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Q-analogues of the number of permutations with k-excedances

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Abstract. Given a permutation $\sigma = \sigma_1 \dots \sigma_n$ in the symmetric group S_n, we say that σ has a k-descent at i if $\sigma_i - \sigma_{i+1} = k$ and σ has a k-succession or a k-excedance at i if $\sigma_i - i = k$. Rakotondrajao [1] studied the numbers $e^k_{n,s}$ of permutations in the symmetric group S_n which have s k-successions and proved a number of simple formulas, generating functions, and recursions for these numbers. The main focus of this paper is to study q-analogues of Rakotondrajao's formulas and recursions. However, we will also provide some new combinatorial interpretations and recursions for $e^k_{n,s}$. Our interpretations will allow us to provide simple combinatorial proofs of the formulas introduced by Rakotondrajao.

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1 Introduction

Given a permutation $\sigma = \sigma_1 \dots \sigma_n$ in the symmetric group S_n and $k \geq 1$, we let

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$$\begin{array}{ll} \mathrm{Des}(\sigma) = \{i : \sigma_i > \sigma_{i+1}\} & \mathrm{des}(\sigma) = |\,\mathrm{Des}(\sigma)| \\ \mathrm{Des}_k(\sigma) = \{i : \sigma_i - \sigma_{i+1} = k\} & \mathrm{des}_k(\sigma) = |\,\mathrm{Des}_k(\sigma)| \\ \mathrm{Exc}(\sigma) = \{i : i < \sigma_i\} & \mathrm{exc}(\sigma) = |\,\mathrm{Exc}(\sigma)| \\ \mathrm{Exc}_k(\sigma) = \{i : \sigma_i - i = k\} & \mathrm{exc}_k(\sigma) = |\,\mathrm{Exc}_k(\sigma)| \end{array}.$$

Elements of $\mathrm{Des}(\sigma)$ are called the *descents* of σ and pairs (σ_i, σ_{i+1}) with $i \in \mathrm{Des}(\sigma)$ are called *descent pairs* of σ. Similarly, elements of $\mathrm{Exc}(\sigma)$ are called the *excedances* of σ and pairs (i, σ_i) with $i \in \mathrm{Exc}(\sigma)$ are called *excedance pairs* of σ.

For $k \geq 0$, Rakotondrajao [1] called elements of $\mathrm{Exc}_k(\sigma)$ k-successions and studied the numbers

$$e^k_{n,s} = |\{\sigma \in S_n : \mathrm{exc}_k(\sigma) = s\}|$$

for $n, s, k \geq 0$. In other words, $e^k_{n,s}$ is the number of permutations in S_n having exactly s k-successions. In analogy with excedances and descents, we prefer to call elements of $\mathrm{Exc}_k(\sigma)$ *k-excedances* of σ and pairs (i, σ_i) with $i \in \mathrm{Exc}_k(\sigma)$ *k-excedance pairs* of σ. Similarly, we shall call elements of $\mathrm{Des}_k(\sigma)$ *k-descents* of σ and pairs (σ_i, σ_{i+1}) with $i \in \mathrm{Des}_k(\sigma)$ *k-descent pairs* of σ. We note that when $k = 0$, $\mathrm{Exc}_0(\sigma) = \{i : i = \sigma_i\}$ so that $\mathrm{exc}_0(\sigma)$ is the number of fixed points of σ. Thus, $e^0_{n,0}$ is the number of permutations $\sigma \in S_n$ which have no fixed points. Such permutations are called derangements and we let $D_n = e^0_{n,0}$ denote the number of derangements of S_n.

Rakotondrajao [1] proved the following basic recursions and formulas for $e^k_{n,s}$.

I. For all $n \geq 2$, $0 \leq k < n$, and $s \geq 1$,

$$e^k_{n,s} = e^k_{n-1,s-1} + (n-s-1)e^k_{n-1,s} + (s+1)e^k_{n-1,s+1}. \tag{1}$$

Here we interpret $e^k_{n,s} = 0$ if $s < 0$.

II. For all $n \geq 1$, $0 \leq k < n$, and $s \geq 0$,

$$e^k_{n+k,s} = \binom{n}{s} e^k_{n+k-s,0}. \tag{2}$$

III. For all $n \geq 1$, $0 \leq k < n$,

$$e^k_{n,0} = (n-1)e^k_{n-1,0} + (n-1-k)e^k_{n-2,0}. \tag{3}$$

Note that when $k = 0$, (3) reduces to the usual recursion for the number of derangements

$$D_n = (n-1)D_{n-1} + (n-1)D_{n-2}.$$

Thus we automatically also have the second recursion for the number of derangements.

IV. For all $n \geq 1$ and $0 \leq k < n$,

$$e^0_{n,0} = ne^0_{n-1,0} + (-1)^n. \tag{4}$$

V. For all $n \geq 2$ and $1 \leq k < n$,

$$e^k_{n,0} = e^{k-1}_{n-1,0} + e^{k-1}_{n,0}. \tag{5}$$

VI. For all $k \geq 0$,

$$E^{(k)}(t,x,u) = \sum_{k\geq 0}\sum_{n\geq 0}\sum_{s=0}^{n} e^k_{n+k,s} t^s \frac{x^k}{k!}\frac{u^n}{n!} = \frac{e^{u(t-1)}}{1-x-u}. \tag{6}$$

VII. It then follows from (6) that for all $n \geq 1$, $0 \leq k < n$, and $s \geq 0$,

$$e^k_{n,s} = \sum_{i=s}^{n-k}(n-i)!\binom{i}{s}\binom{n-k}{i}. \tag{7}$$

In this paper, we shall give two alternative combinatorial interpretations for $e^k_{n,s}$. That is, first it follows from Foata's First Fundamental transformation that for $k \geq 1$,

$$e^k_{n,s} = |\{\sigma \in S_n : Des_k(\sigma) = s\}|.$$

Second, we shall interpret the polynomial

$$E_{n,k}(x) := \sum_{s=0} e^k_{n,s} x^s$$

as a hit polynomial for a certain board $B_{n,k}$ contained in the $[n] \times [n]$ board where $[n] = \{1, \ldots, n\}$. This interpretation allows us to think of $e^k_{n,s}$ as counting certain rook placements in the $[n] \times [n]$ board. This interpretation allows us to give much simpler combinatorial proofs of many of the recursions proved by Rakotondrajao [1]. Moreover, we shall introduce and prove four additional recursions for $e^k_{n,s}$.

VIII. For all $k, n \geq 1$,

$$e^k_{n,0} = ke^{k-1}_{n-1,0} + (n-k)e^k_{n-1,0}. \tag{8}$$

IX. For all $n \geq 2$ and $0 \leq k < n$,

$$e_{n,0}^{k} = k! \sum_{r=0}^{k} \binom{k}{r} \binom{n-k}{k-r} e_{n-k,0}^{k-r}. \tag{9}$$

X. For all $n \geq 2$ and $0 \leq k < n$,

$$e_{n,0}^{k} = e_{n,1}^{k+1} + k e_{n-1,0}^{k}. \tag{10}$$

XI. For all $n \geq 1$ and $0 \leq k < n$,

$$e_{n,0}^{k} = \sum_{r=0}^{k} \binom{k}{r} D_{n-k+r}. \tag{11}$$

The main goal of this paper is to study q-analogues of (1)-(11). In fact, we shall consider two fundamentally different q-analogues of $e_{n,s}^{k}$. That is, for $k \geq 0$ and $0 \leq s \leq n$, we define both $e_{n,s}^{k}(q)$ and $\hat{e}_{n,s}^{k}(q)$ by setting

$$e_{n,s}^{k}(q) := q^{\binom{s}{2}} \sum_{t=s}^{n-k} (-1)^{t-s} [n-t]_q! \begin{bmatrix} t \\ s \end{bmatrix}_q \begin{bmatrix} n-k \\ t \end{bmatrix}_q$$

and

$$\hat{e}_{n,s}^{k}(q) := \sum_{t=s}^{n-k} (-1)^{t-s} q^{\binom{t}{2}} [n-t]_q! \begin{bmatrix} t \\ s \end{bmatrix}_q \begin{bmatrix} n-k \\ t \end{bmatrix}_q.$$

We shall see that $e_{n,s}^{k}(q)$ satisfies natural q-analogues of (1), (2), (3), (4), (6), (8), and (9), but does not seem to satisfy natural analogues of (5), (10), or (11). On the other hand, $\hat{e}_{n,s}^{k}(q)$ satisfies natural q-analogues of (5), (10) and (11), but does not seem to satisfy natural analogues of (1), (2), (3), (4), (6), (8), or (9).

The outline of the this paper is as follows. In Section 2, we shall give our alternative combinatorial interpretations of $e_{n,s}^{k}$. In particular, we show how $E_{n,k}(x)$ is a hit polynomial for a particular board $B_{n,k}$ contained in the $[n] \times [n]$ board. This will allow us to derive (7) as a simple consequence of the relation between the hit numbers and rook numbers of boards contained in $[n] \times [n]$. In Section 3, we shall use rook our theory interpretation to give very simple proofs of some the recursions proved by Rakotondrajao [1] as well as prove the new recursions (8), (9), (10), and (11). In Section 4, we shall show that $e_{n,s}^{k}(q)$ satisfies natural q-analogues of (1), (2), (3), (4), (8), and (9). We shall also show how these recursions allow us to define a statistic stat(σ) such that $e^k_{n,s}(q) = \sum_{\sigma\in S_n,\mathrm{exc}_k(\sigma)=s} q^{\mathrm{stat}(\sigma)}$ and we shall derive a formula for the generating function

$$E^{(k)}(t,u,q) = \sum_{n\geq 0}\sum_{s=0}^{n} e^k_{n+k}, s(q)t^s \frac{u^n}{[n]_q!}$$

where for integers $n \geq 1$, $[n]_q = 1+q+\cdots+q^{n-1} = \frac{q^n-1}{q-1}$ and $[n]_q! = [n]_q[n-1]_q\cdots[1]_q$. Lastly in Section 5, we shall show that $\hat{e}^k_{n,s}(q)$ satisfy natural q-analogues of (5), (10) and (11).

2 Alternative combinatorial interpretations of $e^k_{n,s}$

In this section, we shall show that $e^k_{n,s}$ can be interpreted in a rook theory setting. We shall also show that for $k \geq 1$, $e^k_{n,s}$ can be interpreted as the number of permutations of S_n with s k-descents.

First we shall consider a rook theory interpretation of $e^k_{n,s}$. We label the rows of $[n]\times[n]$ from bottom to top with $1,2,\ldots,n$ and the columns from left to right with $1,2,\ldots,n$ and let (i,j) denote the square in the i-th row and j-th column. Given a board $B \subseteq [n]\times[n]$, we let $\mathcal{N}_k(B)$ denote the set of all placements of k rooks in B such that no two rooks lie in the same row or column. We shall refer to an element $P \in \mathcal{N}_k(B)$ as a *placement of k non-attacking rooks in B*. For $k = 1,\ldots,n$, we define the k-th rook number of B, $r_k(B)$, to be $r_k(B) = |\mathcal{N}_k(B)|$. Given a permutation $\sigma = \sigma_1\ldots\sigma_n \in S_n$, we identify each $\sigma \in S_n$ with the rook placement $\{(\sigma_i, i) : i = 1,\ldots,n\}$ on $[n]\times[n]$. We define the k-th hit number of B relative to n, $h_{k,n}(B)$, by setting $h_{k,n}(B) = |\{\sigma \in S_n : |\sigma\cap B| = k\}|$.

