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Some properties of non minimal permutations

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Abstract. In this paper we present a study on some properties of non minimal permutations of length n with $(n-2)$ descents. After giving the definition and the characterization of minimal permutations of size n with $(n-2)$ descents, we focus on the enumeration and on the study of some statistics of their complementary set, that is the non minimal permutations. We provide a bijective enumeration and two constructive algorithms for non minimal permutations of length n with $(n-2)$ descents. The former algorithm generates the permutations of size n by swapping their entries; the latter generates the permutations of length n by applying a particular "explosion" strategy to the ones of size $(n-1)$.

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1 Introduction

1.1 Preliminary definitions

A permutation of size n is a bijective map from $\{1 \dots n\}$ to itself. We denote by S_n the set of permutations of size n. We consider a permutation $\sigma \in S_n$ as the word $\sigma(1)\sigma(2)\dots\sigma(n)$ of n letters on the alphabet $\{1, 2, \dots, n\}$, containing each letter exactly once (we often use the word *element* or *entry* instead of letter). For example, $6\,2\,4\,3\,5\,1$ represents the permutation $\sigma \in S_6$ such that $\sigma(1) = 6, \sigma(2) = 2, \dots, \sigma(6) = 1$.

In [5] a very well-known statistics on permutations was introduced, i.e. the number of descents.

Definition 1.1 Let σ be a permutation in S_n. We say that σ has a *descent* in position i whenever $\sigma(i) > \sigma(i+1)$. In the same way, we say that σ has an *ascent* in position i whenever $\sigma(i) < \sigma(i+1)$.

Example 1.2 The permutation $\sigma = 6\,9\,8\,4\,1\,3\,7\,2\,5 \in S_9$ has 4 descents, namely in positions 2, 3, 4, 7, and 4 ascents in positions 1, 5, 6 and 8.

It is well known, [1], that the permutations of length n with d descents are enumerated by the Eulerian numbers, $\mathcal{E}(n,d)$ satisfying:

$$\mathcal{E}(n,d) = (n-d)\cdot\mathcal{E}(n-1,d-1) + (d+1)\cdot\mathcal{E}(n-1,d) \tag{1}$$

anchored by $\mathcal{E}(n,0) = 1$ and $\mathcal{E}(n,n-1) = 1$ for $n \geq 1$, [2, 7, 10, 11].

Definition 1.3 A permutation $\pi \in S_k$ is a *pattern* of a permutation $\sigma \in S_n$ if there is a subsequence of σ which is order-isomorphic to π; i.e., if there is a subsequence $\sigma(i_1)\sigma(i_2)\ldots\sigma(i_k)$ of σ (with $1 \leq i_1 < i_2 < \ldots < i_k \leq n$) such that $\sigma(i_\ell) < \sigma(i_m)$ whenever $\pi(\ell) < \pi(m)$.
We also say that π is *involved* in σ and call $\sigma(i_1)\sigma(i_2)\ldots\sigma(i_k)$ an *occurrence* of π in σ.

Example 1.4 For example $1\,2\,3\,4$ is a pattern of $\sigma = 3\,1\,2\,8\,5\,4\,7\,9\,6$ since $\sigma(2)\,\sigma(3)\,\sigma(5)\,\sigma(7)$ is an increasing subsequence of size 4.

We write $\pi \prec \sigma$ to denote that π is a pattern of σ. A permutation σ that does not contain π as a pattern is said to *avoid* π. The class of all permutations avoiding the patterns $\pi_1, \pi_2 \ldots \pi_k$ is denoted as $S(\pi_1, \pi_2, \ldots, \pi_k)$. We say that $S(\pi_1, \pi_2, \ldots, \pi_k)$ is a class of pattern-avoiding permutations of *basis* $\{\pi_1, \pi_2, \ldots, \pi_k\}$. The basis of a class of pattern-avoiding permutations may be finite or infinite. However pattern-avoiding permutation classes considered in the literature (see for example [4, 13] and their references) are often of finite basis.

Example 1.5 For example $\sigma = 1\,4\,2\,5\,6\,3$ is a permutation in $S(321)$ since no subsequence of size 3 of σ is isomorphic to 321.

