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Enumeration of Wilf classes in $S_n \wr C_r$ for two patterns of length 3 *

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Abstract. We study pattern avoidance in colored permutations ($S_n \wr C_r$) using the pattern matching condition developed by Mansour in [4]. We prove results in the enumeration of avoidance classes of two patterns in $S_3 \wr C_2$ and present bijections to other previously studied objects. We will also present conjectures on the number of Wilf classes of these pattern pairs and prove upper and lower bounds on this number. We have modified certain bijections developed by Mansour to prove the Wilf equivalence of colored patterns and colored pattern pairs. We have also enumerated the avoidance of uni-colored patterns of length k in $S_n \wr C_r$. We end with a bijection from the avoidance class in $S_n \wr C_r$ of paired uni-colored patterns of length 3 to symmetric permutations of length $2n$ avoiding a decreasing string of length 5, studied by Egge in [1].

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1 Introduction and Background

Pattern avoidance in the symmetric group has a number of applications. One major application is stack sorting. Other applications relate to the

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theory of Kazhdan-Lusztig polynomials, singularities of Schubert varieties, Chebyshev polynomials, and rook polynomials [4]. Pattern avoidance in the hyperoctahedral group is a useful language in combinatorial statistics defined in type-B non-crossing partitions, enumerative combinatorics, and algebraic combinatorics.

Before we present our results, we first state some definitions.

The group $S_n^{(r)} = S_n \wr C_r$, where C_r is the cyclic group of order r, is an extension of the symmetric group and of the hyperoctahedral group. We will view the elements of the set $S_n^{(r)}$ as colored permutations $\varphi = (\tau_1^{(s_1)}, ..., \tau_n^{(s_n)})$ in which each of the symbols in the set $\{1, 2, ..., n\}$ (a set that we will henceforth denote by $[n]$) appears once as τ_i for some i, and each τ_i is colored by some s_k in the set $[r]$. Thus, $S_n^{(1)} = S_n, S_n^{(2)} = B_n$, and the cardinality of $S_n^{(r)}$ is $n!r^n$. We will use the notation φ^- for the permutation $(\tau_1, ...\tau_n)$ (note the absence of the colors). For example, $\varphi = (1^{(1)}, 3^{(2)}, 2^{(1)})$ is a colored permutation in $S_3^{(2)}$, and $\varphi^- = (1, 3, 2)$.

The following definitions were developed by Mansour [4]:

Let $\varphi = (\tau_1^{(s_1)}, ..., \tau_k^{(s_k)}) \in S_k^{(r)}$, and $\psi = (\sigma_1^{(t_1)}, ..., \sigma_n^{(t_n)})$; we say that ψ *contains* φ (or is φ-containing) if there is a sequence of k indices, $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that the following two conditions hold:

(i) $(\sigma_{i_1}, ..., \sigma_{i_k})$ is order-isomorphic to φ^-;

(ii) $t_{i_j} = s_j$ for all $j = 1, 2, ..., k$.

Otherwise, we say that ψ ***avoids*** φ (or is φ-avoiding). The set of all φ-avoiding colored permutations in $S_n^{(r)}$ is denoted by $\mathrm{Av}_{n,r}(\varphi)$, and in this context φ is called a *colored pattern*. For an arbitrary finite collection of colored patterns T, we say that ψ avoids T if ψ avoids φ for all $\varphi \in T$; the corresponding subset of $S_n^{(r)}$ is denoted by $\mathrm{Av}_{n,r}(T)$. As an example, $\psi = (3^{(1)}, 2^{(2)}, 1^{(2)}) \in S_3^{(2)}$ avoids $(2^{(1)}, 1^{(1)})$; that is, $\psi \in \mathrm{Av}_{3,2}((2^{(1)}, 1^{(1)}))$.

Let T_1, T_2 be two sets of colored patterns; we say that $T_1, T_2 \subset S_p^{(k)}$ are in the same *Wilf class* if $|\mathrm{Av}_{n,r}(T_1)| = |\mathrm{Av}_{n,r}(T_2)|$ for all $n > 0$, $r > 0$.

Multiple authors have investigated pattern avoidance on colored permutations, including Mansour in [4], [5], Egge in [2], as well as Mansour and West in [6], among others. Our work directly builds on the work of Mansour [4]. Mansour presented results involving $S_2^{(r)}$. He proved, for colored patterns φ of length two, that the number of φ-avoiding permutations in $S_n^{(r)}$ is given by the formula $\sum_{j=0}^{n} j!(r-1)^j \binom{n}{j}^2$. He also proved that the number of Wilf classes of restricted colored permutations by two patterns with r

colors in $S_2^{(r)}$ is one for $r = 1$, is four for $r = 2$, and is six for $r \geq 3$. What Mansour achieved for patterns of length two we seek to achieve for patterns of length three. This is also an extension of the work done by Vatter in [9], where uncolored pairs of patterns of length three were studied.

In Section 2, we prove results in the enumeration of avoidance classes of two patterns in $S_3 \wr C_2$ and present a bijection to another previously studied object, the row sums of the square of a certain matrix. Next, in Section 3, we present a conjecture, based on programs written in Maple and Python, on the number of Wilf classes of these pattern pairs and prove upper and lower bounds on this number. In the process of refining our bounds, we also prove a formula for the number of elements which avoid collections of uni-colored patterns. Then, in Section 4, we present a bijection from the elements of $S_n \wr C_2$ which avoid any two uni-colored patterns with distinct colors of length 3 to the elements of S_{2n} invariant under rc and having no decreasing subsequence of length 5, studied by Egge in [1]. Finally, in Section 5, we present various open problems and directions for further research.