Kaplansky and Riordan [6] proved the following fundamental relationship between the rook numbers and the hit numbers of a board $B \subseteq [n]\times[n]$.

THEOREM 2.1 *For any board $B \subseteq [n]\times[n]$,*

$$\sum_{k=0}^{n} h_{k,n}(B)x^k = \sum_{k=0}^{n} r_k(B)(n-k)!(x-1)^k. \quad (12)$$

Let $B_{n,k}$ be the board contained in $[n]\times[n]$ which consists of the diagonal connecting $(1,1+k)$ and $(n-k,n)$. For example, the board $B_{7,2}$ is pictured in Figure 1.

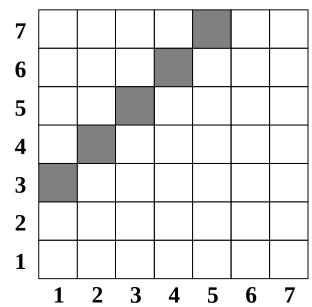

Figure 1: The board $B_{7,2}$.

It is then easy to see that the number of $\sigma \in S_n$ with s k-excedances is the s-th hit number of $B_{n,k}$, i.e. $e^k_{n,s} = h_{s,n}(B_{n,k})$. it is also clear that $r_s(B_{n,k}) = \binom{n-k}{s}$. Thus by Theorem 2.1,

$$\begin{aligned} E_{n,k}(x) &= \sum_{s=0}^{n} e^k_{n,s} x^s = \sum_{s=0}^{n} h_{s,n}(B_{n,k}) x^s \\ &= \sum_{s=0}^{n} r_s(B_{n,k})(n-s)!(x-1)^s \\ &= \sum_{s=0}^{n} \binom{n-k}{s} (n-s)!(x-1)^s. \end{aligned} \tag{13}$$

Taking the coefficient of x^s on both sides, we have the following theorem which also follows from the generating function (6) for $e^k_{n,s}$ proved by Rakotondrajao [1].

THEOREM 2.2 *For $n \geq 1$, $0 \leq k \leq n$, and $s \geq 0$,*

$$e^k_{n,s} = \sum_{t=s}^{n-k} (-1)^{t-s} (n-t)! \binom{t}{s} \binom{n-k}{t}. \tag{14}$$

PROPOSITION 2.3 *For all $n \geq k \geq 0$,*

$$e^k_{n,n-k} = k!.$$

Proof. It is easy to see that if a permutation $\sigma \in S_n$ has $\mathrm{exc}_k(\sigma) = n-k$, then $\sigma_i = i + k$ for $i = 1, \ldots, n-k$. Since the remaining values σ can be arbitrary that there are $k!$ such σ. □

Table 1 gives some small examples of the polynomials $E_{n,k}(x)$.

Table 1: Values of $E_{n,k}(x)$ for select n and k

$E_{0,0}(x) = 1$
$E_{1,0}(x) = x$
$E_{2,0}(x) = 1 + x^2$
$E_{3,0}(x) = 2 + 3x + x^3$
$E_{4,0}(x) = 9 + 8x + 6x^2 + x^4$
$E_{5,0}(x) = 44 + 45x + 20x^2 + 10x^3 + x^5$
$E_{6,0}(x) = 265 + 264x + 135x^2 + 40x^3 + 15x^4 + x^6$
$E_{7,0}(x) = 1854 + 1855x + 924x^2 + 315x^3 + 70x^4 + 21x^5 + x^7$
$E_{8,0}(x) = 14833 + 14832x + 7420x^2 + 2464x^3 + 630x^4 + 112x^5 + 28x^6 + x^8$
$E_{9,0}(x) = 133496 + 133497x + 66744x^2 + 22260x^3 + 5544x^4 + 1134x^5 + 168x^6 + 36x^7 + x^9$

$E_{1,1}(x) = 1$
$E_{2,1}(x) = 1 + x$
$E_{3,1}(x) = 3 + 2x + x^2$
$E_{4,1}(x) = 11 + 9x + 3x^2 + x^3$
$E_{5,1}(x) = 53 + 44x + 18x^2 + 4x^3 + x^4$
$E_{6,1}(x) = 309 + 265x + 110x^2 + 30x^3 + 5x^4 + x^5$
$E_{7,1}(x) = 2119 + 1854x + 795x^2 + 220x^3 + 45x^4 + 6x^5 + x^6$
$E_{8,1}(x) = 16687 + 14833x + 6489x^2 + 1855x^3 + 385x^4 + 63x^5 + 7x^6 + x^7$
$E_{9,1}(x) = 148329 + 133496x + 59332x^2 + 17304x^3 + 3710x^4 + 616x^5 + 84x^6 + 8x^7 + x^8$

$E_{1,2}(x) = 1$
$E_{2,2}(x) = 2$
$E_{3,2}(x) = 4 + 2x$
$E_{4,2}(x) = 14 + 8x + 2x^2$
$E_{5,2}(x) = 64 + 42x + 12x^2 + x^3$
$E_{6,2}(x) = 362 + 256x + 84x^2 + 16x^3 + 2x^4$
$E_{7,2}(x) = 2428 + 1810x + 640x^2 + 140x^3 + 20x^4 + 2x^5$
$E_{8,2}(x) = 18806 + 14568x + 5430x^2 + 1280x^3 + 210x^4 + 24x^5 + 2x^6$
$E_{9,2}(x) = 165016 + 131642x + 50988x^2 + 12670x^3 + 2240x^4 + 294x^5 + 28x^6 + 2x^7$

$E_{1,3}(x) = 1$
$E_{2,3}(x) = 2$
$E_{3,3}(x) = 6$
$E_{4,3}(x) = 18 + 6x$
$E_{5,3}(x) = 78 + 36x + 6x^2$
$E_{6,3}(x) = 426 + 234x + 54x^2 + 6x^3$
$E_{7,3}(x) = 2790 + 1704x + 468x^2 + 72x^3 + 6x^4$
$E_{8,3}(x) = 21234 + 13950x + 4260x^2 + 780x^3 + 90x^4 + 6x^5$
$E_{9,3}(x) = 183822 + 127404x + 41850x^2 + 8520x^3 + 1170x^4 + 108x^5 + 6x^6$

To interpret $e^k_{n,s}$ in terms of k-descents, we need only use Foata's First Transformation [3] which is a bijection $\Phi : S_n \longrightarrow S_n$ that shows that distributions of descents and excedances over S_n are equal. Foata's transformation can most easily be explained with an example.

EXAMPLE 2.4 Let $\omega = 61437258$. This permutation has three excedance pairs: $(1,6),(3,4)$, and $(5,7)$. The first step in Foata's transformation is to write ω in disjoint cycle form: (162)(34)(57)(8). Next, write each cycle with largest element last, and order the cycles by increasing largest element: (34)(216)(57)(8). Finally, to compute $\Phi(\omega)$, reverse each cycle and erase the parentheses: $\Phi(\omega) = 43612758$. In this example the descents of $\Phi(\omega)$ are $43, 61$, and 75. In general, it is not hard to see that (i,j) is a descent pair of $\Phi(\omega)$ if and only if (j,i) is an excedance of ω. To go backwards, given $\sigma = 43612758$, cut before each left-to-right maximum: 43|612|75|8, then reverse each block to get the cycles of $\Phi^{-1}(\sigma)$: (34)(216)(57)(8). Hence for each $k \geq 1$, (i,j) is a k-descent pair of $\Phi(\omega)$ if and only if (j,i) is a k-excedance pair of ω.

Thus Foata's first fundamental transformation shows for all $k,n \geq 1$,

$$\sum_{\sigma \in S_n} x^{\mathrm{des}_k(\sigma)} = \sum_{\sigma \in S_n} x^{\mathrm{exc}_k(\sigma)}. \tag{15}$$

3 Recursions for $e^k_{n,s}$

In this section, we give proofs of the recursions for $e^k_{n,s}$ that were stated in the introduction. In general, we shall see that we can give simple combinatorial proofs of such recursions if we interpret $e^k_{n,s}$ as the number of permutations of S_n with s k-excedances or as the number of placements P of n non-attacking rooks in $[n] \times [n]$ which hit the board $B_{n,k}$ in s places. However, it often seems to be much more challenging to give direct combinatorial proofs of such recursions when we interpret $e^k_{n,s}$ as the number of permutations of S_n with s k-descents.

For completeness, we shall start with a combinatorial proof the basic recursion (1). Our proof when we interpret $e^k_{n,s}$ as the number of permutations of S_n with s k-excedances is essentially the same as that given by Rakotondrajao [1]. In this case, it is also easy to give a direct combinatorial proof when we think of $e^k_{n,s}$ as the number permutations of S_n with s k-descents.

THEOREM 3.1 *For $n \geq 2$, $0 \leq k < n$, and $s \geq 1$,*

$$e_{n,s}^k = (n-s-1)e_{n-1,s}^k + (s+1)e_{n-1,s+1}^k + e_{n-1,s-1}^k. \tag{16}$$

Proof. We shall consider two different ways to insert $n+1$ into a permutation $\sigma = \sigma_1 \dots \sigma_{n-1} \in S_{n-1}$. In our first type of insertion process, when we insert n into the i-th position, we obtain the permutation

$$I_i^1(\sigma) = \sigma_1 \dots \sigma_{i-1}\ n\ \sigma_{i+1} \dots \sigma_{n-1}\sigma_i$$

if $i \leq n-1$ and

$$I_n^1(\sigma) = \sigma_1 \dots \sigma_{n-1}\ n$$

if $i = n$. Thus in our first insertion process, when we *insert n into the i-th position for $i < n$*, n replaces σ_i and bumps σ_i to the end. Our second insertion process is more standard. That is, when we insert n into the i-th position, we obtain the permutation

$$I_i^2(\sigma) = \sigma_1 \dots \sigma_{i-1}\ n\ \sigma_i \dots \sigma_{n-1}$$

if $i < n$ and

$$I_{n+1}^2(\sigma) = \sigma_1 \dots \sigma_{n-1}\ n$$

if $i = n$. Thus in our second insertion process, when we *insert n into the i-th position for $i < n$*, n is inserted immediately in front of σ_i.

Given our first insertion process, it is easy to see that insertion of n can cause one extra k-excedance if we insert n into position $n-k$, we can decrease the excedances by 1 if we insert n into position i where $\sigma_i - i = k$, and we leave the number of k-excedances fixed otherwise. Thus it easily follows that

$$e_{n,s}^k = (n-s-1)e_{n-1,s}^k + (s+1)e_{n-1,s+1}^k + e_{n-1,s-1}^k$$

for all $n \geq 2$, $k \geq 0$, and $s \geq 1$.

Similarly we can use the second insertion process to do the same thing where we interpret $e_{n,s}^k$ as the number of permutations with s k-descents when $k \geq 1$. That is, we can create an extra k-descent if we insert n immediately in front of $n-k$, we can lose a k-descent if we insert n immediately in front of σ_{i+1} where $\sigma_i - \sigma_{i+1} = k$, and we keep the number of k-descents fixed otherwise. □

Next we give a bijective proof of recursion (2). It this case, we get a complete contrast with the proofs of recursion (1). That is, there is a trivial proof of (2) when we interpret $e^k_{n,s}$ as the number of placements P of n non-attacking rooks in $[n] \times [n]$ which hit the board $B_{n,k}$ in s places. The first author [5] found a direct combinatorial of (2) when one interprets $e^k_{n,s}$ as the number of permutations of S_n with s k-descents. However it is considerably more complicated than the direct proof of (2) given below so we will not include it in this paper.

