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Permutation q-enumeration with the Schur row adder

Adriano M. Garsia
Department of Mathematics
University of California, San Diego
United States of America
e-mail: garsia@math.ucsd.edu

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Abstract. We q-enumerate here, by the i-major index, the class of permutations of S_n with largest increasing subsequence of size $n-k$ and increasing first $n-k$ entries. The result is obtained by a surprisingly straightforward use of the Schur row adder. Some closely related open problems are also included.

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1 Introduction

Let us denote by $\mathbf{\Pi}_{n,k}$ the collection of permutations of $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in S_n$ with

$$\sigma_1 < \sigma_2 < \ldots < \sigma_{n-k} \tag{1}$$

and no increasing subsequence of length $> n-k$. Let $\mathbf{C}_{nk}$ denote the collection of signed colored permutations of S_n such that

1) *each entry is red or blue*

2) *the blue entries are increasing*

3) *the first $n-k$ entries are blue*

4) *the sign is $(-1)^{\#blue\ last\ k}$*

We can construct the elements of $\mathbf{C}_{nk}$ by first choosing which r of the last k entries will be red, then filling these entries in all possible ways with r of the integers $1, 2, \ldots, n$ and finally placing the complementary $n-r$ entries with the remaining integers in increasing order. This creates elements of $\mathbf{C}_{nk}$ with $k-r$ blue entries among the last k. Thus follows that we have

$$\sum_{\sigma\in\mathbf{C}_{nk}} sign(\sigma) \;=\; \sum_{r=0}^{k}\binom{k}{r}(-1)^{k-r}n(n-1)\cdots(n-r+1) \qquad (2)$$

Now it was noticed in [2], and proved, using the Theory of Character Polynomials, that

$$\#\mathbf{\Pi}_{nk} \;=\; \sum_{r=0}^{k}\binom{k}{r}(-1)^{k-r}n(n-1)\cdots(n-r+1). \qquad (3)$$

This identity strongly suggests that $\#\mathbf{\Pi}_{nk}$ could be enumerated by using this surprising connection of $\mathbf{\Pi}_{nk}$ with $\mathbf{C}_{nk}$ to obtain (3) as a simple direct application of the inclusion exclusion principle. To this date no such proof has been found.

We will prove here that if we set

$$\Pi_{nk}(q) \;=\; \sum_{\sigma\in\mathbf{\Pi}_{nk}} q^{imaj(\sigma)}$$

where $imaj(\sigma)=maj(\sigma^{-1})$, we also have

$$\Pi_{nk}(q) \;=\; \sum_{r=0}^{k}\binom{k}{r}(-1)^{k-r}[n]_q[n-1]_q\cdots[n-r+1]_q. \qquad (4)$$

In march 2008, in a talk at the MIT combinatorics seminar, the author offered 100$ for an elementary proof of (3). Subsequently a number of tentative solutions were proposed to the author. The winner solution was submitted by Greta Panova who was able to extend her method to obtain also the q-analogue in (4).

We will present here our original symmetric function proof of (4) since the identities and manipulations involved in the proof should be at least as significant as the result itself. We will leave it to Panova to present her proof of (4) in a separate publication.

The crucial ingredient in our proof is the symmetric function operator

$$\mathbf{H}_m \;=\; \sum_{r\geq 0}(-1)^r\, h_{m+r}e_r^{\perp}. \qquad (5)$$

where "$e_r^\perp$" denotes the operator adjoint of multiplication by e_r with respect to the Hall scalar product. In particular we will use here the fact that

$$e_r^\perp s_\lambda = \begin{cases} s_{\lambda/1^r}, & \text{if } r \leq l(\lambda); \\ 0, & \text{otherwise.} \end{cases}$$