2 A single pattern of length two

Let α be the sequence $\alpha = \{1, 7, 17, 31, 49, 71, ...\}$. This sequence can be derived from enumerating $\mathrm{Av}_{2,k}((2^{(1)}, 1^{(1)}))$ for $k > 1$. All but one permutation in any set $S_2 \wr C_k$ avoids $(2^{(1)}, 1^{(1)})$. Thus $\alpha_k = 2k^2 - 1$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 0 & 0 & 0 & \\ 1 & 2 & 2 & 0 & 0 & \\ 1 & 2 & 2 & 2 & 0 & \\ \vdots & & & & & \ddots \end{bmatrix}$$

Figure 1: The infinite lower triangular matrix, A', with 1's in the first column and the rest 2's.

This sequence can be seen elsewhere, as sequence $A056220$ in [7]. It is the image of the square numbers under the "little Hankel" transform. It is also the sequence of row sums of the triangle given in Figure 2, which is the square of the infinite lower triangular matrix with 1's in the first column and the rest 2's shown in Figure 1. We shall call the triangle in Figure 1 A' and its square A, and we will denote the jth entry of the kth row of A by

$A_{k,j}$.

1						=	1
3	4					=	7
5	8	4				=	17
7	12	8	4			=	31
9	16	12	8	4		=	49
11	20	16	12	8	4	=	71
$\vdots$					$\ddots$		$\vdots$

Figure 2: The matrix A, the square of the matrix A'. Its row sums give the sequence α.

Theorem 2.1 *Let A be the matrix defined above. Then*

1. $A_{k,1} = 2k - 1$ *for all* $k > 1$, *and*
2. $A_{k,j} = 4(k - j + 1)$ *for all* $k \geq j \geq 2$
3. $A_{k,j} = 0$ *otherwise.*

Proof. The first statement is easily verified. The entries in the first column of A are the row sums of A'. To verify the second statement, first notice that the kth row of A' has k nonzero entries, and the jth column begins with $j - 1$ zero entries. Clearly, if $k < j$, then $A_{k,j} = 0$ (the third statement). Otherwise the first $j-1$ entries of the kth row are canceled, and, since $j \geq 2$, the remaining products are all 4. Thus, if $k \geq j \geq 2$, $A_{k,j} = 4(k-(j-1)) = 4(k - j + 1)$. $\square$

We could prove algebraically that the sum of the kth row of A is α_k by showing

$$2k^2 - 1 = 2k - 1 + \sum_{j=2}^{k} 4(k - j + 1),$$

but a more interesting method of proof is to partition $\mathrm{Av}_{2,k}((2^{(1)}, 1^{(1)}))$ into sets $B_{k,j}$ such that $|B_{k,j}| = A_{k,j}$ for all k and j. To do this we will define a statistic, *col*, on the elements of $\mathrm{Av}_{2,k}((2^{(1)}, 1^{(1)}))$ and then an equivalence relation, R, from which we form the desired partition.

Before we define the statistic *col* we must first give some short preliminary definitions. If $\pi \in S_n \wr C_k$, then define $inv(\pi)$ to be the number of inversions in π^-. Note that for $\pi \in S_2 \wr C_k$, $inv(\pi)$ is either 0 or 1 depending on whether the underlying permutation is $(1, 2)$ or $(2, 1)$. Also define

$max(\pi)$ to be the largest color assigned to some element of π.

DEFINITION 2.2 If $\pi \in S_n \wr C_k$, then define $col(\pi) = (k+1) - max(\pi) + inv(\pi)$.

LEMMA 2.3 *Let $\pi_1, \pi_2 \in \mathrm{Av}_{2,k}((2^{(1)}, 1^{(1)}))$. Then $col(\pi_1) = col(\pi_2)$ if and only if precisely one of the following holds.*

1. *$max(\pi_1) = max(\pi_2)$ and $inv(\pi_1) = inv(\pi_2)$,*
2. *without loss of generality, $max(\pi_1) = max(\pi_2) + 1$ and $inv(\pi_1) = inv(\pi_2) + 1$.*

Proof. Clearly the statements are mutually exclusive, and the first implies that $col(\pi_1) = col(\pi_2)$.

Since $n = 2$, the values of $inv(\pi_1)$ and $inv(\pi_2)$ are either 0 or 1, as noted above. Thus if $inv(\pi_1) \neq inv(\pi_2)$, then without loss of generality we can say that $inv(\pi_1) = inv(\pi_2) + 1$, which in turn implies that $max(\pi_1) = max(\pi_2) + 1$. The converse is just as simple. □

THEOREM 2.4 *The set $\mathrm{Av}_{2,k}((2^{(1)}, 1^{(1)}))$ can be partitioned into subsets, $B_{k,j}$, such that $B_{k,j} = A_{k,j}$.*

Proof. Define an equivalence relation, R, on the elements of $\mathrm{Av}_{2,k}((2^{(1)}, 1^{(1)}))$ by $\pi_1 R \pi_2$ if and only if $col(\pi_1) = col(\pi_2)$. Denote an equivalence class of π by $B_{k,j}$ where $j = col(\pi)$. The sets $B_{k,j}$ partition $\mathrm{Av}_{2,k}((2^{(1)}, 1^{(1)}))$.

It is important to keep in mind that $1 \leq max(\pi) \leq k$ and $inv(\pi)$ is either 0 or 1.