Theorem 3.2 *For all $n \geq 1$, $0 \leq k < n$, and $s \geq 0$,*

$$e^k_{n,s} = \binom{n-k}{s} e^k_{n-s,0}. \tag{17}$$

Proof. The easiest way to give a combinatorial proof of this theorem is to think of the rook board $B_{n,k}$ introduced in Section 2. Thus $e^k_{n,s}$ is the number of placements of n non-attacking rooks in $[n] \times [n]$ that intersect $B_{n,k}$ in exactly s squares. If we then remove the rows and columns of those rooks which hit the diagonal connecting $(1, 1+k)$ and $(n-k, n)$, it easy to see that we will reduce ourselves to a permutation that is counted by $e^k_{n-s,0}$. This process is pictured in Figure 2.

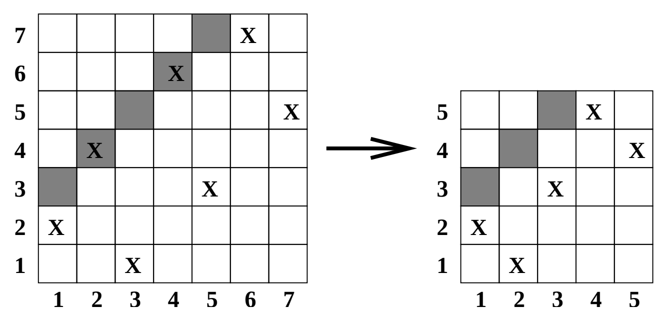

Figure 2: Removing rooks that lie in $B_{n,k}$.

□

The rook theory point of view also allows for a simple proof of recursion (3).

Theorem 3.3 *For all $n \geq 2$ and $0 \leq k < n$,*

$$e^k_{n,0} = (n-1)e^k_{n-1,0} + (n-1-k)e^k_{n-2,0}. \tag{18}$$

Proof. Think of reversing the first insertion process described in Theorem 3.1. That is, given a permutation $\sigma = \sigma_1 \ldots \sigma_n \in S_n$, we let

$$\bar{\sigma} = \sigma_1 \ldots \sigma_{i-1} \; \sigma_n \; \sigma_{i+1} \ldots \sigma_{n-1}$$

if $\sigma_i = n$ for $i < n$ and let

$$\bar{\sigma} = \sigma_1 \ldots \sigma_{n-1}$$

if $\sigma_n = n$. There are now two cases.

Case 1 $\bar{\sigma}$ had no k-excedances.
Note $\bar{\sigma}$ will have no k-excedances if either $\sigma_n = n$ or $\sigma_i = n$ with $i < n$ and $\sigma_n \neq i + k$. Vice versa, given a $\tau \in S_{n-1}$ with no k-excedances, we can use the first insertion process and insert n into τ in every position except $n - k$ to obtain a permutation in S_n with no k-excedances. Thus there are a $(n-1)e^k_{n-1,0}$ permutations in case 1.

Case 2 $\bar{\sigma}$ has 1 k-excedance.
This happens only if there is an $i \leq n-1-k$ such that $\sigma_i = n$ and $\sigma_n = i+k$. However in this case, we let $\tilde{\sigma}$ denote the permutation of S_{n-2} which results by removing the i-column and $i+k$-th row from the rook placement that corresponds to $\bar{\sigma}$ in $B_{n-1,k}$ as we did in the proof of Theorem 3.2. It is easy to see that $\tilde{\sigma}$ has no k-excedances. Vice versa, it is easy to see that we can start with any permutation $\tilde{\tau} \in S_{n-2}$ that has no k-excedances and $i \leq n-1-k$ and obtain a permutation $\bar{\tau} \in S_{n-1}$ so that its corresponding rook placement has a rook in column i and row $i+k$ and reduces to the rook placement corresponding to $\tilde{\tau}$ when we remove the i-th and $i+k$-th row. Then we can obtain a $\sigma \in S_n$ with no k-excedances by inserting n into position i in $\bar{\tau}$ and moving $i+k$ to the end. Thus there are $(n-1-k)e^k_{n-2,0}$ permutations in Case 2. $\square$

We note that Theorem 3.3 is a nice generalization of the basic theorem for the number of derangements. That is, $e^0_{n,0} = D_n$ where D_n is the number of derangements of S_n and (18) reduces to

$$D_n = (n-1)D_{n-1} + (n-1)D_{n-2}. \tag{19}$$

In fact, a moments thought will convince one that our proof reduces to the usual proof of this recursion in this case. That is, the permutations in Case 2 are the derangements where n is in 2-cycle and the permutations in Case 1 are derangements when n is not in a 2-cycle.

As pointed out in the introduction, our previous three recursion were proved by Rakotondrajao [1]. Our next three recursions are new.

THEOREM 3.4 *For all $n \geq 2$ and $0 \leq k < n$,*

$$e^k_{n,0} = ke^{k-1}_{n-1,0} + (n-k)e^k_{n-1,0}. \tag{20}$$

Proof. This recursion is simply the result of classifying the rook placements corresponding to permutations $\sigma \in S_n$ with no k-excedances by the position of the rook in row 1. That is, there are two cases.

Case 1. The rook in the bottom row lies in column i where $i \leq n-k$.
In this case, we consider the rook placement that results by removing column i and row $i+k$. This will result in a placement of $n-2$ non-attacking rooks such that no rook lies in $B_{n-1,k}$. We need to add a rook in bottom row and we do this so that the resulting rook placement is non-attacking. This will leave us with a rook placement of $n-1$ rooks in $[n-1]\times[n-1]$ that does not intersect $B_{n-1,k}$. Hence there are $(n-k)e^k_{n-1,0}$ such placements. This type of reduction is pictured at the top of Figure 3.

Case 2. The rook in the bottom row lies in column i where $i > n-k$.
In this case, removing row 1 and column i will result in a rook placement that does not intersect $B_{n-1,k-1}$. Thus there are $ke^{k-1}_{n-1,0}$ such rook placements. This type of reduction is pictured at the bottom of Figure 3

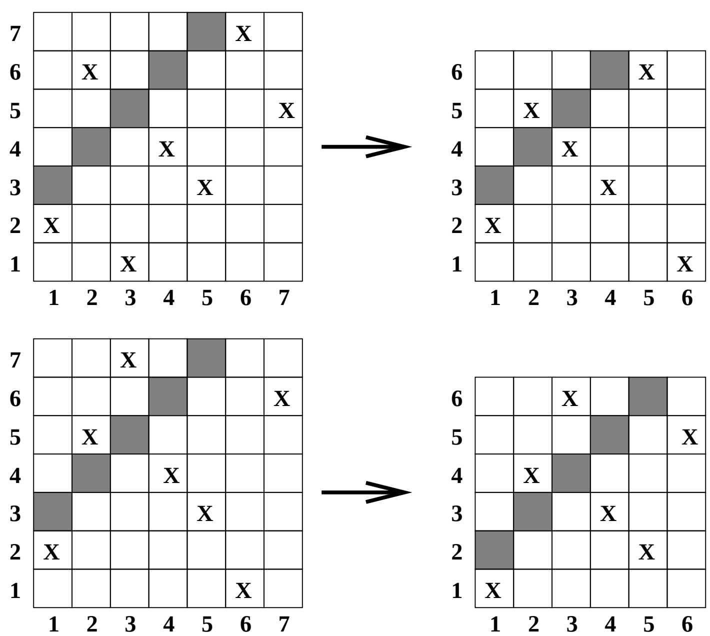

Figure 3: Reducing rook placements corresponding to $e^k_{n,0}$ by the position of the rook in row 1.

□

THEOREM 3.5 *For all $n \geq 1$ and $0 \leq k < n$,*

$$e_{n,0}^{k} = k! \sum_{r=0}^{k} \binom{k}{r} \binom{n-k}{k-r} e_{n-k,0}^{k-r}. \tag{21}$$

Proof. Note that $e_{n,0}^{k}$ the number of placements of n non-attacking rooks on the $n \times n$ grid that never hit $B_{n,k}$.

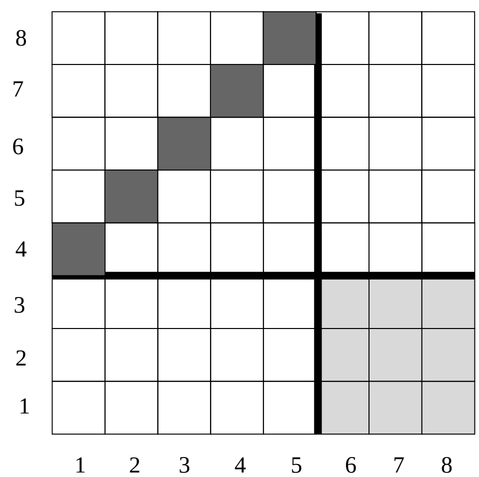

Figure 4: The board, $B_{8,3}$, with some lightly shaded cells.

Consider the lightly shaded cells in the lower right hand corner of the board shown in Figure 4. There can be anywhere from 0 to k rooks placed in this square area. Suppose that we choose to place r rooks in this area. First we choose the r rows which will contain the rooks in this area in $\binom{k}{r}$ ways. Since there must be k rooks in the last k columns, there must be $k-r$ rooks in the rectangular region above the lightly shaded cells and we can choose the $k-r$ rows which will contain these rooks in $\binom{n-k}{k-r}$ ways. Having picked the k rows that contain the rooks in the last k columns, there are $k!$ ways to place the rooks in the last k columns. Thus there $k!\binom{k}{r}\binom{n-k}{k-r}$ ways to pick a placement P of k non-attacking rooks in the last k columns so that r rooks fall in the lightly shaded area. Finally, we must count the number of ways to extend such a placement P to a placement Q of n non-attacking rooks in $[n] \times [n]$ so that no rook lies in $B_{n,k}$. If one thinks of removing the rows and columns of the rooks in P, it is easy to see that we are left with the board $B_{n-k,k-r}$ so that are $e_{n-k,0}^{k-r}$ ways to pick Q. Summing over all possible values of r yields the result. □

THEOREM 3.6 *For all $n \geq 2$ and $0 \leq k < n$,*

$$e_{n,0}^{k} = e_{n,1}^{k+1} + ke_{n-1,0}^{k}. \tag{22}$$

Proof. If $\sigma \in S_n$ has no k-excedances, then its corresponding rook placement in the $[n] \times [n]$ board does not intersect $B_{n,k}$ so there can not be a rook in the top row in cell $n-k$. We will classify these permutations by the location of the rook in the top row.

Case 1. The rook in the top row is in cell i where $n-k+1 \leq i \leq n$. Removing the row and column containing the top rook from this rook placement leaves us with a placement on the $[n-1] \times [n-1]$ board that does not intersect $B_{n-1,k}$. Thus there are $ke_{n-1,0}^{k}$ such permutations in this case. This type of reduction is pictured at the top of Figure 5.

Case 2. The rook in the top row is in cell i where $1 \leq i \leq n-k-1$. If move the top row to the bottom of this rook placement and then swap this new bottom row with row $i+k+1$, we are left with a placement on the $[n] \times [n]$ board that intersects $B_{n,k+1}$ in exactly one place, namely in cell $(i, i+k+1)$. Thus there are $e_{n,1}^{k+1}$ such permutations in Case 2. This type of reduction is pictured at the bottom of Figure 5. □

Next we consider another recursion proved by Rakotondrajao [1]. In this case, we give a rook theory proof which is much simpler than Rakotondrajao's original proof.