It is well known [10] that if $\mu \vdash k$ and $n-k$ is at least as large as the first part of μ we have

$$s_{(n-k,\mu)}(x) = \mathbf{H}_{n-k} s_\mu(x). \tag{6}$$

In other words we can use $\mathbf{H}_m$ to add a row of length m to the index of a Schur function. For this reason $\mathbf{H}_m$ is sometimes referred to as the "*Schur row adder*". It also goes by such fancy names as a "*vertex operator*". All this not withstanding, the identity in (6) is easily seen to be but an immediate consequence of the well known Jacobi Trudi identities expressing a skew Schur function in terms of the homogeneous symmetric function basis. To see this simply note that we have

$$\begin{aligned} s_{3,3,2} &= \det \begin{pmatrix} h_3 & h_4 & h_5 \\ h_2 & h_3 & h_4 \\ 1 & 1 & h_2 \end{pmatrix} \\ &= h_3 \det \begin{pmatrix} h_3 & h_4 \\ h_1 & h_2 \end{pmatrix} - h_4 \det \begin{pmatrix} h_2 & h_4 \\ 1 & h_2 \end{pmatrix} + h_5 \det \begin{pmatrix} h_2 & h_3 \\ 1 & h_1 \end{pmatrix}. \end{aligned} \tag{7}$$

Now it follows again from the Jacobi Trudi identities that

$$\det \begin{pmatrix} h_3 & h_5 \\ h_1 & h_2 \end{pmatrix} = s_{32}\,, \quad \det \begin{pmatrix} h_2 & h_5 \\ 1 & h_2 \end{pmatrix} = s_{32/1}\,,$$

$$\det \begin{pmatrix} h_2 & h_3 \\ 1 & h_1 \end{pmatrix} = s_{32/11}$$

Thus (7) is none other than

$$s_{3,3,2} = h_3\, s_{32} - h_4\, s_{32/1} + h_5\, s_{32/11}$$

which is an instance of (6). This kind of proof of (7) may be more digestible to most practicioners of symmetric function theory. Actually, (6) is quite a bit more elementary than the Jacobi Trudi identities. Indeed, it can be shown (see [2]) that it is an immediate consequence of Frobenius original construction of the irreducible characters of S_n.

We will show in the end that the operator $\mathbf{H}_m$ can be also used to prove the existence of several q,t analogues of (4). We will leave as an open problem to discover the nature of the permutation statistics that are involved in the corresponding q,t-enumerations.

2 The q-enumeration

Before we can proceed with our proof, we need to make preliminary observations and establish some auxiliary identities.

Recall that a theorem of Schensted [11] states that the permutations of S_n with maximal increasing subsequences of length m are precisely those whose pairs of tableaux under the Robinson Schensted correspondence have a first row of length m. Moreover, if $\sigma \to (P,Q)$ under RS then the condition in (1) holds if and only if Q has

$$1,2,\dots n-k$$

in its first row. It follows then that to construct an element $\sigma \in \mathbf{\Pi}_{n,k}$ we simply take a pair of standard tableaux P,Q of shape $(n-k,\mu)$ with Q having $1,2,\dots n-k$ in the first row and then get σ by reversing the Robinson Schensted correspondence on the pair (P,Q). More precisely, we have

$$\begin{aligned} \mathbf{\Pi}_{n,k} \;&=\; RS^{-1}\bigcup_{\mu\vdash k}\Big\{(P,Q)\,:\,P,Q\in ST(n-k,\mu)\ \&\\ &\qquad \text{the first row of } Q \text{ is } 1,2,\dots n-k\Big\} \end{aligned} \tag{8}$$

Note that to obtain a standard tableau of shape $(n-k,\mu)$ with $1,2,\dots,n-k$ in its first row we need only take a standard tableau of shape μ, add $n-k$ to its entries and prepend to the resulting tableau a row of length $n-k$ containing $1,2,\dots n-k$ from left to right. It follows from this simple observation, that, at least for $n \geq 2k^{(\dagger)}$, the polynomial $\Pi_{n,k}(q)$ may be computed by means of the following identity.

Proposition 2.1

$$\Pi_{n,k}(q) \;=\; \sum_{\mu\vdash k} f_\mu \sum_{P\in ST(n-k,\mu)} q^{maj(P)}. \tag{9}$$

$^{(\dagger)}$To assure that for each $\mu \vdash k$, $(n-k,\mu)$ is a partition

Proof. It is well known that if, under the RS correspondence, a permutation σ maps to the pair (P,Q) then the descent set of the inverse of σ is given by the descent set of P. The latter being the set of i such that $i+1$ is NW of in P (in the french depiction of P). Thus (9) is an immediate consequence of (8). □

This given, the proof of (4) is reduced to symmetric function manipulations by the following auxiliary result.