If $j = 1$, then $1 = col(\pi) = (k+1) - max(\pi) + inv(\pi)$. The only way that this could occur is if $max(\pi) = k$ and $inv(\pi) = 0$. In $\mathrm{Av}_{2,k}((2^{(1)}, 1^{(1)}))$ there are $2k-1$ such elements. All must have underlying permutation $(1,2)$. $k-1$ have color k only in the first position, $k-1$ have color k only in the second position, and one is colored k in both positions. The set of all such elements is $B_{k,1}$ since color $k+1$ does not exist. Thus $|B_{k,1}| = 2(k-1) + 1 = 2k - 1 = A_{k,1}$.

If $k \geq j \geq 2$ then $B_{k,j}$ contains some elements with no inversions and some elements with one inversion. The elements, π_1, with one inversion must satisfy $j = (k+1) - max(\pi_1) + 1$. Thus $max(\pi_1) = k - j + 2$. In $\mathrm{Av}_{2,k}((2^{(1)}, 1^{(1)}))$ there are $2(k-j+1)+1$ such elements. All have underlying permutation $(2,1)$. $k-j+1$ have color $k-j+2$ only in the first position,

$k-j+1$ have color $k-j+2$ only in the second position, and one is colored $k-j+2$ in both positions. The elements, π_0, with no inversions must satisfy $j = (k+1) - max(\pi_0)$. Thus $max(\pi_0) = k-j+1$. In $\mathrm{Av}_{2,k}((2^{(1)}, 1^{(1)}))$ there are $2(k-j+1)-1$ such elements. All have underlying permutation $(1, 2)$. $k-j$ have color $k-j+1$ only in the first position, $k-j$ have color $k-j+1$ only in the second position, and one is colored $k-j+1$ in both positions. Thus $|B_{j,k}| = 2(k-j+1)+1+2(k-j+1)-1 = 4(k-j+1) = A_{k,j}$.

In $S_2 \wr C_k$, we cannot have $k < j$ since colors greater than k do not exist. Thus all sets $B_{k,j}$ are accounted for, and we see that in all cases, $|B_{k,j}| = A_{k,j}$. □

To visualize the sets $B_{k,j}$ we could list all the elements of $\mathrm{Av}_{2,k}((2^{(1)}, 1^{(1)}))$, grouping the equivalence classes together. Then we could outline the equivalence classes and one could count them by hand. An example can be seen in Figure 3. We have listed the elements of $\mathrm{Av}_{2,5}((2^{(1)}, 1^{(1)}))$ and outlined the sets $B_{5,j}$.

3 Paired patterns of length three

We study colored permutations avoiding two patterns of length three. Ultimately we seek to enumerate avoidance classes for these paired patterns, but another goal is simply to prove Wilf equivalence of pairs in order to partition all pairs into Wilf classes. To formulate our conjectures we created algorithms in Maple and Python. We chose these languages for a few reasons. Maple was familiar and is a powerful mathematical tool; however, Maple is rather slow when constructing the wreath products. This is where Python becomes useful. Python was used to construct the wreath product and find the sizes of the avoidance classes. Our programs calculate the size of avoidance classes in wreath products. The process is a brute force calculation, we generate the wreath product and check containment element-wise. Since we are usually interested in avoiding more than one element, our programs are designed to work with an arbitrary number of avoidance elements and arbitrary size of the wreath product. To date, using the Python programs, we can compute avoidance of short patterns in wreath products of size up to that of $S_8 \wr C_2$. Once the sizes of the avoidance classes were computed, another algorithm grouped elements of equal size. Thus these programs give a lower bound on the number of Wilf classes, and we conclude that there are a minimum of 74. In addition we conjecture that the bound is tight. We suspect it is tight because for the largest wreath product groups created ($S_8 \wr C_2$), the Wilf classes did not split any further

	~~$2^{(1)}1^{(1)}$~~	$1^{(1)}2^{(1)}$	
	$1^{(1)}2^{(2)}$	$2^{(1)}1^{(2)}$	$\|B_{5,5}\| = 4$
	$1^{(2)}2^{(1)}$	$2^{(2)}1^{(1)}$	
	$1^{(2)}2^{(2)}$	$2^{(2)}1^{(2)}$	
$\|B_{5,4}\| = 8$	$2^{(1)}1^{(3)}$	$1^{(1)}2^{(3)}$	
	$2^{(2)}1^{(3)}$	$1^{(2)}2^{(3)}$	
	$2^{(3)}1^{(1)}$	$1^{(3)}2^{(1)}$	
	$2^{(3)}1^{(2)}$	$1^{(3)}2^{(2)}$	
	$2^{(3)}1^{(3)}$	$1^{(3)}2^{(3)}$	
	$1^{(1)}2^{(4)}$	$2^{(1)}1^{(4)}$	$\|B_{5,3}\| = 12$
	$1^{(2)}2^{(4)}$	$2^{(2)}1^{(4)}$	
	$1^{(3)}2^{(4)}$	$2^{(3)}1^{(4)}$	
	$1^{(4)}2^{(1)}$	$2^{(4)}1^{(1)}$	
	$1^{(4)}2^{(2)}$	$2^{(4)}1^{(2)}$	
	$1^{(4)}2^{(3)}$	$2^{(4)}1^{(3)}$	
	$1^{(4)}2^{(4)}$	$2^{(4)}1^{(4)}$	
$\|B_{5,2}\| = 16$	$2^{(1)}1^{(5)}$	$1^{(1)}2^{(5)}$	
	$2^{(2)}1^{(5)}$	$1^{(2)}2^{(5)}$	
	$2^{(3)}1^{(5)}$	$1^{(3)}2^{(5)}$	
	$2^{(4)}1^{(5)}$	$1^{(4)}2^{(5)}$	
	$2^{(5)}1^{(1)}$	$1^{(5)}2^{(1)}$	$\|B_{5,1}\| = 9$
	$2^{(5)}1^{(2)}$	$1^{(5)}2^{(2)}$	
	$2^{(5)}1^{(3)}$	$1^{(5)}2^{(3)}$	
	$2^{(5)}1^{(4)}$	$1^{(5)}2^{(4)}$	
	$2^{(5)}1^{(5)}$	$1^{(5)}2^{(5)}$	

Figure 3: Here $k = 5$. The sets $B_{5,j}$ are outlined and their sizes are listed adjacent to the table.

from those generated in the smaller wreath product groups.