THEOREM 3.7 *For all $n \geq 2$ and $1 \leq k < n$,*

$$e_{n,0}^{k} = e_{n,0}^{k-1} + e_{n-1,0}^{k-1}. \tag{23}$$

Proof. If $\sigma \in S_n$ has no k-excedances, then its corresponding rook placement in the $[n] \times [n]$ board does not intersect $B_{n,k}$. Now consider the rook placement that results by taking the rook placement for σ and moving the bottom row to the top and all the other rows down by one as is pictured in Figure 6.

We see that there are two cases.

Case 1. The bottom rook is not in position $n-k+1$. This case is pictured at the top of Figure 6. In this case, notice that the resulting rook placement does not intersect $B_{n,k-1}$. Thus there are $e_{n,0}^{k-1}$ such permutations in this case.

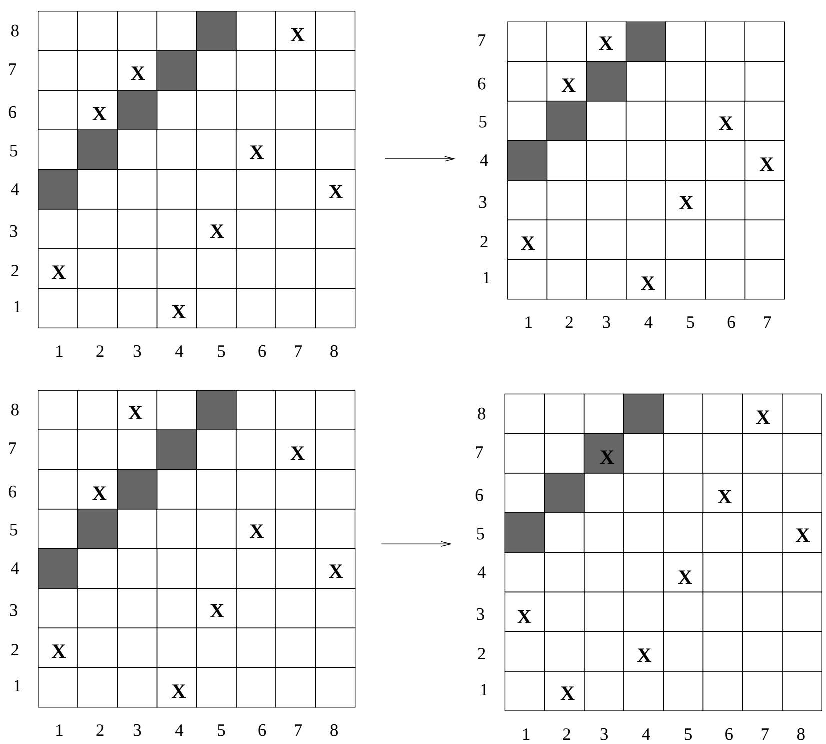

Figure 5: Reducing rook placements corresponding to $e^k_{n,0}$ by the position of the rook in top row.

Case 2. The bottom rook is in position $n-k+1$.
This case is pictured at the bottom of Figure 6. In this case, the resulting rook placement intersect $B_{n,k-1}$ in column $n-k+1$ and row n. However if we remove that row and column, then we end up with a rook placement in $[n-1]\times[n-1]$ that does not intersect $B_{n-1,k-1}$. Thus there are $e^{k-1}_{n-1,0}$ such permutations in Case 2. □

Next we shall show that iterating the recursion (21) gives us an expression for $e^k_{n,0}$ entirely in terms of derangement numbers D_n.

THEOREM 3.8 *For all $n \geq 1$ and $0 \leq k < n$,*

$$e^k_{n,0} = \sum_{r=0}^{k} \binom{k}{r} D_{n-k+r}. \tag{24}$$

Proof. We can prove this theorem using induction on k. When $k=0$ and $n \geq 0$, it is clear that $e^0_{n,0} = D_n$. Now assume that $e^i_{n,0} = \sum_{r=0}^{i} \binom{i}{r} D_{n-i+r}$,

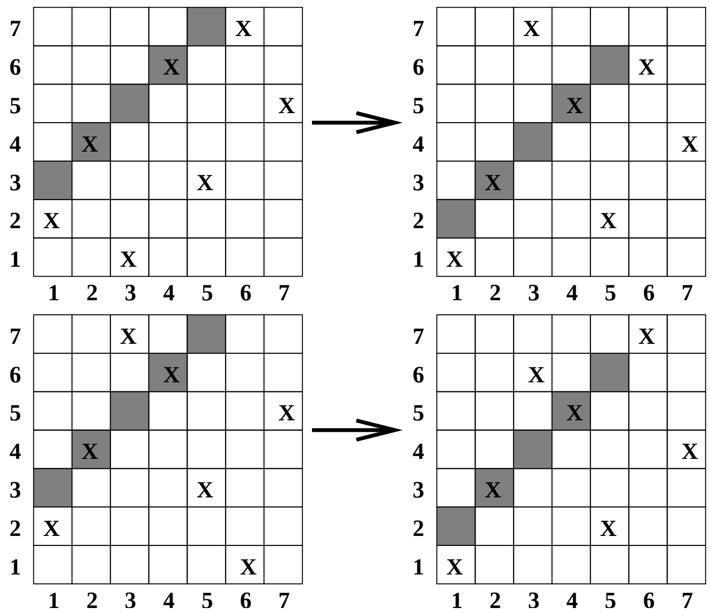

Figure 6: Moving the bottom row to the top for $B_{n,k}$.

for all $n \geq 0$ and all $i < k$. Our goal is now to show that $e^k_{n,0} = \sum_{r=0}^{k} \binom{k}{r} D_{n-k+r}$. Firstly, using (23) we have that

$$e^k_{n,0} = e^{k-1}_{n,0} + e^{k-1}_{n-1,0}.$$

Thus by induction we have that

$$\begin{aligned} e^k_{n,0} &= \sum_{r=0}^{k-1} \binom{k-1}{r} D_{n-(k-1)+r} + \sum_{r=0}^{k-1} \binom{k-1}{r} D_{(n-1)-(k-1)+r} \\ &= \sum_{r=0}^{k-1} \binom{k-1}{r} D_{n+1-k+r} + \sum_{r=0}^{k-1} \binom{k-1}{r} D_{n-k+r}. \end{aligned}$$

Then, peeling off the $k-1$ term of the first sum and the 0 term of the second sum gives

$$\begin{aligned}
&\binom{k-1}{k-1}D_n + \sum_{r=0}^{k-2}\binom{k-1}{r}D_{n+1-k+r} \\
&+\binom{k-1}{0}D_{n-k} + \sum_{r=1}^{k-1}\binom{k-1}{r}D_{n-k+r} \\
&=\binom{k}{k}D_n + \sum_{r=1}^{k-1}\left(\binom{k-1}{r-1}+\binom{k-1}{r}\right)D_{n-k+r} + \binom{k}{0}D_{n-k} \\
&=\binom{k}{k}D_n + \sum_{r=1}^{k-1}\binom{k}{r}D_{n-k+r} + \binom{k}{0}D_{n-k} \\
&=\sum_{r=0}^{k}\binom{k}{r}D_{n-k+r},
\end{aligned}$$

which was to be shown. □

We note that the task of computing an arbitrary $e^k_{n,s}$, can be reduced to computing $e^k_{n-s,0}$ using Theorem 3.2. Then, the task of computing $e^k_{n-s,0}$ can be reduced to computing $D_{n-s-k+r}$ for specific values of r using Theorem 3.8. Thus the task of computing $e^k_{n,s}$ can be obtained by the following sum expressed entirely in terms of the number of derangements for various r.

Corollary 3.9 *For $n \geq 1$, $0 \leq k < n$ and $s \geq 0$,*

$$e^k_{n,s} = \binom{n-k}{s}\sum_{r=0}^{k}\binom{k}{r}D_{n-s-k+r}. \tag{25}$$

We end this section by explaining some patterns in the polynomials $E_{n,k}(x)$. For example, Table 1 suggests that for $k \geq 1$, $e^k_{n,s}$ decreases as s increases. This is true and can be easily proved as follows. By recursion (2), have that for $s \geq 0$,

$$\begin{aligned}
e^k_{n,s} &= \binom{n-k}{s}e^k_{n-s,0} \text{ and} \\
e^k_{n,s-1} &= \binom{n-k}{s-1}e^k_{n-s+1,0}.
\end{aligned}$$

By recursion (8), we have that

$$e^k_{n-s+1,0} = ke^{k-1}_{n-s,0} + (n-k-s+1)e^k_{n-s,0}.$$

Thus it follows that

$$\begin{aligned} & e^k_{n,s-1} - e^k_{n,s} \\ &= \left((n-k-s+1)\binom{n-k}{s-1} - \binom{n-k}{s} \right) e^k_{n-s,0} + k\binom{n-k}{s-1} e^{k-1}_{n-s,0}. \quad (26) \end{aligned}$$

However for $s \geq 1$, $(n-k-s+1)\binom{n-k}{s-1} - \binom{n-k}{s} = \frac{n!}{(s-1)!(n-k-s)!}(1-\frac{1}{s}) \geq 0$. Thus $e^k_{n,s-1} > e^k_{n,s}$.

Similarly, Table 1 suggests $e^0_{n,s}$ decreases as s increases for $2 \leq s \leq n-2$. This is also easily proved. That is, by recursion (2), we have that for $s \geq 1$,

$$\begin{aligned} e^0_{n,s} &= \binom{n}{s} e^0_{n-s,0} = D_{n-s} \text{ and} \\ e^0_{n,s-1} &= \binom{n}{s-1} e^0_{n-s+1,0} = D_{n-s+1}. \end{aligned}$$

But $D_{n-s+1} = (n-s+1)D_{n-s} + (-1)^{n-s+1}$ so that for $s \geq 1$,

$$\begin{aligned} e^0_{n,s-1} - e^0_{n,s} &= \left((n-s+1)\binom{n}{s-1} - \binom{n}{s} \right) D_{n-s} + (-1)^{n-s+1}\binom{n}{s-1} \\ &= \binom{n}{s-1}\left((n-s+1)(1-\frac{1}{s})D_{n-s} + (-1)^{n-s+1} \right). \quad (27) \end{aligned}$$

Clearly, when $s = 1$, $e^0_{n,s-1} - e^0_{n,s} = (-1)^n$ which explains the patterns of the coefficients of x^0 and x^1 in the polynomials $E_{n,0}(x)$ in Table 1. But for $n \geq 3$ and $2 \leq s \leq n-2$, $n-s+1 \geq 3$ and $1 - \frac{1}{s} \geq \frac{1}{2}$. Thus $(n-s+1)(1-\frac{1}{s})D_{n-s} > 1$ and hence, the right hand side of (27) is positive. Finally, we should observe that it is easy to see that $e^0_{n,n} = 1$ since the only permutation $\sigma \in S_n$ with n fixed points is the identity permutation $\sigma = 12\ldots n$, $e^0_{n,n-1} = 0$ because there are no $\sigma \in S_n$ with exactly $n-1$ fixed points, and $e^0_{n,n-2} = \binom{n}{2}$ since any permutation $\sigma \in S_n$ with $n-2$ fixed points consists of a 2-cycle plus $n-2$ 1-cycles and we clearly have $\binom{n}{2}$ ways to pick the 2-cycle.

4 Two q-Analogues of $e^k_{n,s}$

In this section, we shall define a pair of closely related q-analogues of $e^k_{n,s}$. Our basic starting point is a q-analogue of our expression for $e^k_{n,s}$ given in (14).