THEOREM 2.2 *Setting*

$$\phi_{n,k} \;=\; \sum_{\mu\vdash k} f_\mu s_{n-k,\mu} \tag{10}$$

we have, for $n \geq 2k$

$$\Pi_{n,k}(q) \;=\; \big\langle \phi_{n,k}\,,\, \tilde{H}_n \big\rangle \tag{11}$$

where $\langle\, ,\, \rangle$ denotes the Hall scalar product of symmetric functions and $\tilde{H}_n$ denotes the graded Frobenius characteristic of the Harmonics of S_n.

$$\tilde{H}_n(x;q) \;=\; (q,q)_n \sum_{\lambda\vdash n} s_\lambda(x) s_\lambda\big[\tfrac{1}{1-q}\big] \;=\; \sum_{\lambda\vdash n} s_\lambda(x) \sum_{P\in ST(\lambda)} q^{maj(P)} \tag{12}$$

with

$$(q,q)_n \;=\; (1-q)(1-q^2)\cdots(1-q^n).$$

In particular, using the the identity $e_1^n = \sum_{\lambda\vdash n} f_\lambda s_\lambda$ ([10] p. 114 ,(7.6)) we also derive that

$$\#\mathbf{\Pi}_{n,k} \;=\; \Pi_{n,k}(1) \;=\; \big\langle \phi_{n,k}\,,\, e_1^n \big\rangle. \tag{13}$$

Proof. Using the expansions in (10) and (12), the orthonormality of the Schur function basis with respect to the Hall scalar product gives

$$\big\langle \phi_{n,k}\,,\, \tilde{H}_n \big\rangle \;=\; \sum_{\mu\vdash k} f_\mu \big\langle s_{n-k,\mu}\,,\, \tilde{H}_n \big\rangle \;=\; \sum_{\mu\vdash k} f_\mu \sum_{P\in ST(n-k,\mu)} q^{maj(P)}$$

Thus (11) is only another way of writing (9). □

To compute the scalar product in (11) we will use the “row adding ” formula in (6). In fact, this enables us to convert (11) and (13) into the following identities.

Proposition 2.3

$$\Pi_{n,k}(q) \;=\; \sum_{s=0}^{k} \binom{k}{s} (-1)^{k-s} \left\langle h_{n-s} e_1^s \,,\, \tilde{H}_n \right\rangle \tag{14}$$

$$\#\mathbf{\Pi}_{n,k} \;=\; \sum_{s=0}^{k} \binom{k}{s} (-1)^{k-s} \left\langle h_{n-s} e_1^s \,,\, e_1^n \right\rangle \tag{15}$$

Proof. Using (6) we may write

$$\phi_{n,k} \;=\; \sum_{\mu\vdash k} f_\mu s_{n-k,\mu} \;=\; \sum_{\mu\vdash k} f_\mu \, \mathbf{H}_{n-k}\, s_\mu \;=\; \mathbf{H}_{n-k} \sum_{\mu\vdash k} f_\mu \, s_\mu \;=\; \mathbf{H}_{n-k} e_1^k. \tag{16}$$

But (5) gives

$$\mathbf{H}_{n-k} e_1^k \;=\; \sum_{r\geq 0} (-1)^r \, h_{n-k+r} e_r^{\perp} e_1^k$$

and since $e_1 = p_1$ we have (by the linearity of the "$\perp$" operator)

$$e_r^{\perp} e_1^k \;=\; \sum_{\rho\vdash r} \frac{p_\rho^{\perp} p_1^k}{z_\rho} \;=\; \frac{1}{r!} \partial_{p_1}^r p_1^k \;=\; \frac{1}{r!}\, k(k-1)\cdots(k-r+1)\, p_1^{k-r} \;=\; \binom{k}{r} e_1^{k-r}$$

(16) becomes

$$\phi_{n,k} \;=\; \sum_{r\geq 0} \binom{k}{r} (-1)^r \, h_{n-k+r} \, e_1^{k-r}. \tag{17}$$

Using this in (11) and (13) gives

$$\Pi_{n,k}(q) \;=\; \sum_{r\geq 0} \binom{k}{r} (-1)^r \left\langle h_{n-k+r} \, e_1^{k-r} \,,\, \tilde{H}_n \right\rangle$$

and

$$\#\mathbf{\Pi}_{n,k} \;=\; \sum_{r\geq 0} \binom{k}{r} (-1)^r \left\langle h_{n-k+r} \, e_1^{k-r} \,,\, e_1^n \right\rangle$$

and the proof of (14) and (15) is completed by the change of summation index $r \to k - s$. □