To prove conclusively that pattern pairs are Wilf equivalent, we use four maps, which we subsequently programmed. These maps are based on similar ones developed by Mansour in [4] and applied to S_n.

To prove that these maps preserve Wilf equivalence we require a regularity condition.

Definition 3.1 Suppose $g : S \to T$, where S and T are sets of words.

Then g is said to be *structure preserving* if for every word $\psi \in S$ and every subword φ of ψ then $g(\varphi)$ is a subword of $g(\psi)$.

EXAMPLE 3.2 Since all the bijections in this paper are structure preserving we provide a non-example. Consider $g : S_5 \to S_5$ where g exchanges the 1 and 2, for example

$$g(32541) = 31542.$$

Then g is not structure preserving since $g(541) = 541$ is not contained in 31542.

LEMMA 3.3 *If g is structure preserving bijection, then g preserves Wilf equivalence.*

Proof. Assume g is structure preserving and that ψ avoids φ. Since g is structure preserving we must have $g(\psi)$ avoids $g(\varphi)$, if $g(\varphi)$ were contained in $g(\psi)$ then the structure preserving condition on g would be contradicted. Thus $|\operatorname{Av}(\varphi)| = |\operatorname{Av}(g(\varphi))|$. Moreover, it can be seen that $\operatorname{Av}(g(\varphi)) = g(\operatorname{Av}(\varphi))$. Thus g preserves Wilf equivalence. □

Using this tool we show that the four bijections defined below, preserve Wilf equivalence.

DEFINITION 3.4 **(Reversal)** The bijection entitled reversal (R) maps an element $\varphi = (\tau_1^{(s_1)}, ..., \tau_k^{(s_k)})$ to $R(\varphi) = (\tau_k^{(s_k)}, ..., \tau_1^{(s_1)})$. For example, Let $\varphi = (2^{(1)}, 3^{(2)}, 1^{(2)}) \in S_3 \wr C_2$. Then

$$\varphi = (2^{(1)}, 3^{(2)}, 1^{(2)}) \longleftrightarrow R(\varphi) = (1^{(2)}, 3^{(2)}, 2^{(1)}).$$

THEOREM 3.5 *The reversal map preserves Wilf equivalence.*

Proof. Let $\psi \in S_n \wr C_r$ and φ be a subword of ψ. Write $\psi = (\sigma_1^{(t_1)}, ..., \sigma_n^{(t_n)})$ and $\varphi = (\sigma_{i_1}^{(t_{i_1})}, ..., \sigma_{i_k}^{(t_{i_k})})$ where $i_1 < i_2 < \cdots < i_k$. Then $R(\varphi) = (\sigma_{i_k}^{(t_{i_k})}, \sigma_{i_{k-1}}^{(t_{i_{k-1}})}, ..., \sigma_{i_1}^{(t_{i_1})})$ remains of subword of $R(\psi) = (\sigma_n^{(t_n)}, \sigma_{n-1}^{(t_{n-1})}, ..., \sigma_1^{(t_1)})$. Therefore the reversal map is structure preserving and by Lemma 3.3 it preserves Wilf equivalence. □

DEFINITION 3.6 **(Permute Colors)** The bijection entitled permute colors (PC_π) does not modify the underlying permutation of S_n. Instead, it modifies the colors. These colors are permuted in some way as to identify a new pattern. Let $\varphi = (\tau_1^{(s_1)}, ..., \tau_k^{(s_k)})$. If π is any permutation of appropriate length, then $PC_\pi(\varphi) = \tau_1^{(\pi(s_1))}, \tau_2^{(\pi(s_2))}, ..., \tau_k^{(\pi(s_k))}$. For example, let the permutation be $\pi = 21$. Let $\varphi = (2^{(1)}, 3^{(2)}, 1^{(2)}) \in S_3 \wr C_2$. Then

$$\varphi = (2^{(1)}, 3^{(2)}, 1^{(2)}) \longleftrightarrow PC_{21}(\varphi) = (2^{(2)}, 3^{(1)}, 1^{(1)}).$$

THEOREM 3.7 *The permute colors map preserves Wilf equivalence.*

The proof is similar to Theorem 3.5.

DEFINITION 3.8 **(Complement Permutation)** The bijection entitled complement permutation (CP) does not modify the colors nor their order. It deals only with the underlying permutation in S_n. The bijection maps τ_i to $n+1-\tau_i$. Thus an element $\varphi = (\tau_1^{(s_1)}, ..., \tau_k^{(s_k)})$ is mapped to $CP(\varphi) = ((k+1-\tau_1)^{(s_1)}, ..., (k+1-\tau_k)^{(s_k)})$ For example, let $\varphi = (2^{(1)}, 3^{(2)}, 1^{(2)}) \in S_3 \wr C_2$. Then

$$\varphi = (2^{(1)}, 3^{(2)}, 1^{(2)}) \longleftrightarrow CP(\varphi) = (2^{(1)}, 1^{(2)}, 3^{(2)}).$$

THEOREM 3.9 *The complement permutation map preserves Wilf equivalence.*

The proof of Theorem 3.9 and 3.7 are similar to that of Theorem 3.5. However, the next bijection requires a slightly more complicated method of proof.