For $n \geq 1$, $0 \leq k < n$ and $s \geq 0$, define

$$e_{n,s}^{k}(q) := \sum_{t=s}^{n-k} (-1)^{t-s} q^{\binom{s}{2}} [n-t]_q! \begin{bmatrix} t \\ s \end{bmatrix}_q \begin{bmatrix} n-k \\ t \end{bmatrix}_q \tag{28}$$

and

$$\overline{e}_{n,s}^{k}(q) := \sum_{t=s}^{n-k} (-1)^{t-s} q^{\binom{s+1}{2}+\binom{t}{2}+t(k-s)} [n-t]_q! \begin{bmatrix} t \\ s \end{bmatrix}_q \begin{bmatrix} n-k \\ t \end{bmatrix}_q. \tag{29}$$

If $k \geq n$, then we simply define

$$e_{n,s}^{k}(q) = \overline{e}_{n,s}^{k}(q) := \begin{cases} 0, & \text{if } s > 0 \\ [n]_q!, & \text{if } s = 0 \end{cases}.$$

It is also simple to verify that

$$e_{n,s}^{k}(q) = q^{\binom{n}{2}} \overline{e}_{n,s}^{k}(q^{-1}). \tag{30}$$

We let

$$E_{n,k}(x,q) = \sum_{s=0}^{n-k} e_{n,s}^{k}(q) x^s. \tag{31}$$

Our goal in this section is to show that $e_{n,s}^{k}(q)$'s satisfy natural q-analogues of many of the recursions given in the previous section. We can then use (30) to prove similar formulas involving $\overline{e}_{n,s}^{k}(q)$. We will present the corresponding formulas for $\overline{e}_{n,s}^{k}(q)$ in Table 2.

The proofs in this section will be mainly algebraic. We will present the q-analogues in a different order than the previous section where $q = 1$. This is because the proofs in this section build on each other in a required order.

THEOREM 4.1 *For $n \geq 2$ and $0 \leq k < n$,*

$$e_{n,0}^{k}(q) = q [n-1]_q \, e_{n-1,0}^{k}(q) + [n-1-k]_q \, e_{n-2,0}^{k}(q). \tag{32}$$

Proof. By our definition of $e_{n,s}^{k}(q)$, the RHS of (32) equals

$$q [n-1]_q \sum_{t=0}^{n-k-1} (-1)^t [n-t-1]_q! \begin{bmatrix} n-k-1 \\ t \end{bmatrix}_q +$$
$$[n-1-k]_q \sum_{t=0}^{n-k-2} (-1)^t [n-t-2]_q! \begin{bmatrix} n-k-2 \\ t \end{bmatrix}_q.$$

The $t = 0$ and $t = 1$ terms from the first sum contribute

$$\begin{aligned}
& q\,[n-1]_q\,[n-1]_q! - q\,[n-1]_q\,[n-2]_q! \begin{bmatrix} n-k-1 \\ 1 \end{bmatrix}_q \\
= & \left([n]_q - [1]_q\right)[n-1]_q! - q\,[n-1]_q! \begin{bmatrix} n-k-1 \\ 1 \end{bmatrix}_q \\
= & [n]_q! - [n-1]_q!(1 + q \begin{bmatrix} n-k-1 \\ 1 \end{bmatrix}_q) \\
= & [n]_q! - [n-1]_q! \begin{bmatrix} n-k \\ 1 \end{bmatrix}_q.
\end{aligned}$$

The LHS of (32) equals

$$\begin{aligned}
& \sum_{t=0}^{n-k} (-1)^t\,[n-t]_q! \begin{bmatrix} n-k \\ t \end{bmatrix}_q \\
= & [n]_q! - [n-1]_q! \begin{bmatrix} n-k \\ 1 \end{bmatrix}_q + \sum_{t=2}^{n-k} (-1)^t\,[n-t]_q! \begin{bmatrix} n-k \\ t \end{bmatrix}_q \\
= & [n]_q! - [n-1]_q! \begin{bmatrix} n-k \\ 1 \end{bmatrix}_q + \sum_{t=0}^{n-k-2} (-1)^t\,[n-2-t]_q! \begin{bmatrix} n-k \\ t+2 \end{bmatrix}_q.
\end{aligned}$$

Thus, to verify the theorem, it remains to show that

$$\begin{aligned}
& q\,[n-1]_q \sum_{t=2}^{n-k-1} (-1)^t\,[n-t-1]_q! \begin{bmatrix} n-k-1 \\ t \end{bmatrix}_q + \\
& [n-1-k]_q \sum_{t=0}^{n-k-2} (-1)^t\,[n-t-2]_q! \begin{bmatrix} n-k-2 \\ t \end{bmatrix}_q \\
= & \sum_{t=0}^{n-k-2} (-1)^t\,[n-2-t]_q! \begin{bmatrix} n-k \\ t+2 \end{bmatrix}_q.
\end{aligned}$$

The LHS of the above becomes

$$
\begin{aligned}
& q\,[n-1]_q \sum_{t=0}^{n-k-3} (-1)^t\,[n-t-3]_q! \begin{bmatrix} n-k-1 \\ t+2 \end{bmatrix}_q + \\
& \sum_{t=0}^{n-k-2} (-1)^t\,[n-t-2]_q! \begin{bmatrix} n-k-1 \\ t+1 \end{bmatrix}_q [t+1]_q \\
= \; & \sum_{t=0}^{n-k-3} (-1)^t\,[n-t-3]_q! \begin{bmatrix} n-k-1 \\ t+2 \end{bmatrix}_q (q\,[t+1]_q + q^{t+2}\,[n-2-t]_q) + \\
& \sum_{t=0}^{n-k-2} (-1)^t\,[n-t-2]_q! \begin{bmatrix} n-k-1 \\ t+1 \end{bmatrix}_q (1+q\,[t]_q).
\end{aligned}
$$

Writing this as four separate sums gives

$$
\begin{aligned}
& \sum_{t=0}^{n-k-3} (-1)^t\,[n-t-3]_q! \begin{bmatrix} n-k-1 \\ t+2 \end{bmatrix}_q q\,[t+1]_q + \\
& \sum_{t=0}^{n-k-3} (-1)^t\,[n-t-2]_q! \begin{bmatrix} n-k-1 \\ t+2 \end{bmatrix}_q q^{t+2} + \\
& \sum_{t=0}^{n-k-2} (-1)^t\,[n-t-2]_q! \begin{bmatrix} n-k-1 \\ t+1 \end{bmatrix}_q + \\
& \sum_{t=1}^{n-k-2} (-1)^t\,[n-t-2]_q! \begin{bmatrix} n-k-1 \\ t+1 \end{bmatrix}_q q\,[t]_q\,.
\end{aligned}
$$

Notice that the first sum and the fourth sum are off by a factor of -1 and thus they cancel. Combining the remaining two sums gives

$$\begin{aligned}
&\sum_{t=0}^{n-k-3}(-1)^t\,[n-t-2]_q!q^{t+2}\begin{bmatrix}n-k-1\\t+2\end{bmatrix}_q+\\
&(-1)^{n-2-k}\,[k]_q!+\sum_{t=0}^{n-k-3}(-1)^t\,[n-t-2]_q!\begin{bmatrix}n-k-1\\t+1\end{bmatrix}_q\\
=&(-1)^{n-k}\,[k]_q!+\sum_{t=0}^{n-k-3}(-1)^t\,[n-t-2]_q!\left(q^{t+2}\begin{bmatrix}n-k-1\\t+2\end{bmatrix}_q+\begin{bmatrix}n-k-1\\t+1\end{bmatrix}_q\right)\\
=&(-1)^{n-k}\,[k]_q!+\sum_{t=0}^{n-k-3}(-1)^t\,[n-t-2]_q!\begin{bmatrix}n-k\\t+2\end{bmatrix}_q\\
=&\sum_{t=0}^{n-k-2}(-1)^t\,[n-t-2]_q!\begin{bmatrix}n-k\\t+2\end{bmatrix}_q.
\end{aligned}$$

This is what needed to be shown. □

THEOREM 4.2 *For $n \geq 1$, $0 \leq k < n$ and $s \geq 0$,*

$$e^k_{n,s}(q) = q^{\binom{s}{2}}\begin{bmatrix}n-k\\s\end{bmatrix}_q e^k_{n-s,0}(q). \tag{33}$$

Proof. By definition, $e^k_{n,s}(q)$ equals

$$\begin{aligned}
&\sum_{t=s}^{n-k}(-1)^{t-s}q^{\binom{s}{2}}\,[n-t]_q!\begin{bmatrix}t\\s\end{bmatrix}_q\begin{bmatrix}n-k\\t\end{bmatrix}_q\\
=\;& q^{\binom{s}{2}}\sum_{t=s}^{n-k}(-1)^{t-s}\,[n-t]_q!\frac{[t]_q!}{[s]_q!\,[t-s]_q!}\frac{[n-k]_q!}{[t]_q!\,[n-k-t]_q!}\\
=\;& q^{\binom{s}{2}}\begin{bmatrix}n-k\\s\end{bmatrix}_q\sum_{t=s}^{n-k}(-1)^{t-s}\,[n-t]_q!\begin{bmatrix}n-k-s\\t-s\end{bmatrix}_q\\
=\;& q^{\binom{s}{2}}\begin{bmatrix}n-k\\s\end{bmatrix}_q\sum_{t=0}^{n-s-k}(-1)^{t}\,[n-s-t]_q!\begin{bmatrix}n-k-s\\t\end{bmatrix}_q\\
=\;& q^{\binom{s}{2}}\begin{bmatrix}n-k\\s\end{bmatrix}_q e^k_{n-s,0}(q).
\end{aligned}$$

□

THEOREM 4.3 *For $n \geq 2$, $0 \leq k < n$, and $s \geq 1$,*

$$e^k_{n,s}(q) = q^{s+1}\,[n-s-1]_q\, e^k_{n-1,s}(q) + [s+1]_q\, e^k_{n-1,s+1}(q) + q^{s-1} e^k_{n-1,s-1}(q). \tag{34}$$

Proof. We can use Theorem 4.2 to show that the LHS of (34) is

$$q^{\binom{s}{2}} \begin{bmatrix} n-k \\ s \end{bmatrix}_q e^k_{n-s,0}(q).$$

To prove the theorem, we will now show that the RHS of (34) divided by $q^{\binom{s}{2}}$ is equal to $\begin{bmatrix} n-k \\ s \end{bmatrix}_q e^k_{n-s,0}(q)$. Applying Theorem 4.2 to the RHS of (34) and dividing by $q^{\binom{s}{2}}$ yields

$$\begin{aligned}
&q^{s+1}\,[n-s-1]_q \begin{bmatrix} n-1-k \\ s \end{bmatrix}_q e^k_{n-1-s,0}(q) + \\
&q^{s+1}\,[s+1]_q \begin{bmatrix} n-1-k \\ s+1 \end{bmatrix}_q e^k_{n-s-2,0}(q) + \begin{bmatrix} n-1-k \\ s-1 \end{bmatrix}_q e^k_{n-s,0}(q) \\
=&q^s \begin{bmatrix} n-1-k \\ s \end{bmatrix}_q \Big(q\,[n-s-1]_q\, e^k_{n-1-s,0}(q) + [n-s-1-k]_q\, e^k_{n-s-2,0}(q) \Big) \\
&+ \begin{bmatrix} n-1-k \\ s-1 \end{bmatrix}_q e^k_{n-s,0}(q).
\end{aligned}$$