This brings us in a position to give our

Proof of (4). As a warm up let us show how we can quickly dispose of (3). In fact, (15) gives

$$\begin{aligned}\#\mathbf{\Pi}_{n,k} &= \sum_{s=0}^{k}\binom{k}{s}(-1)^{k-s}\left\langle h_{n-s}e_1^s\,,\,e_1^n\right\rangle\\ &= \sum_{s=0}^{k}\binom{k}{s}(-1)^{k-s}\left\langle h_{n-s}\,,\,(e_1^s)^{\perp}e_1^n\right\rangle\\ &= \sum_{s=0}^{k}\binom{k}{s}(-1)^{k-s}\,n(n-1)\cdots(n-s+1)\big\langle h_{n-s}\,,\,e_1^{n-s}\big\rangle\end{aligned}$$

and (9) follows since the orthonormality of the Schur functions gives

$$\big\langle h_{n-s}\,,\,e_1^{n-s}\big\rangle = \sum_{\mu\vdash n-s} f_\mu\big\langle s_{n-s}\,,\,s_\mu\big\rangle = 1.$$

The derivation of (4) is a little more elaborate. To begin we must recall that we have (see [10] Ch. 5)

$$\sum_{P\in ST(\lambda)} q^{maj(P)} = (q,q)_n s_\lambda(1,q,q^2,\ldots) = (q,q)_n s_\lambda\big[\tfrac{1}{1-q}\big].$$

Thus, using the Cauchy formula $h_n[AB]=\sum_{\lambda\vdash n}s_\lambda[A]s_\lambda[B]$ with $A=X$ and $B=\frac{1}{1-q}$, the expansion in (12) may also be rewritten in the simple plethystic form

$$\tilde{H}_n(x;q) = (q,q)_n\, h_n\big[\tfrac{X}{1-q}\big] \tag{18}$$

and we can write

$$\big\langle h_{n-s}e_1^s\,,\,\tilde{H}_n\big\rangle = (q,q)_n\big\langle h_{n-s}e_1^s\,,\,h_n\big[\tfrac{X}{1-q}\big]\big\rangle. \tag{19}$$

To compute the latter scalar product, we have a cute trick that considerably simplifies our calculations. Namely, it follows from the definition of the Hall scalar product that for any two power basis elements p_α, p_β we have

$$\begin{aligned}\big\langle p_\alpha[X]\,,\,p_\beta\big[\tfrac{X}{1-q}\big]\big\rangle &= \chi(\alpha=\beta)z_\beta p_\beta\big[\tfrac{1}{1-q}\big] = \chi(\alpha=\beta)z_\alpha p_\alpha\big[\tfrac{1}{1-q}\big]\\ &= \big\langle p_\alpha\big[\tfrac{X}{1-q}\big]\,,\,p_\beta[X]\big\rangle.\end{aligned}$$

Using this (19) becomes

$$\begin{aligned}\left\langle h_{n-s}e_1^s\,,\,\tilde H_n\right\rangle &= (q,q)_n\left\langle h_{n-s}\left[\tfrac{X}{1-q}\right]e_1^s\left[\tfrac{X}{1-q}\right]\,,\,h_n\right\rangle\\ &= \frac{(q,q)_n}{(1-q)^s}\left\langle h_{n-s}\left[\tfrac{X}{1-q}\right]e_1^s\,,\,h_n\right\rangle\\ &= \frac{(q,q)_n}{(1-q)^s}\left\langle h_{n-s}\left[\tfrac{X}{1-q}\right]\,,\,(e_1^s)^\perp h_n\right\rangle\\ &= \frac{(q,q)_n}{(1-q)^s}\left\langle h_{n-s}\left[\tfrac{X}{1-q}\right]\,,\,h_{n-s}\right\rangle \qquad (20)\end{aligned}$$

and since (18) and (12) yield

$$(q,q)_{n-s}\,h_{n-s}\left[\tfrac{X}{1-q}\right] \;=\; \tilde H_{n-s}(x;q) \;=\; \sum_{\lambda\vdash n-s} s_\lambda(x)\sum_{P\in ST(\lambda)} q^{maj(P)}$$