DEFINITION 3.10 **(Inverse)** The bijection entitled inverse (I) is defined as follows. If $\varphi \in S_n \wr C_r$ and $\varphi = \varphi_1^{\alpha_1} \dots \varphi_n^{\alpha_n}$ then

$$I(\varphi) = i_1^{\alpha_{i_1}} \dots i_n^{\alpha_{i_n}},$$

where $\varphi_{i_1} < \varphi_{i_2} < \dots < \varphi_{i_n}$.

If we consider this definition of inverse on $S_n \wr C_1$, then it will coincide with the algebraic definition of inverse on S_n. This is easy to see, if we think of a permutation as $i \to \varphi_i$ then the inverse is "flipping the arrows", so that $\varphi_i \to i$ and then sorting the φ_i's. The only difference is that in $S_n \wr C_r$ the colors travel along each arrow.

EXAMPLE 3.11 Consider $\varphi = (3^{(1)}, 1^{(2)}, 2^{(2)}) \in S_3 \wr C_2$, then $I(\varphi)$ is as follows,

$$I\begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 3^{(1)} & 1^{(2)} & 2^{(2)} \end{pmatrix} = \begin{matrix} 1^{(1)} & 2^{(2)} & 3^{(2)} \\ \uparrow & \uparrow & \uparrow \\ 3 & 1 & 2 \end{matrix} = \begin{matrix} 2^{(2)} & 3^{(2)} & 1^{(1)} \\ \uparrow & \uparrow & \uparrow \\ 1 & 2 & 3 \end{matrix}.$$

Where the first step flipped the arrows and transferred the colors and the final step sorted the bottom.

Another way to understand this inverse map is to consider the element as a string of colored dominos, with the top number on the domino representing the upper entry in two-line notation, and the lower number representing its image. Then to find the inverse, simply flip each domino through 180 degrees, and then rearrange the dominos so the upper numbers increase left to right. We give an example in Figure 4, where we let color 1 be grey and color 2 be white, showing that $I((3^{(1)}, 1^{(2)}, 2^{(2)})) = (2^{(2)}, 3^{(2)}, 1^{(1)})$.

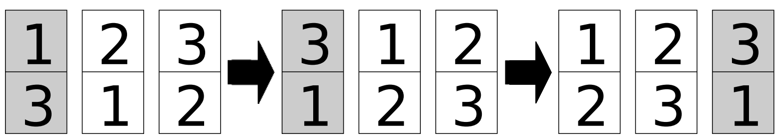

Figure 4: An example of the colored domino method for finding the inverse of a colored permutation, showing that $I((3^{(1)}, 1^{(2)}, 2^{(2)})) = (2^{(2)}, 3^{(2)}, 1^{(1)})$.

THEOREM 3.12 *The inverse map preserves Wilf equivalence.*

Proof. Let $\psi \in S_n \wr C_r$ and suppose φ is a subword of ψ. Write $\psi = \psi_1^{\alpha_1} \cdots \psi_n^{\alpha_n}$ and $\varphi = \psi_{i_1}^{\alpha_{i_1}} \cdots \psi_{i_m}^{\alpha_{i_m}}$ where $i_1 < i_2 < \cdots < i_m$. Then $\psi^{-1} = j_1^{\alpha_{j_1}} \cdots j_n^{\alpha_{j_n}}$ where

$$\psi_{j_1} < \cdots < \psi_{j_n} \tag{1}$$

and $\varphi^{-1} = i_{a_1}^{\alpha_{i_{a_1}}} \cdots i_{a_m}^{\alpha_{i_{a_m}}}$ where

$$\psi_{i_{a_1}} < \psi_{i_{a_2}} < \cdots < \psi_{i_{a_m}} \tag{2}$$

But, $\psi_{j_k} = \psi_{i_{a_p}}$ precisely when $j_k = i_{a_p}$. Thus it follows from (1) and (2) that φ^{-1} is a subword of ψ^{-1}. Therefore the inverse map is structure preserving and by Lemma 3.3 the inverse map preserves Wilf equivalence. □