We can now apply Theorem 4.1 to the expression in parentheses to obtain

$$\begin{aligned}
& q^s \begin{bmatrix} n-1-k \\ s \end{bmatrix}_q e^k_{n-s,0}(q) + \begin{bmatrix} n-1-k \\ s-1 \end{bmatrix}_q e^k_{n-s,0}(q) \\
= & \left(q^s \begin{bmatrix} n-1-k \\ s \end{bmatrix}_q + \begin{bmatrix} n-1-k \\ s-1 \end{bmatrix}_q \right) e^k_{n-s,0} \\
= & \begin{bmatrix} n-k \\ s \end{bmatrix}_q e^k_{n-s,0},
\end{aligned}$$

which verifies the theorem. □

THEOREM 4.4 *For $n \geq 2$ and $0 < k < n$,*

$$e^k_{n,0}(q) = [k]_q\, e^{k-1}_{n-1,0}(q) + q^k\,[n-k]_q\, e^k_{n-1,0}(q). \tag{35}$$

Proof. Note that $\frac{[k]_q+q^k[n-k-t]_q}{[n-t]_q} = 1$. We will use this fact in the second step of the proof. First we have that $e^k_{n,0}(q)$ equals

$$\begin{aligned}
& \sum_{t=0}^{n-k}(-1)^t\,[n-t]_q!\begin{bmatrix} n-k \\ t \end{bmatrix}_q \\
= & \sum_{t=0}^{n-k}(-1)^t\,[n-t]_q!\begin{bmatrix} n-k \\ t \end{bmatrix}_q \left(\frac{[k]_q}{[n-t]_q} + \frac{q^k\,[n-k-t]_q}{[n-t]_q}\right) \\
= & [k]_q \sum_{t=0}^{n-k}(-1)^t\,[n-t-1]_q!\begin{bmatrix} n-k \\ t \end{bmatrix}_q \\
& +q^k\,[n-k]_q \sum_{t=0}^{n-k-1}(-1)^t\,[n-t-1]_q!\begin{bmatrix} n-k-1 \\ t \end{bmatrix}_q \\
= & [k]_q\, e^{k-1}_{n-1,0}(q) + q^k\,[n-k]_q\, e^k_{n-1,0}(q).
\end{aligned}$$

□

Theorem 4.4 gives us the following interesting result.

THEOREM 4.5 *For $n \geq 1$ and $0 < k < n$,*

$$e^k_{n,0}(q) = [k]_q! \sum_{r=0}^{k} q^{(k-r)^2} \begin{bmatrix} k \\ r \end{bmatrix}_q \begin{bmatrix} n-k \\ k-r \end{bmatrix}_q e^{k-r}_{n-k,0}(q). \qquad (36)$$

Proof. To prove this fact, we just need to apply the recursion from Theorem 4.4 to $e^k_{n,0}(q)$ a total of k times. Consider how we would get a term of the form $e^{k-r}_{n-k,0}(q)$ for some r satisfying $0 \leq r \leq k$. We can track terms that come out of the recursion by whether it came out of the first term of the recursion or the second. In this situation, we needed to use the first term of the recursion a total of r times and the second term of the recursion a total of $k-r$ times. Thus, we will obtain a factor of $[k]_q \cdot [k-1]_q \cdots [k-r+1]_q$ from the first terms and a factor of $[n-k]_q \cdot [n-k-1]_q \cdots [n-2k-r+1]_q$ from the second. Multiplying these together gives a factor of $[k]_q! \begin{bmatrix} n-k \\ k-r \end{bmatrix}_q$. We interpret the $\begin{bmatrix} k \\ r \end{bmatrix}_q$ as choosing the r times we use the first term of the recursion from the total of k steps and record this as a rearrangement of r 1's and $k-r$ 0's and we count q raised to the power of the number of coinversions in the rearrangement. Notice that each coinversion means that we used the second term of the recursion before the first at some other step. Thus, the factor of q that comes along when using the second term will be one higher. The smallest possible factor of q that you could obtain would be by using the first term in the recursion r times consecutively and then using the second term in the recursion $k-r$ times consecutively and this would yield $q^{(k-r)^2}$. Thus, we have accounted for every term in the formula. □

THEOREM 4.6 *For $n \geq 2$,*

$$e^0_{n,0}(q) = [n]_q\, e^0_{n-1,0}(q) + (-1)^n. \tag{37}$$

Proof.

$$\begin{aligned} e^0_{n,0}(q) &= \sum_{t=0}^{n} (-1)^t\, [n-t]_q! \begin{bmatrix} n \\ t \end{bmatrix}_q \\ &= [n]_q! \sum_{t=0}^{n} \frac{(-1)^t}{[t]_q!} \\ &= [n]_q \left([n-1]_q! \sum_{t=0}^{n-1} \frac{(-1)^t}{[t]_q!} + [n-1]_q! \frac{(-1)^n}{[n]_q!} \right) \\ &= [n]_q\, e^0_{n-1,0}(q) + (-1)^n. \end{aligned}$$

□

Using (30), we can get q-analogues of $\overline{e}^k_{n,s}(q)$ which are presented in Table 2.

We note that Theorem 4.6 shows that $e^0_{n,0}(q)$ satisfies a natural q-analogue of a recursion for the number of derangements and (37) in Table 2 shows that $\overline{e}^0_{n,0}(q)$ satisfies a second q-analogue of the recursion for the number of derangements. Garsia and Remmel [4] introduced two such q-analogues of the derangement numbers D_n as follows. Given a $\sigma \in \mathbb{D}_n$, arrange the cycles of σ so that the second smallest element in each cycle is on the right and the cycles are ordered from left to right by increasing second smallest elements. Briggs and Remmel [2] refer to such an arrangement of cycles of $\sigma \in \mathbb{D}_n$ as the 1-*standard order* of σ. For example, $\sigma = (3,1,11,2)(10,4,5)(9,8,12,6,13,7)$ is in 1-standard order. Having written σ in 1-standard order, Garsia and Remmel then set $\overline{\sigma}$ to be the permutation in one line notation that results from the 1-standard order of σ by erasing the parentheses and commas. Thus in our case,

Table 2: Corresponding q-analogues for $\overline{e}^k_{n,s}(q)$

Equation	Corresponding q-analogue for $\overline{e}$
(32)	$\overline{e}^k_{n,0}(q) = [n-1]_q\, \overline{e}^k_{n-1,0}(q) + q^{n+k-1}\,[n-1-k]_q\, \overline{e}^k_{n-2,0}(q)$
(33)	$\overline{e}^k_{n,s}(q) = q^{ks} \genfrac{}{}{0pt}{}{n-k}{s}_q \overline{e}^k_{n-s,0}(q)$
(34)	$\overline{e}^k_{n,s}(q) = [n-s-1]_q\, \overline{e}^k_{n-1,s}(q) + q^{n-s-1}\,[s+1]_q\, \overline{e}^k_{n-1,s+1}(q) + q^{n-s}\overline{e}^k_{n-1,s-1}(q)$
(35)	$\overline{e}^k_{n,0}(q) = q^{n-k}\,[k]_q\, \overline{e}^{k-1}_{n-1,0}(q) + [n-k]_q\, \overline{e}^k_{n-1,0}(q)$
(36)	$\overline{e}^k_{n,0}(q) = [k]_q! \sum_{r=0}^k q^{r(n+r-2k)} \genfrac{}{}{0pt}{}{k}{r}_q \genfrac{}{}{0pt}{}{n-k}{k-r}_q \overline{e}^{k-r}_{n-k,0}(q)$
(37)	$\overline{e}^0_{n,0}(q) = [n]_q\, \overline{e}^0_{n-1,0}(q) + (-1)^n q^{\genfrac{}{}{0pt}{}{n}{2}}$

$\overline{\sigma} = 3\ 1\ 11\ 2\ 10\ 4\ 5\ 9\ 8\ 12\ 6\ 13\ 7$. Garsia and Remmel defined their q-analogue of the derangement numbers by setting

$$D_n(q) = \sum_{\sigma \in \mathbb{D}_n} q^{\mathrm{inv}(\overline{\sigma})}, \tag{38}$$

where for any $\sigma = \sigma_1 \dots \sigma_n \in S_n$, $\mathrm{inv}(\sigma) = |\{1 \leq i < j \leq n \mid \sigma_i > \sigma_j\}|$ denotes the number of inversions of σ. Note that this definition implies $D_1(q) = 0$ and $D_2(q) = 1$. Garsia and Remmel then proved the following q-analogues of the basic recursions for the number of derangements.

$$D_{n+1}(q) = q[n]_q D_n(q) + [n]_q D_{n-1}(q), \quad \text{for } n \geq 2 \tag{39}$$

and

$$D_{n+1}(q) = [n+1]_q D_{n,1}(q) + (-1)^{n+1}, \quad \text{for } n \geq 1. \tag{40}$$

Garsia and Remmel defined a second q-analogue of the derangement numbers by setting

$$\overline{D}_n(q) = \sum_{\sigma \in \mathbb{D}_n} q^{\mathrm{coinv}(\overline{\sigma})}, \tag{41}$$

where for $\sigma = \sigma_1 \dots \sigma_n \in S_n$, $\mathrm{coinv}(\sigma)$ equals the number of pairs $1 \leq i < j \leq n$ such that $\sigma_i < \sigma_j$. In this case, $\overline{D}_1(q) = 0$ and $\overline{D}_{2,1}(q) = q$. Then Garsia and Remmel showed that for $n \geq 2$,

$$\overline{D}_{n+1}(q) = [n]_q \overline{D}_n(q) + q^n [n]_q \overline{D}_{n-1}(q), \tag{42}$$

and for $n \geq 1$,

$$\overline{D}_{n+1}(q) = [n+1]_q \overline{D}_n(q) + (-1)^{n+1} q^{\binom{n+1}{2}}. \tag{43}$$

Wachs [9] gave another interpretation of $D_n(q)$ by showing that $D_n(q) = \sum_{\sigma \in \mathbb{D}_n} q^{maj(\sigma)}$.

Next we will show how one can use that Garsia-Remmel definition of $D_n(q)$ to define a permutation statistic $\mathrm{stat}_k(\sigma)$ such that

$$\sum_{\sigma \in S_n} x^{\mathrm{exc}_k(\sigma)} q^{\mathrm{stat}_k(\sigma)} = E_{n,k}(x,q).$$

Our idea is to use our combinatorial proofs of the $q = 1$ versions of (33), (35), and (37) to determine $\mathrm{stat}_k(\sigma)$.

It will be easiest to describe the statistic on the rook placement associated with a given permutation. Fix $k \geq 0$ and for any $\sigma \in S_n$, we let $P(\sigma)$ denote the placement of n rooks in $[n] \times [n]$ determined by σ. We will recursively define $\mathrm{stat}_k(P(\sigma))$ as follows.