it follows that

$$\left\langle h_{n-s}\left[\tfrac{X}{1-q}\right]\,,\,h_{n-s}\right\rangle \;=\; \frac{1}{(q,q)_{n-s}}\left\langle \tilde H_{n-s}\,,\,h_{n-s}\right\rangle \;=\; \frac{1}{(q,q)_{n-s}}$$

and (20) becomes

$$\begin{aligned}\left\langle h_{n-s}e_1^s\,,\,\tilde H_n\right\rangle &= \frac{(q,q)_n}{(1-q)^s}\frac{1}{(q,q)_{n-s}} \;=\; \frac{1}{(1-q)^s}\frac{\prod_{i=1}^n(1-q^i)}{\prod_{i=1}^{n-s}(1-q^i)}\\ &= [n]_q[n-1]_q\cdots[n-s+1]_q.\end{aligned}$$

Using this in (14) completes our proof of (4). □

3 A q,t-enumeration problem

Garsia-Haiman in [4] introduced rescaled versions $\tilde H_\mu(X;q,t)$ of the Macdonald polynomials and conjectured that, for $\mu\vdash n$ they are the Frobenius characteristics of certain q,t-analogues of the left regular representation of S_n. In particular it follows that

$$\tilde H_\mu(X;q,t)\Big|_{t,q=1} \;=\; e_1^n. \qquad (21)$$

Moreover it is also shown in [4] that for μ a single row or a single column we have

$$\tilde H_n = (q,q)_n h_n\left[\tfrac{X}{1-q}\right], \qquad \tilde H_{1^n} = (t,t)_n h_n\left[\tfrac{X}{1-t}\right]. \qquad (22)$$

In view of our preceding considerations, these two facts suggest that we should also study, for each $\mu \vdash n$ the following q,t-analogues of the polynomial in (11), namely

$$\Pi_{\mu,k}(q,t) \;=\; \big\langle \phi_{n,k}\,,\, \tilde{H}_\mu \big\rangle. \tag{23}$$

Note that from (21) and our proof of (4) it follows that for all $\mu \vdash n$ we have

$$\Pi_{\mu,k}(q,t)\Big|_{t,q=1} \;=\; \sum_{r=0}^{k} \binom{k}{r} (-1)^{k-r} n(n-1)\cdots(n-r+1). \tag{24}$$

Moreover, it is also shown in [4] that we have the Schur function expansions

$$\tilde{H}_\mu(X;q,t) \;=\; \sum_{\lambda \vdash n} s_\lambda(X) \tilde{K}_{\lambda\mu}(q,t) \tag{25}$$

where

$$\tilde{K}_{\lambda\mu}(q,t) \;=\; T_\mu K_{\lambda\mu}(q,1/t)$$

with $K_{\lambda\mu}(q,t)$ the original [10] Macdonald q,t Kostka coefficients and $T_\mu = t^{n(\mu)} q^{n(\mu')(\dagger)}$. Since the Garsia-Haiman conjectures were eventually proved by Mark Haiman in [9] it is now a theorem that for each pair λ, μ the polynomial $\tilde{K}_{\lambda\mu}(q,t)$ has positive integer coefficients. Now using (25) in (23), the definition in (10) (with μ replaced by ν) gives

$$\Pi_{\mu,k}(q,t) \;=\; \sum_{\nu \vdash k} f_\nu \tilde{K}_{(n-k,\nu),\mu}(q,t). \tag{26}$$

Thus it is also a theorem (albeit not at all trivial) that this polynomial has also positive integer coefficients. So the question then arises:

What is $\Pi_{\mu,k}(q,t)$ q,t*-enumerating and by what statistics?*

A strong clue in trying to answer this question is the result of Haglund-Haiman-Loehr in [8] yielding a combinatorial formula for the polynomials $\tilde{H}_\mu(X;q,t)$. To be more explicit, we need to give a version of the Haglund-Haiman-Loehr construction in a notation that is more suitable to the present context.