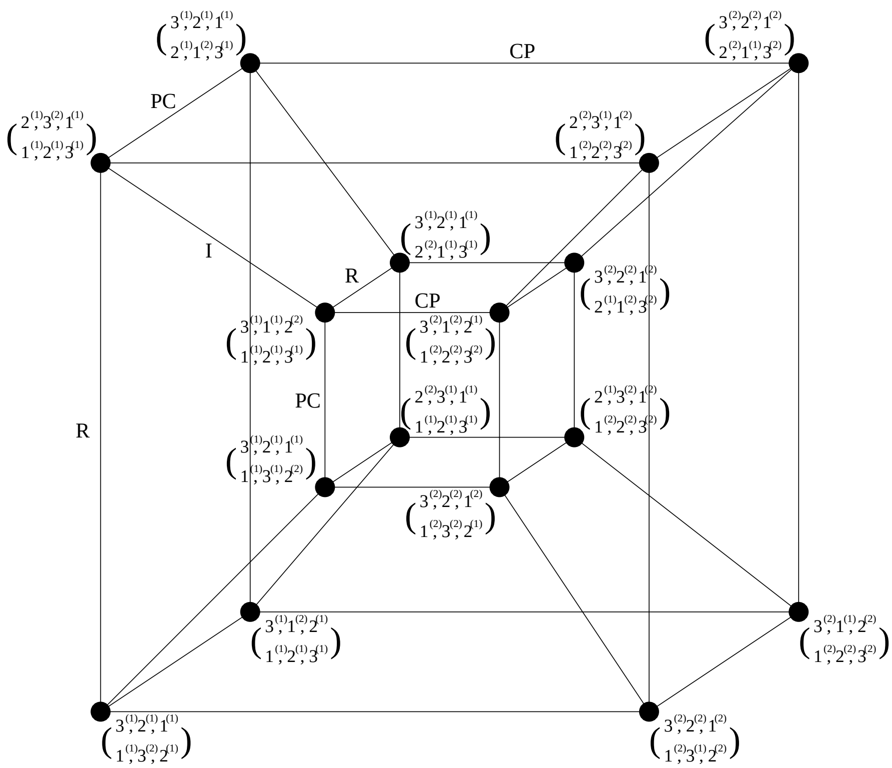

Figure 5: Note that in the outer cube, the vertical and diagonal connections are made via the maps R and PC respectively, while in the inner cube these maps are switched.

There are 48 patterns of length three in $S_n^{(2)}$ and thus there are $\binom{48}{2} = 1128$ pairs of patterns. Using these four maps, we can collect these patterns into 112 Wilf classes which we will refer to as "blocks" (one such block is depicted in Figure 5). Our programs, however, identified 74 potential Wilf classes. This leads us to suspect that there are fewer than 112. We therefore have 28 conjectures about Wilf equivalence. The following is an example of one such conjecture.

Conjecture. We conjecture that the pattern pairs $\begin{pmatrix} 3^{(1)}, 2^{(1)}, 1^{(2)} \\ 3^{(1)}, 2^{(2)}, 1^{(1)} \end{pmatrix}$, $\begin{pmatrix} 2^{(1)}, 3^{(2)}, 1^{(1)} \\ 2^{(2)}, 3^{(1)}, 1^{(1)} \end{pmatrix}$, and $\begin{pmatrix} 2^{(1)}, 3^{(2)}, 1^{(1)} \\ 2^{(2)}, 1^{(1)}, 3^{(2)} \end{pmatrix}$ are Wilf equivalent.

Our next approach involves enumerating $\mathrm{Av}_{n,r}(T)$ directly. We are able to prove formulas for any collection of patterns T as long as the individual elements in T are uni-colored.

DEFINITION 3.13 In $S_p \wr C_k$, if every element σ_i of a pattern σ is colored with a constant s_i, we will say that σ is *uni-colored.*

If T is a collection of uni-colored patterns in $S_p \wr C_k$ then T_j denotes the subset of elements in T which are uni-colored with the color j. Then for a general element in $S_n \wr C_r$, $n \geq p$, let i_h, $h \in \mathbb{N}$, denote the number of permutation elements colored h where h appears in the pattern for $1 \leq h \leq k$. Let i_{k+1} denote the number of elements in the permutation colored some color not included in the patterns in T. Using these definitions we can state the following theorem.

THEOREM 3.14 *Let T denote a set of uni-colored patterns, and $S_p(T_j^-)$ be the set of permutations in the symmetric group of order p that avoid the uncolored patterns T_j^-. Then, for $r \geq k$,*

$$|\operatorname{Av}_{n,r}(T)| = \sum_{\substack{i_j \geq 0 \\ i_1+\cdots+i_{k+1}=n}} (r-k)^{i_{k+1}} \binom{n}{i_1, ..., i_k, i_{k+1}}^2 (i_{k+1})! \prod_{j=1}^{k} |S_{i_j}(T_j^-)|.$$

Proof. We provide a combinatorial proof. We will show that the above formula enumerates the elements of $S_n \wr C_r$ that avoid all uni-colored patterns in T. The factor $\binom{n}{i_1,...,i_k,i_{k+1}}^2$ is the number of ways to choose the positions of the elements colored j, $1 \leq j \leq k$ in the wreath product element. It is also the number of ways to choose which numbers from $[n]$ will occur in the underlying permutation at those positions. The factor $|S_{i_j}(T_j^-)|$ is the number of ways to then order these numbers in these positions in such a way as to ensure that they avoid T_j^-. Finally, the factor $(i_{k+1})!$ is the number of ways to arrange the elements of the permutation colored some number not included in the patterns, and the factor $(r-k)^{i_{k+1}}$ is the number of ways to choose colors for those remaining elements. □

EXAMPLE 3.15 Say we want to find $|\operatorname{Av}(T)|$ in $S_4 \wr C_3$ when

$$T = \{(1^{(1)}, 2^{(1)}, 3^{(1)}), (1^{(2)}, 3^{(2)}, 2^{(2)})\}.$$

Then $n = 4$, $r = 3$, $p = 3$, and $k = 2$. Our formula calculates this amount as

$$|\operatorname{Av}_{4,3}(T)| = \sum_{\substack{i_1,i_2,i_3\geq 0\\ i_1+i_2+i_3=4}} (3-2)^{i_3}\binom{4}{i_1,i_2,i_3}^2 (i_3)!\prod_{j=1}^{2}|S_{i_j}(T_j^-)|.$$

If we restrict Theorem 3.14 to two colors a_1 and a_2 such that $T_{a_1}\cup T_{a_2} = T$, then we have the following:

Corollary 3.16

$$\begin{aligned}&|\operatorname{Av}_{n,r}(T)|\\ =&\sum_{i=0}^{n}\sum_{j=0}^{n-i}\binom{n}{i}\binom{n}{i}|S_i(T_{a_1}^-)|\binom{n-i}{j}\binom{n-i}{j}|S_j(T_{a_2}^-)|(n-i-j)!(r-2)^{n-i-j}.\end{aligned}$$

We can further restrict Theorem 3.14 to two single uni-colored patterns in $S_3\wr C_2$ with different colors. Note that because our patterns are of length three, $|S_i(T_1^-)| = C_i$ and $|S_j(T_2^-)| = C_j$ where C_i and C_j denote the ith and jth Catalan numbers respectively.

