tr\phi$ since T satisfies (CR). Therefore, $st\widetilde{\mathcal{L}}(S)sr$. That is, S satisfies (CR). □

Now, we construct IC quasi-semiadequate semigroups satisfying the congruence condition by Hall semigroups and weakly ample semigroups.

Let B be a band and T be a weakly ample semigroup with semilattice Y of idempotents isomorphic to $B/\mathcal{J}$. Let $\psi : T \to W_B/\gamma$ be an idempotent–separating good homomorphism, whose range contains all the idempotents of W_B/γ where γ is the minimum inverse semigroup congruence on W_B.

THEOREM 5.11 *The spined product*

$$K = K(B, T, \psi) = \{(x, t) \in W_B \times T : x\gamma = t\psi\}$$

is an IC quasi–semiadequate semigroup satisfying the congruence condition with band of idempotents isomorphic to B and $K/\eta \cong T$.

Proof. We begin by pointing out that if $t\psi$ is an idempotent in W_B/γ, then because ψ is good, $t\psi = (t\psi)^+ = t^+\psi$ so that $t\psi$ is the image of an idempotent in T. Hence the band of idempotents of $K = K(B, T, \psi)$ is

$$\tilde{B} = \{((\rho_e, \lambda_e), i) : e \in B, i \in Y \text{ and } (\rho_e, \lambda_e)\gamma = i\psi\}.$$

Further, since the range of ψ contains all the idempotents of W_B/γ, the mapping $\kappa : \tilde{B} \to B^*$ defined by

$$((\rho_e, \lambda_e), i)\kappa = (\rho_e, \lambda_e)$$

is surjective. Also, if $((\rho_e, \lambda_e), i), ((\rho_e, \lambda_e), j) \in K$, then $i\psi = (\rho_e, \lambda_e)\gamma = j\psi$. Since ψ is idempotent–separating, we have $i = j$ so that κ is injective. Clearly, κ is a homomorphism and thus $\tilde{B}$ is isomorphic to B^* and hence to B.

We now show that K is an IC quasi-semiadequate satisfying the congruence condition. Let $(w, t) \in K$ and $w = (\rho_e\alpha_l, \lambda_f\alpha_r^{-1})$. Since ψ is good, $t^*\psi = (t\psi)^* = (w\gamma)^* = (\rho_f, \lambda_f)\gamma$. Hence, $a = ((\rho_f, \lambda_f), t^*) \in K$ and a routine calculation shows that $(w, t)\widetilde{\mathcal{L}}a$. Similarly, (w, t) is $\widetilde{\mathcal{R}}$–related to $b = ((\rho_e, \lambda_e), t^+)$. Since both T and W_B satisfy the congruence condition, we know that K also satisfies the congruence condition. Next, we show that K is IC. Define a mapping $\tau : b\tilde{B}b \to a\tilde{B}a$ by

$$((\rho_x, \lambda_x), i)\tau = ((\rho_{x\alpha}, \lambda_{x\alpha}), (it)^*)$$

where $c = ((\rho_x, \lambda_x), i) \in b\tilde{B}b$. From $c \in b\tilde{B}b$, we deduce that $x \in eBe$ and $i \in t^+E(T)$. From $(\rho_x, \lambda_x)\gamma = i\psi$ and $w\gamma = t\psi$, we obtain

$$(it)\psi = ((\rho_x, \lambda_x)w)\gamma = (\rho_x\beta_l, \lambda_{x\alpha}\beta_r^{-1})\gamma,$$

where $\beta = \alpha \mid x\tilde{B}x$. Since ψ is good, $(it)^*\psi = (\rho_{x\alpha}, \lambda_{x\alpha})\gamma$. Thus, $x\alpha \in fBf$ and $(it)^* \in t^*E$. It follows that $c\tau$ is in $a\tilde{B}a$.

Similarly, if we define $\theta : a\tilde{B}a \to b\tilde{B}b$ by

$$((\rho_y, \lambda_y), j)\theta = ((\rho_{y\alpha^{-1}}, \lambda_{y\alpha^{-1}}), (tj)^+),$$

then θ is actually a mapping. It is straightforward to check that τ, θ are mutually inverse. Further, for all $z \in B$

$$L_z\rho_x\rho_e\alpha_l = L_{(exzxe)\alpha} = L_{(xxzxx)\alpha} = L_{(x\alpha)(xzx)\alpha(x\alpha)} = L_z\rho_x\alpha_l\rho_{x\alpha}$$

and

$$R_z\lambda_f\alpha_r^{-1}\lambda_x = R_{x(fzf)\alpha^{-1}x} = R_{((x\alpha)(fzf)(x\alpha))\alpha^{-1}} = R_{(f(x\alpha)z(x\alpha)f)\alpha^{-1}} = R_z\lambda_{x\alpha}\lambda_f\alpha_r^{-1}.$$