Case 1. $|P(\sigma) \cap B_{n,k}| = s > 0$.
In this case, we scan along the diagonal going from $(1, 1+k)$ to $(n-k, n)$ and record a sequence $t = t_1 \dots t_{n-k}$ of 0's and 1's where $t_i = 0$ if there is no rook in the i-th square of $B_{n,k}$ and $t_i = 1$ if there is a rook in the i-th square of $B_{n,k}$. We let N denote the rook placement that results from removing the rows and columns of the rooks in $B_{n,k}$. Thus, N is a rook placement on the $[n-s] \times [n-s]$ board which does not intersect $B_{n-s,k}$. We then define $\mathrm{stat}_k(P) = \binom{s}{2} \mathrm{inv}(t) + \mathrm{stat}_k(N)$. This definition reflects the recursion that

$$e^k_{n,s}(q) = q^{\binom{s}{2}} \begin{bmatrix} n-k \\ s \end{bmatrix}_q e^k_{n-s,0}(q).$$

Case 2. $k > 0$ and $P(\sigma)$ does not intersect $B_{n,k}$.
In this case, if the rook in the bottom row of $P(\sigma)$ is in position i, then we define a new placement N of rooks on the $[n-1] \times [n-1]$ grid obtained by removing the row 1 and column i if $i > n-k$ or by removing row $i+k$ and column i if $i \leq n-k$. In this case we define

$$\mathrm{stat}_k(P(\sigma)) = \begin{cases} (n-i) + \mathrm{stat}_{k-1}(N) & \text{if } i > n-k \\ (n-i) + \mathrm{stat}_k(N) & \text{if } i \leq n-k. \end{cases}$$

This definition reflects the recursion

$$e^k_{n,0}(q) = [k]_q \, e^{k-1}_{n-1,0}(q) + q^k \, [n-k]_q \, e^k_{n-1,0}(q).$$

Case 3. $k = 0$ and $P(\sigma)$ does not intersect $B_{n,0}$.
In this case, $P(\sigma)$ is a derangement and we simply let $\mathrm{stat}_0(\sigma) = \mathrm{inv}(\overline{\sigma})$ as Garsia and Remmel did. This reflects the recursion that

$$e^0_{n,0}(q) = [n]_q \, e^0_{n-1,0}(q) + (-1)^n.$$

EXAMPLE 4.7 Suppose that $\sigma = 8\ 3\ 2\ 5\ 6\ 1\ 4\ 7$ and $k = 1$. The corresponding rook placement is pictured in figure 7 on the top left. The first reduction comes from Case 1. The sequence $t = 0101100$ so that the contribution to $q^{\mathrm{stat}_1(\sigma)}$ is $q^{\binom{3}{2}} q^{inv(0101100)} = q^{10}$ and we are reduced to finding $q^{\mathrm{stat}_1(\tau)}$ where $\tau = 5\ 2\ 1\ 3\ 4$. For τ, we are in Case 2 so that we get a contribution of $q^{5-3} = q^2$ to $q^{\mathrm{stat}_1(\tau)}$ and we are reduced to computing $q^{\mathrm{stat}_1(\alpha)}$ where $\alpha = 4\ 2\ 3\ 1$. For α, we are again in Case 2 and we get a contribution of $q^{4-4} = q^0$ to $q^{\mathrm{stat}_1(\alpha)}$ and we are reduced to computing $q^{\mathrm{stat}_0(\beta)}$ where $\beta = 3\ 1\ 2$. For β, we are in Case 3. The 1-standard order of β is (1,3,2) so that $q^{\mathrm{stat}_0(\beta)} = q^{\mathrm{inv}(132)} = q$. Thus $q^{\mathrm{stat}_1(\sigma)} = q^{10+2+0+1} = q^{13}$.

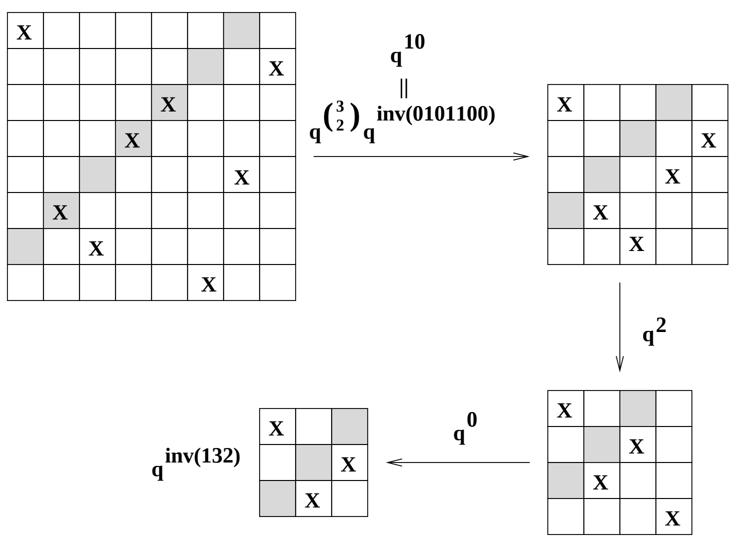

Figure 7: Demonstrating the statistic $\mathrm{stat}_k(P)$

Rakotondrajao [1] proved that

$$E^{(k)}(t,u) = \sum_{n \geq 0} \sum_{s=0}^{n} e^k_{n+k,s} t^s \frac{u^n}{n!} = k! \frac{e^{u(t-1)}}{(1-u)^{k+1}}. \qquad (44)$$

Next we prove a q-analogue of this formula.

THEOREM 4.8 *For all $k \geq 0$,*

$$E^{(k)}(t,u,q) = \sum_{n\geq 0}\sum_{s=0}^{n} e^{k}_{n+k,s}(q)\frac{t^s u^n}{[n]_q!} = \frac{[k]_q!\exp_q(t,u)}{\prod_{j=0}^{k}(1-q^j u)} \qquad (45)$$

where $\exp_q(t,u) = \sum_{m\geq 0}\frac{u^m\prod_{j=0}^{m-1}(q^j t-1)}{[m]_q!}$.

Proof. We need only show that the coefficient of $t^s u^{n-k}$ in $\frac{[n-k]_q!\,[k]_q!\exp_q(t,u)}{\prod_{j=0}^{k}(1-q^j u)}$ is in fact equal to the formula for $e^k_{n,s}(q)$ that was defined in (28). Using the fact that $\prod_{j=0}^{k}\frac{1}{(1-q^j u)} = \sum_{m\geq 0}\begin{bmatrix} m+k \\ k\end{bmatrix}_q u^m$, we find that

$$\begin{aligned}
&\frac{[n-k]_q!\,[k]_q!\exp_q(t,u)}{\prod_{j=0}^{k}(1-q^j u)}\\
&= [n-k]_q!\,[k]!\left(\sum_{m\geq 0}\frac{u^m\prod_{j=0}^{m-1}(q^j t-1)}{[m]_q!}\right)\left(\sum_{m\geq 0}\begin{bmatrix} m+k \\ k\end{bmatrix}_q u^m\right)\\
&= [n-k]_q!\,[k]_q!\left(\sum_{m\geq 0}\frac{u^m}{[m]_q!}\sum_{j=0}^{m} q^{\binom{j}{2}}\begin{bmatrix} m \\ j\end{bmatrix}_q(-1)^{m-j}t^j\right)\left(\sum_{m\geq 0}\begin{bmatrix} m+k \\ k\end{bmatrix}_q u^m\right).
\end{aligned}$$

Now taking the coefficient of u^{n-k} gives

$$[n-k]_q!\,[k]_q!\sum_{i=0}^{n-k}\sum_{j=0}^{i}\begin{bmatrix} i \\ j\end{bmatrix}_q\frac{(-1)^{i-j}}{[i]_q!}t^j q^{\binom{j}{2}}\begin{bmatrix} n-i \\ k\end{bmatrix}_q.$$

Finally, we take the coefficient of t^s and arrive at

$$\begin{aligned}
e^k_{n,s}(q) &= [n-k]_q!\,[k]_q!\sum_{i=s}^{n-k}\begin{bmatrix} i \\ s\end{bmatrix}_q\frac{(-1)^{i-s}}{[i]_q!}q^{\binom{s}{2}}\begin{bmatrix} n-i \\ k\end{bmatrix}_q\\
&= \sum_{i=s}^{n-k}(-1)^{i-s}q^{\binom{s}{2}}\begin{bmatrix} i \\ s\end{bmatrix}_q\frac{[n-i]_q!\,[n-k]_q!}{[n-i-k]_q!\,[i]_q!}\\
&= \sum_{i=s}^{n-k}(-1)^{i-s}q^{\binom{s}{2}}\,[n-i]!\begin{bmatrix} i \\ s\end{bmatrix}_q\begin{bmatrix} n-k \\ i\end{bmatrix}_q
\end{aligned}$$

which matches formula (28). □

We note that with our definitions $e^k_{n,s}(q)$ and $\overline{e}^k_{n,s}(q)$, we were able to find q-analogues of all of the recursions proved in Section 3 except for (5), (10) and (11). In the next section, we introduce another q-analogue of $e^k_{n,s}$, $\hat{e}^k_{n,s}(q)$ which does satisfy natural q-analogues of (5), (10) and (11), but does not seem to satisfy q-analogues of the other recursions in Section 3.

5 A third q-Analogue of $e^k_{n,s}$

For $n \geq 1$, $0 \leq k < n$ and $s \geq 0$, we define

$$\hat{e}^k_{n,s}(q) := \sum_{t=s}^{n-k} (-1)^{t-s} q^{\binom{t}{2}} [n-t]_q! \begin{bmatrix} t \\ s \end{bmatrix}_q \begin{bmatrix} n-k \\ t \end{bmatrix}_q. \tag{46}$$

Again, if $n \geq k$ we define

$$\hat{e}^k_{n,s}(q) := \begin{cases} 0, & \text{if } s > 0 \\ [n]_q!, & \text{if } s = 0. \end{cases}$$

Here are some recursions satisfied by $\hat{e}^k_{n,s}(q)$.

Theorem 5.1 *For $n \geq 2$ and $0 < k < n$,*

$$\hat{e}^k_{n,0}(q) = \hat{e}^{k-1}_{n,0}(q) + q^{n-k} \hat{e}^{k-1}_{n-1,0}(q). \tag{47}$$

Proof. By (46), $\hat{e}^{k-1}_{n,0}(q)$ equals

$$\sum_{t=0}^{n+1-k} (-1)^t q^{\binom{t}{2}} [n-t]_q! \begin{bmatrix} n+1-k \\ t \end{bmatrix}_q.$$

Now, using the fact that $\begin{bmatrix} n+1-k \\ t \end{bmatrix}_q = \begin{bmatrix} n-k \\ t \end{bmatrix}_q + q^{n+1-k-t} \begin{bmatrix} n-k \\ t-1 \end{bmatrix}_q$, the above becomes

$$\begin{aligned}
&\hat{e}^{k-1}_{n,0}(q)\\
&= \sum_{t=0}^{n+1-k} (-1)^t q^{\binom{t}{2}} [n-t]_q! \left(\begin{bmatrix} n-k \\ t \end{bmatrix}_q + q^{n+1-k-t} \begin{bmatrix} n-k \\ t-1 \end{bmatrix}_q \right)\\
&= \sum_{t=0}^{n-k} (-1)^t q^{\binom{t}{2}} [n-t]_q! \begin{bmatrix} n-k \\ t \end{bmatrix}_q + \sum_{t=1}^{n+1-k} (-1)^t q^{\binom{t}{2}} [n-t]_q! q^{n+1-k-t} \begin{bmatrix} n-k \\ t-1 \end{bmatrix}_q\\
&= \sum_{t=0}^{n-k} (-1)^t q^{\binom{t}{2}} [n-t]_q! \begin{bmatrix} n-k \\ t \end{bmatrix}_q + \sum_{t=1}^{n+1-k} (-1)^t q^{\binom{t-1}{2}} q^{n-k} [n-t]_q! \begin{bmatrix} n-k \\ t-1 \end{bmatrix}_q\\
&= \sum_{t=0}^{n-k} (-1)^t q^{\binom{t}{2}} [n-t]_q! \begin{bmatrix} n-k \\ t \end{bmatrix}_q + \sum_{t=0}^{n-k} (-1)^{(t+1)} q^{\binom{t}{2}} q^{n-k} [n-t-1]_q! \begin{bmatrix} n-k \\ t \end{bmatrix}_q.
\end{aligned}$$