Corollary 3.17

$$\begin{aligned}&|\operatorname{Av}_{n,r}(T)|\\ =&\sum_{i=0}^{n}\sum_{j=0}^{n-i}\binom{n}{i}\binom{n}{i}C_i\binom{n-i}{j}\binom{n-i}{j}C_j(n-i-j)!(r-2)^{n-i-j}.\end{aligned}$$

The formula in Corollary 3.17 can be written in terms of a hypergeometric series.

$$|\operatorname{Av}_{n,r}(T)| = \sum_{i=0}^{n}(r-2)^{n-i}\binom{n}{i}^2(n-i)!|S_i(T_1^-)|\ \ {}_2F_2\left(\begin{matrix}\frac{1}{2},-n+i\\ 1,2\end{matrix};\frac{-4}{r-2}\right).$$

Using these formulas we are able to prove that certain blocks are indeed within the same Wilf class. One particular Wilf class contains 7 blocks. Here is a list of representative patterns from these blocks:

$$\begin{pmatrix}1^{(1)},2^{(1)},3^{(1)}\\ 1^{(2)},2^{(2)},3^{(2)}\end{pmatrix}\begin{pmatrix}1^{(1)},2^{(1)},3^{(1)}\\ 1^{(2)},3^{(2)},2^{(2)}\end{pmatrix}\begin{pmatrix}1^{(1)},2^{(1)},3^{(1)}\\ 2^{(2)},3^{(2)},1^{(2)}\end{pmatrix}\begin{pmatrix}1^{(1)},2^{(1)},3^{(1)}\\ 3^{(2)},2^{(2)},1^{(2)}\end{pmatrix}$$

$$\begin{pmatrix}1^{(1)},3^{(1)},2^{(1)}\\ 2^{(2)},1^{(2)},3^{(2)}\end{pmatrix}\begin{pmatrix}1^{(1)},3^{(1)},2^{(1)}\\ 3^{(2)},1^{(2)},2^{(2)}\end{pmatrix}\begin{pmatrix}1^{(1)},3^{(1)},2^{(1)}\\ 1^{(2)},3^{(2)},2^{(2)}\end{pmatrix}$$

Note that we cannot form bijections between these pattern pairs using any of the defined maps. However, the patterns of length three are all Wilf equivalent in S_n and are enumerated by the Catalan numbers. Therefore the equation from Corollary 3.17 enumerates the avoidance classes for each of these pattern pairs, thereby proving their Wilf equivalence. With these formulas we have therefore lowered the upper bound on the number of Wilf classes from 112 to 106.

4 A bijection to uncolored symmetric permutations avoiding decreasing subsequences of length 5

The formula enumerating the set $\mathrm{Av}_{n,2}(\pi_1, \pi_2)$ where π_1 and π_2 are unicolored elements of $S_3 \wr C_2$ with distinct colors is the following:

$$|\operatorname{Av}_{n,2}(\pi_1, \pi_2)| = \sum_{i=0}^{n} \binom{n}{i}\binom{n}{i} C_i C_{n-i}$$

according to corollary 3.15. This formula is identical to that enumerating elements of S_{2n} invariant under the reverse-complement map and avoiding a decreasing pattern of length five in the usual sense. This latter set was studied by Egge in [1]. We present a bijection between these two sets, by first obtaining an element of $\mathrm{Av}_{n,2}((3^{(1)}, 2^{(1)}, 1^{(1)}), (3^{(2)}, 2^{(2)}, 1^{(2)}))$, and then forming a bijection to $\mathrm{Av}_{n,2}(\pi_1, \pi_2)$.

Our bijection is formulated by altering the method of enumeration used by Egge. His method involves a series of bijections, the last of which can be manipulated and redefined to identify elements of $\mathrm{Av}_{n,2}\left(\left(\begin{array}{c} 3^{(1)}, 2^{(1)}, 1^{(1)} \\ 3^{(2)}, 2^{(2)}, 1^{(2)} \end{array}\right)\right)$. We include only the definitions crucial to our purpose. For a thorough description of Egge's method, see [1].

Let $S_n^{rc}(54321)$ denote the set of permutations in S_n invariant under the reverse complement map and avoiding the pattern 54321. Let R_n denote the set of all pairs of ordered pairs of standard tableaux, $\{(P_o, P_e), (Q_o, Q_e)\}$, satisfying the following conditions.

1. Each element of $[n]$ appears in exactly one of P_o, P_e and again in exactly one of Q_o, Q_e.

2. P_o, P_e, Q_o, and Q_e each have exactly 2 rows, but in each case one or both of these rows may have length 0.

3. P_o and Q_o are of shape λ_1 while P_e and Q_e are of shape λ_2.

The first two conditions are [1, Proposition 5.3], and the third condition is another result of Egge's construction.