Hence,

$$\begin{aligned} c(w,t) &= ((\rho_x,\lambda_x),i)((\rho_e\alpha_l,\lambda_f\alpha_r^{-1}),t)\\ &= ((\rho_x\rho_e\alpha_l,\lambda_f\alpha_r^{-1}\lambda_x),it)\\ &= ((\rho_e\alpha_l\rho_{x\alpha},\lambda_{x\alpha}\lambda_f\alpha_r^{-1}),t(it)^*) \quad \text{(since } T \text{ is weakly ample)}\\ &= ((\rho_e\alpha_l,\lambda_f\alpha_r^{-1}),t)((\rho_{x\alpha},\lambda_{x\alpha}),(it)^*)\\ &= ((\rho_e\alpha_l,\lambda_f\alpha_r^{-1}),t)(((\rho_x,\lambda_x),i)\tau)\\ &= (w,t)(c\tau), \end{aligned}$$

so that K is IC. $((\rho_{x\alpha},\lambda_{x\alpha}),(it)^*)$

Finally, we shows that $K/\eta \cong T$. Define a mapping $\pi : K \to T$ by $(w,t)\pi = t$. Since $E((w,t)^+)(w,t)E((w,t)^*) = E(w^+)wE(w^*) \times \{t\}$, $(w,t)\eta(v,u)$ if and only if $w\gamma = v\gamma$ and $t = u$. But $t = u$ implies that $w\gamma = t\psi = u\psi = v\gamma$. Hence $(w,t)\eta(v,u)$ if and only if $t = u$. Since, for any $t \in T$, $t\psi \in W_B/\gamma$ so that $t\psi = w\gamma$ for some $w \in W_B$, π is an epimorphism with $ker\pi = \eta$. Thus, $K/\eta \cong T$. □

Let S be an IC quasi-semiadequate semigroup satisfying the congruence condition and B be its band of idempotents. From Theorem 5.6 we have a homomorphism $\xi : S \to W_B$ defined by $a\xi = (\rho_a,\lambda_a)$ and then we obtain a homomorphism $\varphi : S \to W_B \times S/\eta$ defined by $a\varphi = (a\xi, a\eta)$. Now $a\xi = b\xi$ and $a\eta = b\eta$ imply that $(a,b) \in \mu \cap \eta \subseteq \widetilde{\mathcal{H}} \cap \eta$. By Proposition 4.10 we deduce $a = b$. That is, φ is injective.

We define $\psi : S/\eta \to W_B/\gamma$ by $(a\eta)\psi = (a\xi)\gamma$.

Lemma 5.12 *ψ is an idempotent–separating good homomorphism whose range contains all the idempotents of W_B/γ.*

Proof. Notice that, for $a, b \in S$,

$$\begin{aligned} & a\eta = b\eta\\ \Longrightarrow\ & E(a^+)aE(a^*) = E(b^+)bE(b^*)\\ \Longrightarrow\ & \{(eaf)\xi : e \in E(a^+), f \in E(a^*)\} = \{(gbh)\xi : g \in E(b^+), h \in E(b^*)\}\\ \Longrightarrow\ & E(a^+\xi)a\xi E(a^*\xi) = E(b^+\xi)b\xi E(b^*\xi)\\ \Longrightarrow\ & (a\xi)\gamma = (b\xi)\gamma. \end{aligned}$$

It means ψ is a mapping. Since

$$(a\eta b\eta)\psi = (ab\eta)\psi = (ab\xi)\gamma = (a\xi)\gamma(b\xi)\gamma = (a\eta)\psi(b\eta)\psi$$

and η, ξ, γ are all good homomorphisms, ψ is also a good homomorphism.

Let $(e\eta)\psi = (f\eta)\psi$ for $e\eta, f\eta \in S/\eta$, where $e, f \in B$. Then $(e\xi)\gamma = (f\xi)\gamma$; i.e. $e\xi$ and $f\xi$ are $\mathcal{J}-$equivalent in $E(W_B)$. Hence e and f are $\mathcal{J}-$equivalent in $E(B)$. Consequently, $e\eta = f\eta$. Therefore, ψ is idempotent–separating.

Notice that the set of idempotents of W_B/γ is $\{(\rho_e, \lambda_e)\gamma : e \in B\}$. Since $S\xi$ is a full subsemigroup of W_B, $e\xi = (\rho_e, \lambda_e)$ implies that $(\rho_e, \lambda_e)\gamma = (e\eta)\psi$. Thus all idempotents of W_B/γ are in the range of ψ. □

Remark: By Lemma 5.9 and Proposition 5.10, S/η satisfies the congruence condition since W_B/γ clearly satisfies the congruence condition. In view of Lemma 5.2, S/η is weakly ample.

THEOREM 5.13 $S \cong K = K(B, S/\eta, \psi)$.