Chosen a $\mu \vdash n$, for each permutation $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in S_n$, let $T_\mu(\sigma)$ be the injective tableau obtained by filling the cells of the (french)

(†) For $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ we set $n(\mu) = \sum_{i=1}^{k} (i-1)\mu_i$

Ferrers diagram of μ with the entries $\sigma_1, \sigma_2, \ldots, \sigma_n$, by rows proceeding from top to bottom and from left to right. Now, denoting by $T_{i,j}$ the entry of a tableau T that is in the i^{th} row (starting from the bottom) and in the j^{th} column (from left to right), Jim Haglund introduced, for $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$, the statistic

$$inv_\mu(T) \;=\; \sum_{1 \le i < j \le \mu_1} \chi\big(T_{1,i} > T_{1,j}\big) + \sum_{r=2}^{k} \sum_{1 \le i < j \le \mu_r} E(T_{r-1,i}, T_{r,i}, T_{r,j}). \tag{27}$$

Where for three distinct integers a, b, c he sets

$$E(a,b,c) \;=\; \chi(a < c < b) + \chi(b < a < c) + \chi(c < b < a).$$

More visually, (27) simply consists adding to the number of inversions in the first row of T, the number of triplets $\left[\begin{smallmatrix} T_{r,i} & \cdots & T_{r,j} \\ T_{r-1,i} & & \end{smallmatrix}\right]$ of T that admit an increasing counterclockwise cyclic rearrangement. Denoting by $C_j(T)$ the word obtained by reading the j^{th} column of T from top to bottom, Jim Haglund also sets

$$maj_\mu(T) \;=\; \sum_{j=1}^{\mu_1} maj\big(C_j(T)\big). \tag{28}$$

This given, Haglund-Haiman-Loehr in [8] prove that

$$\tilde{H}_\mu(X;q,t) \;=\; \sum_{\sigma \in S_n} q^{inv_\mu\left(T(\sigma)\right)} t^{maj_\mu\left(T(\sigma)\right)} Q_{ides(\sigma)}[X] \tag{29}$$

where for a subset $S \subseteq [1, 2, \ldots, n-1]$ the symbol $Q_S[X]$ denotes the corresponding Gessel quasi-symmetric polynomial and "$ides(\sigma)$" denotes the descent set of σ^{-1}. For our needs here we do not need the explicit definition of $Q_S[X]$ since Gessel in [7] showed that if the polynomial

$$P[X] \;=\; \sum_{S \subseteq [1,2,\ldots,n-1]} C_S Q_S[X]$$

is symmetric then for any $p = (p_1, p_2, \ldots, p_k)$ composition of n, then the Hall scalar product

$$\big\langle P\, ,\, h_{p_1} h_{p_2} \cdots h_{p_{k-1}} h_{p_k} \big\rangle$$

evaluates to the sum

$$\sum_{S \subseteq \{p_1, p_1+p_2, \ldots, p_1+p_2+\cdots+p_{k-1}\}} C_S.$$

In particular, using this fact together with (29), we can immediately derive a combinatorial interpretation for the scalar product

$$\left\langle \tilde{H}_\mu \, , \, h_{n-s} e_1^s \right\rangle$$

as being none other than the sum

$$\begin{aligned} & \left\langle \tilde{H}_\mu \, , \, h_{n-s} h_1^s \right\rangle \\ = & \sum_{\sigma \in S_n} q^{inv_\mu\left(T(\sigma)\right)} t^{maj_\mu\left(T(\sigma)\right)} \chi(ides(\sigma) \subseteq \{n-s+1, \ldots, n-1, n\}) \\ = & \sum_{\sigma \in S_n} q^{inv_\mu\left(T(\sigma^{-1})\right)} t^{maj_\mu\left(T(\sigma^{-1})\right)} \chi(\sigma_1 < \sigma_2 < \ldots < \sigma_{n-s}) \end{aligned} \tag{30}$$

This given, it is easy to see that, starting from the definition in (23) and using (17), the same calculations we carried out in the proof of Proposition (9) yield us the identity

$$\begin{aligned} & \Pi_{\mu,k}(q,t) \\ = & \sum_{s=0}^{k} \binom{k}{s} (-1)^{k-s} \sum_{\sigma \in S_n} q^{inv_\mu\left(T(\sigma^{-1})\right)} t^{maj_\mu\left(T(\sigma^{-1})\right)} \chi(\sigma_1 < \sigma_2 < \ldots < \sigma_{n-s}) \end{aligned} \tag{31}$$

which is precisely a q,t-analogue of the right-hand side of (2). That is we have

$$\Pi_{\mu,k}(q,t) \;=\; \sum_{\sigma \in \mathbf{C}_{nk}} sign(\sigma) q^{inv_\mu\left(T(\sigma)\right)} t^{maj_\mu\left(T(\sigma)\right)} \tag{32}$$

Since we have noted that the resulting polynomial has positive coefficients we are again led to conclude that there must be another inclusion-exclusion mechanism underlying this identity.