Egge defined a series of bijections based on the RSK correspondence and Schutzenberger's jeu de taquin, which start with an element of $S_n^{rc}(54321)$ and result in an element of R_n. Once this pair of ordered pairs of tableaux, $\{(P_o, P_e), (Q_o, Q_e)\}$, has been identified, our work diverges from that of Egge.

We have two pairs of standard tableaux, (P_o, P_e) and (Q_o, Q_e). Because of Egge's construction of these tableaux, P_o and Q_o are of shape λ_1 while P_e and Q_e are of shape λ_2. We rearrange the ordered pairs according to their shape so that we now have (P_o, Q_o) and (P_e, Q_e). Since the tableaux in each ordered pair are now of the same shape and size, we can consider them as input for the inverse of the RSK correspondence. Note that since each tableau has no more than 2 rows, the permutation obtained via inverse RSK has no decreasing subsequence of length 3 or greater, i.e. avoids the pattern 321.

Let n be fixed but arbitrary. Define a map,

$$b : R_n \to \mathrm{Av}_{n,2}\left(\begin{pmatrix} 3^{(1)}, 2^{(1)}, 1^{(1)} \\ 3^{(2)}, 2^{(2)}, 1^{(2)} \end{pmatrix}\right)$$

by $b(\{(P_o, Q_o), (P_e, Q_e)\}) = \sigma$ if the following hold.

1. The entries of P_o are the elements of σ^- that are colored 1 in σ.

2. The entries of P_e are the elements of σ^- that are colored 2 in σ.

3. The position, in σ, of the entries of P_o are determined by the entries of Q_o via the inverse of the RSK correspondence.

The inverse of the RSK correspondence, applied to the tableaux, would normally result in two permutations, but since each element of $[n]$ appears only once in one of Q_o and Q_e, we instead allow these numbers to identify positions in a longer permutation containing the first two as subpermutations. One may find it helpful to imagine weaving the two permutations together according to the instructions given by Q_o and Q_e.

THEOREM 4.1 *The map b is a bijection.*

Proof. This bijection is essentially the inverse of the RSK correspondence. The difference is that we have two uni-colored permutations that we are weaving together to form a single colored permutation. The RSK correspondence is a bijection and b inherits this property.

To convince ourselves that the elements of the codomain avoid $(3^{(1)}, 2^{(1)}, 1^{(1)})$ and $(3^{(2)}, 2^{(2)}, 1^{(2)})$, we appeal to a result of Schensted in [8]. The number of rows in a tableau obtained through the RSK correspondence is exactly the length of the longest decreasing subsequence of the initial permutation. Since we are starting with tableaux with two or less rows, the longest decreasing subsequence colored either 1 or 2 is of length two or less. Therefore the resulting permutation avoids $(3^{(1)}, 2^{(1)}, 1^{(1)})$ and $(3^{(2)}, 2^{(2)}, 1^{(2)})$. □

As an example to illustrate this bijection, consider the permutation $\pi = 82463571$. Inspection will verify that π is invariant under the reverse complement map and avoids 54321. Thus $\pi \in S_8^{rc}(54321)$. The set $\{(P_o, Q_o), (P_e, Q_e)\}$ is associated with π via Egge's bijections. These tableaux are pictured in Figure 6.

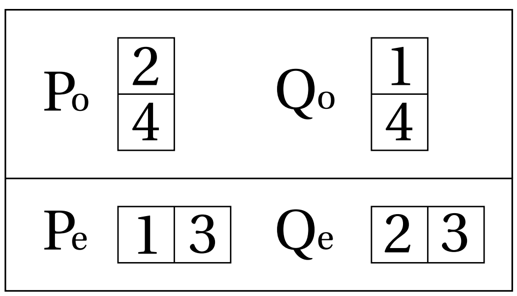

Figure 6: The tableaux P_o and Q_o (shown above) and P_e and Q_e (shown below) associated with the permutation $82463571 \in S_n^{rc}(54321)$.

The image will be a colored permutation of length 4 so we start with 4 empty positions: _ _ _ _. We then look to Q_o and use the RSK correspondence to determine where the entries of P_o will be placed. We color them with 1s:

$$\underline{4^{(1)}}\ \underline{\quad}\ \underline{\quad}\ \underline{2^{(1)}}.$$

Next we look to Q_e and use the RSK correspondence to determine where the entries of P_e will be placed. We color them with 2s.

$$\underline{4^{(1)}}\ \underline{1^{(2)}}\ \underline{3^{(2)}}\ \underline{2^{(1)}}.$$

From the pairs of tableaux given in Figure 6, we obtain the permutation $(4^{(1)}, 1^{(2)}, 3^{(2)}, 2^{(1)})$ which avoids $((3^{(1)}, 2^{(1)}, 1^{(1)}), (3^{(2)}, 2^{(2)}, 1^{(2)}))$. If we want to obtain an element that avoids another pair of uni-colored patterns with distinct colors, we simply apply the bijections between uncolored patterns of length 3 to the subsequences of each color.

5 Further Research and Acknowledgments

It would seem natural to ask for the generating function for the sequence generated by these avoidance classes for various n described in Section 4, but such a function has not been derived. Nor has Egge derived a function for his set of symmetric permutations enumerated by the same formula. Thus the matter of finding a generating function, ordinary or exponential, is an open question stemming from our work.

There are other open questions on finding the number of elements which avoid non-uni-colored patterns and pattern pairs. Finally, some work remains on the task of tightening the bounds on the number of Wilf classes. Another set of open questions are, for each Wilf class, how many colored permutations avoid that pair of patterns?

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