Proof. We have a one–one homomorphism $\varphi : S \to W_B \times S/\eta$ defined by $a\varphi = (a\xi, a\eta)$. By the definition of ψ, $(a\xi)\gamma = (a\eta)\psi$ so that $a\varphi \in K$.

If $(x, a\eta) \in K$, then

$$x\gamma = (a\eta)\psi = (a\xi)\gamma = (\rho_a, \lambda_a)\gamma.$$

Since $(\rho_a, \lambda_a)\gamma = E((\rho_a, \lambda_a)^+)(\rho_a, \lambda_a)E((\rho_a, \lambda_a)^*)$, there exist (ρ_e, λ_e) in $E((\rho_a, \lambda_a)^+)$ and (ρ_f, λ_f) in $E((\rho_a, \lambda_a)^*)$ such that

$$x = (\rho_e, \lambda_e)(\rho_a, \lambda_a)(\rho_f, \lambda_f)$$

Further, since ξ is good and $\xi \mid B$ is an isomorphism from B onto B^*, it follows that $e \in E(a^+), f \in E(a^*)$. Now, put$b = eaf$. Then $b\eta a$ and so $b\xi = (eaf)\xi = x$. It means that

$$b\varphi = (b\xi, b\eta) = (x, a\eta)$$

and φ is surjective. Finally, for $a, b \in S$,

$$(ab)\varphi = ((ab)\xi, (ab)\eta) = (a\xi, a\eta)(b\xi, b\eta) = (a)\varphi(b)\varphi.$$

Therefore, $S \cong K$. □

From Theorem 5.11 and Theorem 5.13 we conclude our main result as follows:

THEOREM 5.14 *Let B be a band and T be a weakly ample semigroup with semilattice Y of idempotents isomorphic to $B/\mathcal{J}$. Let $\psi : T \to W_B/\gamma$ be an idempotent–separating good homomorphism, whose range contains all the* *idempotents of W_B/γ where γ is the minimum inverse semigroup congruence on W_B. The spined product*

$$K = K(B, T, \psi) = \{(x, t) \in W_B \times T : x\gamma = t\psi\}$$

is an IC quasi-semiadequate semigroup satisfying the congruence condition.

Conversely, every IC quasi-semiadequate semigroup satisfying the congruence condition can be constructed in this way.

References

[1] A. El–Qallali, *Structure theory for abundant and related semigroups*, Ph. D. Thesis, Yok, 1980.

[2] A. El–Qallali and J. B. Fountain, *Idempotent–connected abundant semigroups*, Proc. Roy. Soc. Edinburgh A, 91 (1981) 79–90.

[3] A. El–Qallali and J. B. Fountain, *Quasi–adequate semigroups*, Proc. Roy. Soc. Edinburgh A, 91 (1981) 91–99.

[4] A. El–Qallali, *Quasi–adequate semigroups II*, Semigroup Forum, 44 (1992) 273–282.

[5] J. B. Fountain, *Adequate semigroups*, Proc. Edinburgh Math. Soc., 22 (1979) 113–125.

[6] J. B. Fountain, *Abandant semigroups*, Proc. London Math. Soc., 44 (1982) 103–129.

[7] J. M. Howie, *An Introduction to Semigroup Theory*, Academic Press, 1976.

[8] J. B. Foutain, G. M. S. Gomes and V. Gould, *A Munn type representation for a class of E–semiadequate semigroups*, Journal of Algebra, 218(1999) 693–714.

[9] G. M. S. Gomes and V. Gould, *Fundanmental Ehresmann semigroups*, Semigroup Forum, 63(2001) 11–33.

[10] X. J. Guo, *F–abundant semigroups*, Glasgow Math. J., 43 (2001) 153–163.

[11] T. E. Hall, *Orthodox semigroups*, Pacific J. Math., 39 (1971) 677–686.

[12] M. V. Lawson, *The naturally partial order on abundant semigroups*, Proceedings of the Edinburgh Mathematical Society, 30 (1987) 169–186.

[13] X. M. Ren and K. P. Shum, *The structure of superabundant semigroups*, Science in China ser. A Mathematics, 47 (2004) 756–771.

[14] D. B. McAlister, *One–to–one partial right translations of a right cancellative semi–group*, J. Algebra, 43 (1976) 231-251.

[15] F. Pastijn, *A represention of a semigroup by a semigroup of matrices over a group with zero*, Semigroup Forum, 10 (1975) 238–249.

[16] K. P. Shum, Aihua Du and Y. Q. Guo, *Good quasi–adequate semigroups*, submitted.

[17] K. S. S. Nambooripad, *The structure of regular semigroups*, Men. Amer. Math. Soc. 22 (1979).

[18] M. Yamada, *On a regular semigroups in which the idempotents form a band*, Pacific J. Math. 33 (1970) 261–272.