The first impulse is to suspect that (31) should simply reduce to

$$\Pi_{\mu,k}(q,t) \;=\; \sum_{\sigma \in \mathbf{\Pi}_{n,k}} q^{inv_\mu\left(T(\sigma)\right)} t^{maj_\mu\left(T(\sigma)\right)}. \tag{33}$$

In fact, computer experimentations, have revealed that this is true in a variety of cases but not true in full generality. We are grateful to Greta Panova for pointing out this fact. This circumstance adds a further puzzling aspect to our q,t-enumeration problem.

Representing a polynomial

$$\sum_{i=0}^{2}\sum_{j=0}^{3} c_{i,j} t^i q^j$$

by the matrix

$$\begin{bmatrix} c_{2,0} & c_{2,1} & c_{2,2} & c_{2,3} \\ c_{1,0} & c_{1,1} & c_{1,2} & c_{1,3} \\ c_{0,0} & c_{0,1} & c_{0,2} & c_{0,3} \end{bmatrix}$$

Here are a few samples:

$$\Pi_{[2,2],2}(q,t) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \Pi_{[3,2],2}(q,t) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\Pi_{[4,2],2}(q,t) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 3 & 1 & 0 \\ 0 & 0 & 1 & 2 & 3 & 1 \end{bmatrix}$$

$$\Pi_{[5,2],2}(q,t) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 4 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 & 4 & 2 & 1 \end{bmatrix}$$

$$\Pi_{[3,3],2}(q,t) = \begin{bmatrix} 1 & 2 & 2 & 1 & 0 \\ 0 & 2 & 3 & 3 & 1 \\ 0 & 0 & 1 & 2 & 1 \end{bmatrix}, \quad \Pi_{[4,3],2}(q,t) = \begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 2 & 4 & 5 & 3 & 1 \\ 0 & 0 & 1 & 2 & 3 & 2 \end{bmatrix}$$

$$\Pi_{[5,3],2}(q,t) = \begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 6 & 5 & 3 & 1 & 0 \\ 0 & 0 & 1 & 2 & 3 & 4 & 3 & 1 \end{bmatrix}$$

$$\Pi_{[6,3],2}(q,t) = \begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 6 & 6 & 5 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 & 4 & 5 & 4 & 2 & 0 \end{bmatrix}$$

$$\Pi_{[5,3],3}(q,t) = \begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 9 & 14 & 14 & 9 & 4 & 1 & 0 & 0 \\ 0 & 0 & 3 & 9 & 18 & 23 & 20 & 12 & 4 & 1 \\ 0 & 0 & 0 & 1 & 3 & 6 & 9 & 11 & 8 & 3 \end{bmatrix}$$

Note that, even if we identify what is being q,t-counted, to complete the picture we should try to get an explicit formula for the polynomial

$$\Gamma_{\mu,s}(q,t) \;=\; \langle \tilde{H}_\mu \,,\, h_{n-s}h_1^s \rangle \tag{34}$$

which, if none other, should be a q,t-analogue of $n(n-1)\cdots(n-s+1)$.

It is interesting to note that some of the Macdonald polynomial machinery developed in [1] and [3], may be used to solve also this portion of our puzzle. To see how this comes about, we begin by writing $\Gamma_{\mu,s}(q,t)$ in the form

$$\Gamma_{\mu,s}(q,t) \;=\; \langle e_1^\perp \tilde{H}_\mu \,,\, h_{n-s}h_1^{s-1} \rangle \tag{35}$$

then use the "*dual Pieri Rule*" (see [5])

$$e_1^\perp \tilde{H}_\mu(X;q,t) \;=\; \sum_{\nu\to\mu} c_{\mu\nu}(q,t)\tilde{H}_\nu \;\;^{(\dagger)} \tag{36}$$

and obtain the recursion

$$\Gamma_{\mu,s}(q,t) \;=\; \sum_{\nu\to\mu} c_{\mu\nu}(q,t)\,\Gamma_{\mu,s-1}(q,t). \tag{37}$$