Now the first sum of this last expression is the definition of $\hat{e}^k_{n,0}(q)$ so that (47) holds if we can prove that

$$\sum_{t=0}^{n-k} (-1)^{(t+1)} q^{\binom{t}{2}} q^{n-k} [n-t-1]_q! \begin{bmatrix} n-k \\ t \end{bmatrix}_q + q^{n-k} \hat{e}^{k-1}_{n-1,0}(q) = 0. \qquad (48)$$

But (48) is equivalent to

$$\hat{e}^{k-1}_{n-1,0}(q) = \sum_{t=0}^{n-k} (-1)^t q^{\binom{t}{2}} [n-t-1]_q! \begin{bmatrix} n-k \\ t \end{bmatrix}_q$$

which is just the definition of $\hat{e}^{k-1}_{n-1,0}(q)$. □

THEOREM 5.2 *For $n \geq 2$ and $0 \leq k < n$,*

$$\hat{e}^k_{n,0}(q) = q\hat{e}^{k+1}_{n,1}(q) + q^{n-k} [k]_q \hat{e}^k_{n-1,0}(q). \qquad (49)$$

Proof. Note that

$$\begin{aligned}\hat{e}_{n,0}^{k}(q) =& \sum_{t=0}^{n-k}(-1)^t q^{\binom{t}{2}}[n-t]_q! \begin{bmatrix} n-k \\ t \end{bmatrix}_q \\ =&[n]_q! - [n-1]_q![n-k]_q + \sum_{t=2}^{n-k}(-1)^t q^{\binom{t}{2}}[n-t]_q! \begin{bmatrix} n-k \\ t \end{bmatrix}_q \\ =&[n-1]_q!([n]_q - [n-k]_q) + \sum_{t=1}^{n-k-1}(-1)^{t-1} q^{\binom{t+1}{2}}[n-t-1]_q! \begin{bmatrix} n-k \\ t+1 \end{bmatrix}_q \\ =&q^{n-k}[k]_q([n-1]_q!) + \sum_{t=1}^{n-k-1}(-1)^{t-1} q^{\binom{t+1}{2}}[n-t-1]_q! \begin{bmatrix} n-k \\ t+1 \end{bmatrix}_q \end{aligned} \tag{50}$$

On the other hand,

$$q\hat{e}_{n,1}^{k+1}(q) = \sum_{t=1}^{n-k-1}(-1)^{t-1} q^{\binom{t}{2}}[n-t]_q! q \begin{bmatrix} n-k-1 \\ t \end{bmatrix}_q \tag{51}$$

and

$$\begin{aligned} q^{n-k}[k]_q\hat{e}_{n-1,0}^{k}(q) =& \sum_{t=0}^{n-k-1}(-1)^t q^{\binom{t}{2}}[n-t-1]_q! \begin{bmatrix} n-k-1 \\ t \end{bmatrix}_q q^{n-k}[k]_q \quad (52) \\ =&q^{n-k}[k]([n-1]_q!) + \\ & \sum_{t=1}^{n-k-1}(-1)^t q^{\binom{t}{2}}[n-t-1]_q! \begin{bmatrix} n-k-1 \\ t \end{bmatrix}_q q^{n-k}[k]_q. \end{aligned}$$

Comparing (50), (51), and (52), we see that to prove (49), we need only show that

$$\begin{aligned} &\sum_{t=1}^{n-k-1}(-1)^{t-1} q^{\binom{t+1}{2}}[n-t-1]_q! \begin{bmatrix} n-k \\ t+1 \end{bmatrix}_q = \\ &\sum_{t=1}^{n-k-1}(-1)^{t-1} q^{\binom{t}{2}}[n-t-1]_q! \begin{bmatrix} n-k-1 \\ t \end{bmatrix}_q (q[n-t]_q[t]_q - q^{n-k}[k]_q). \end{aligned} \tag{53}$$

Now it is easy to see that

$$\begin{aligned} q[n-t]_q[t]_q - q^{n-k}[k]_q &= q[n-t]_q[t-1]_q + q^t[n-t] - q^{n-k}[k] \\ &= q[n-t]_q[t-1]_q + q^t[n-t-k] \end{aligned}$$

so that we can rewrite (53) as

$$\sum_{t=1}^{n-k-1} (-1)^{t-1} q^{\binom{t+1}{2}} [n-t-1]_q! \begin{bmatrix} n-k \\ t+1 \end{bmatrix}_q =$$
$$\sum_{t=1}^{n-k-1} (-1)^{t-1} q^{\binom{t}{2}} [n-t]_q! \begin{bmatrix} n-k-1 \\ t \end{bmatrix}_q [t-1]_q +$$
$$\sum_{t=1}^{n-k-1} (-1)^{t-1} q^{\binom{t+1}{2}} [n-t-1]_q! \begin{bmatrix} n-k-1 \\ t \end{bmatrix}_q [n-k-t]_q \qquad (54)$$

Now the first term of the first sum on the RHS of (54) is 0 and the last term of the second sum on the RHS of (54) is equal to the last term of the sum on the LHS of (54). Thus if we cancel the last terms of those two sums and we re-index the first sum on the RHS of (54) to start at $t = 1$, we are reduced to showing that

$$\sum_{t=1}^{n-k-2} (-1)^{t-1} q^{\binom{t+1}{2}} [n-t-1]_q! \begin{bmatrix} n-k \\ t+1 \end{bmatrix}_q =$$
$$\sum_{t=1}^{n-k-2} (-1)^{t-1} q^{\binom{t+1}{2}} [n-t-1]_q! \left(\begin{bmatrix} n-k-1 \\ t \end{bmatrix}_q [n-k-t]_q - q[t]_q \begin{bmatrix} n-k-1 \\ t+1 \end{bmatrix}_q \right). \qquad (55)$$

It is easy to see that (55) holds if we can show that

$$\begin{bmatrix} n-k \\ t+1 \end{bmatrix}_q = \begin{bmatrix} n-k-1 \\ t \end{bmatrix}_q [n-k-t]_q - q[t]_q \begin{bmatrix} n-k-1 \\ t+1 \end{bmatrix}_q. \qquad (56)$$

This is straightforward to prove. That is, clearly

$$q[t+1]_q \begin{bmatrix} n-k-1 \\ t+1 \end{bmatrix}_q = q[n-k-t-1]_q \begin{bmatrix} n-k-1 \\ t \end{bmatrix}_q. \qquad (57)$$

Thus

$$q[t]_q \begin{bmatrix} n-k-1 \\ t+1 \end{bmatrix}_q + q^t \begin{bmatrix} n-k-1 \\ t+1 \end{bmatrix}_q =$$
$$[n-k-t]_q \begin{bmatrix} n-k-1 \\ t \end{bmatrix}_q - \begin{bmatrix} n-k-1 \\ t \end{bmatrix}_q.$$

Hence

$$[n-k-t]_q \begin{bmatrix} n-k-1 \\ t \end{bmatrix}_q - q[t]_q \begin{bmatrix} n-k-1 \\ t+1 \end{bmatrix}_q =$$
$$q^t \begin{bmatrix} n-k-1 \\ t+1 \end{bmatrix}_q + \begin{bmatrix} n-k-1 \\ t \end{bmatrix}_q =$$
$$\begin{bmatrix} n-k \\ t \end{bmatrix}_q .$$

□

THEOREM 5.3 *For $n \geq 1$ and $0 \leq k < n$,*

$$\hat{e}^k_{n,0}(q) = \sum_{r=0}^{k} q^{r(n-k)} \begin{bmatrix} k \\ r \end{bmatrix}_q \hat{e}^0_{n-r,0}(q). \tag{58}$$

Proof. We can prove this theorem using induction on k. When $k=0$ and $n \geq 0$, it is clearly true. Now assume that $\hat{e}^i_{n,0}(q) = \sum_{r=0}^{i} q^{r(n-i)} \begin{bmatrix} i \\ r \end{bmatrix}_q \hat{e}^0_{n-r,0}(q)$, for all $n \geq 0$ and all $i < k$. Our goal is now to show that $\hat{e}^k_{n,0}(q) = \sum_{r=0}^{k} q^{r(n-k)} \begin{bmatrix} k \\ r \end{bmatrix}_q \hat{e}^0_{n-r,0}(q)$. Firstly, using Theorem 5.1 we have that

$$\hat{e}^k_{n,0}(q) = \hat{e}^{k-1}_{n,0}(q) + q^{n-k} \hat{e}^{k-1}_{n-1,0}(q).$$

We then apply the induction hypothesis to the RHS of the above to get

$$\sum_{r=0}^{k-1} q^{r(n-k+1)} \begin{bmatrix} k-1 \\ r \end{bmatrix}_q \hat{e}^0_{n-r,0}(q) + q^{n-k} \sum_{r=0}^{k-1} q^{r(n-k)} \begin{bmatrix} k-1 \\ r \end{bmatrix}_q \hat{e}^0_{n-1-r,0}(q)$$
$$= \sum_{r=0}^{k-1} q^{r(n-k+1)} \begin{bmatrix} k-1 \\ r \end{bmatrix}_q \hat{e}^0_{n-r,0}(q) + \sum_{r=0}^{k-1} q^{(r+1)(n-k)} \begin{bmatrix} k-1 \\ r \end{bmatrix}_q \hat{e}^0_{n-1-r,0}(q).$$

Then, peeling off the 0 term of the first sum and the $k-1$ term of the second sum gives

$$\begin{aligned}
& \begin{bmatrix} k-1 \\ 0 \end{bmatrix}_q \hat{e}^0_{n,0}(q) + \sum_{r=1}^{k-1} q^{r(n-k+1)} \begin{bmatrix} k-1 \\ r \end{bmatrix}_q \hat{e}^0_{n-r,0}(q) \\
& + q^{k(n-k)} \begin{bmatrix} k-1 \\ k-1 \end{bmatrix}_q \hat{e}^0_{n-k,0}(q) + \sum_{r=0}^{k-2} q^{(r+1)(n-k)} \begin{bmatrix} k-1 \\ r \end{bmatrix}_q \hat{e}^0_{n-1-r,0}(q) \\
= \; & q^{k(n-k)} \begin{bmatrix} k-1 \\ k-1 \end{bmatrix}_q \hat{e}^0_{n-k,0}(q) + \begin{bmatrix} k-1 \\ 0 \end{bmatrix}_q \hat{e}^0_{n,0}(q) \\
& + \sum_{r=1}^{k-1} q^{r(n-k)} \left(q^r \begin{bmatrix} k-1 \\ r \end{bmatrix}_q + \begin{bmatrix} k-1 \\ r-1 \end{bmatrix}_q \right) \hat{e}^0_{n-r,0}(q) \\
= \; & q^{k(n-k)} \begin{bmatrix} k-1 \\ k-1 \end{bmatrix}_q \hat{e}^0_{n-k,0}(q) + \sum_{r=1}^{k-1} q^{r(n-k)} \begin{bmatrix} k \\ r \end{bmatrix}_q \hat{e}^0_{n-r,0}(q) + \begin{bmatrix} k-1 \\ 0 \end{bmatrix}_q \hat{e}^0_{n,0}(q) \\
= \; & \sum_{r=0}^{k} q^{r(n-k)} \begin{bmatrix} k \\ r \end{bmatrix}_q \hat{e}^0_{n-r,0}(q),
\end{aligned}$$

which was to be shown. Note, that when going from the second line to the third, we used the fact that $q^r \left[\begin{smallmatrix} k-1 \\ r \end{smallmatrix}\right]_q + \left[\begin{smallmatrix} k-1 \\ r-1 \end{smallmatrix}\right]_q = \left[\begin{smallmatrix} k \\ r \end{smallmatrix}\right]_q$. □

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