Explicit combinatorial expressions for the coefficients $c_{\mu\nu}(q,t)$ were derived in [6] from the original [10] Stanley-Macdonald Pieri rules. Since they turned out to be some pretty fierce rational functions, several identities were developed in [3] (see also [1]) for the evaluations of such sums as in (37). One of the summation formulas can be stated as follows

THEOREM 3.1 *Let ∇ be the operator whose action on symmetric functions is defined by setting for all μ*

$$\nabla \tilde{H}_\mu \;=\; T_\mu \tilde{H}_\mu \tag{38}$$

and for each partition μ set

$$B_\mu(q,t) \;=\; \sum_{(i,j)\in\mu} t^{j-1}q^{i-1} \tag{39}$$

where "$(i,j)\in\mu$" is to signify that the sum is over the cells of the Ferrers diagram of μ with i,j the cartesian coordinates of the NE corner of a cell. Then for any symmetric function F we have

$$\sum_{\nu\to\mu} c_{\mu\nu}(q,t)\,F[MB_\nu - 1] \;=\; G[MB_\mu - 1]$$

$$\text{(with } M=(1-t)(1-q)\text{)},$$

(†) "$\nu \to \mu$" is to signify that ν is obtained from μ by removing a corner square *where*

$$G[X] \;=\; \nabla^{-1}\tfrac{e_1+1}{M}\nabla F[X]. \tag{40}$$

Now it was shown in [4] that we have

$$\sum_{\nu\to\mu} c_{\mu\nu}(q,t) \;=\; B_\mu(q,t) \tag{41}$$

and since we also have for $\mu \vdash n$

$$\Gamma_{\mu,0}(q,t) \;=\; \langle \tilde{H}_\mu\,,\, h_n\rangle \;=\; 1,$$

using (41) in (37) for $s=1$ we derive that

$$\Gamma_{\mu,1}(q,t) \;=\; B_\mu(q,t)$$

which is as beautiful a q,t-analogue of n as we could desire. Using this, the recursion in (37), for $s=2$ gives

$$\begin{aligned}
\Gamma_{\mu,2}(q,t) &= \sum_{\nu\to\mu} c_{\mu\nu}(q,t)B_\nu(q,t)\\
&= \sum_{\nu\to\mu} c_{\mu\nu}(q,t)\tfrac{e_1+1}{M}\big[MB_\nu - 1\big]\\
\text{(by Theorem 3.1)}\quad &= \nabla^{-1}\tfrac{e_1+1}{M}\nabla\tfrac{e_1+1}{M}\big[MB_\mu - 1\big]\\
&= \nabla^{-1}\left(\tfrac{e_1+1}{M}\right)^2\big[MB_\mu - 1\big]
\end{aligned}$$

since the definition in (38) gives

$$\nabla\tfrac{e_1+1}{M} \;=\; \tfrac{e_1+1}{M}$$

and we can easily see that iterating this process yields the general formula

$$\Gamma_{\mu,s}(q,t) \;=\; \nabla^{-1}\left(\tfrac{e_1+1}{M}\right)^s\big[MB_\mu - 1\big]. \tag{42}$$

This can be made more explicit by eliminating the presence of ∇ by means of the identity (see [3])

$$\nabla^{-1}e_1\nabla \;=\; D_1^* \tag{43}$$

where D_1^* is but one instance of the family of symmetric function operators defined by the plethystic formula

$$D_k^*\,F[X] = \Big(F\big[X - \tfrac{M}{qtz}\big]\sum_{m\geq 0} z^m h_m[X]\Big)\Big|_{z^k}. \tag{44}$$

In fact, since $\nabla 1 = 1$, using (43), we can write

$$\nabla^{-1} e_1^r \;=\; \nabla^{-1} e_1^r \, \nabla\, 1 \;=\; D_1^{*r}$$

and (42) becomes

$$\Gamma_{\mu,s}(q,t) \;=\; \Big(\tfrac{D_1^*+1}{M}\Big)^s \, 1\big[MB_\mu - 1\big] \tag{45}$$

yielding as a q,t-analogue of (4) the formula

$$\Pi_{\mu,k}(q,t) \;=\; \sum_{r=0}^{k} \binom{k}{r} (-1)^{k-r} \Big(\tfrac{D_1^*+1}{M}\Big)^s \, 1\big[MB_\mu - 1\big]. \tag{46}$$

We will leave it to further work to work out, if possible, a more explicit version of this identity.

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