1.2 Minimal permutations with d descents

Minimal permutations with d descents arise from biological motivations, [5, 6]. Among the many models for genome evolution, the *whole genome duplication - random loss model* represents genomes with permutations, that can evolve through *duplication-loss steps* representing the biological phenomenon that duplicates fragments of genomes, and then loses one copy of every duplicated gene, [3]. In [5] it was shown that the class of permutations obtained in this model after a given number p of steps is a class of pattern-avoiding permutations of finite basis; in particular the following theorem was proved.

THEOREM 1.6 *The class of permutations obtainable by at most p steps in the whole genome duplication - random loss model is a class of pattern-avoiding permutations whose basis $\mathcal{B}_d$ is finite and is composed of the minimal permutations with $d = 2^p$ descents, minimal being intended in the sense of $\prec$.*

In this section, we focus on the basis $\mathcal{B}_d$ of excluded patterns appearing in Theorem 1.6. More generally, we also study the case where d is not a power of 2.

DEFINITION 1.7 A permutation $\sigma \in S_n$ is *minimal with d descents* if

1. σ has exactly d descents
2. σ does not contain a pattern $\pi \in S_k$, $k < n$, having d descents.

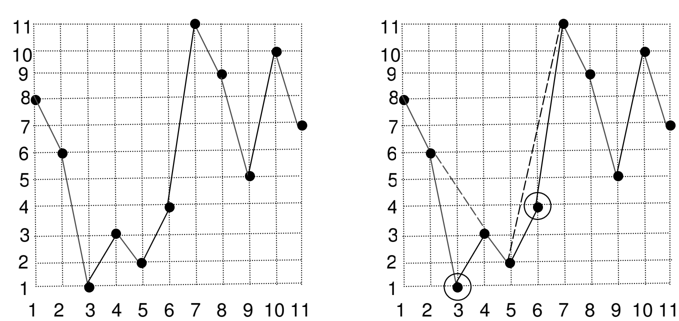

Figure 1: Permutations σ and π of Example 1.8

Example 1.8 Let $\sigma = 8\,6\,1\,3\,2\,4\,11\,9\,5\,10\,7$ be a permutation with 6 descents; σ is not minimal with 6 descents. Indeed, removing from σ the elements 1 and 4 (that are circled in Figure 1) we obtain permutation $\pi = 6\,4\,2\,1\,9\,7\,3\,8\,5$ with the same number of descents. π is minimal with 6 descents: it is impossible to remove an element from it while keeping the number of descents equal to 6.

A first characterization of minimal permutations with d descents is given in Proposition 1.9, whose proof is given in [3].

Proposition 1.9 *If σ is a minimal permutation of size n with d descents, then σ does not contain consecutive ascents and n satisfies $d+1 \leq n \leq 2d$.*

The condition provided by Proposition 1.9 is not sufficient to give a characterization of minimal permutations with d descents.

Example 1.10 The permutation $\sigma = 7\,5\,1\,3\,2\,10\,8\,4\,9\,6$, with 6 descents, does not contain consecutive ascents (see Figure 2). However, σ is not minimal as it contains pattern $\pi = 6\,4\,2\,1\,9\,7\,3\,8\,5$, which is minimal with 6 descents.

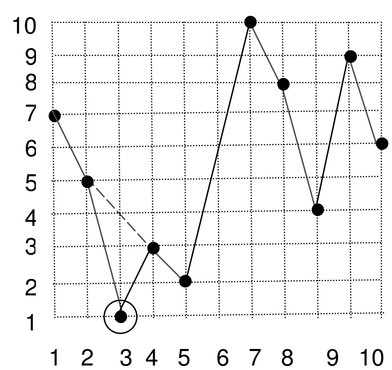

Figure 2: A non minimal permutation without consecutive ascents

An exhaustive characterization of minimal permutations, as given in [3], can be summarized in the following theorem, giving a local characterization of minimal permutations with d descents.

THEOREM 1.11 *A permutation $\sigma \in S_n$ is minimal with d descents if and only if it has exactly d descents and its ascents $\sigma(i)\sigma(i+1)$ are such that $2 \leq i \leq n-2$ and the subsequence $\sigma(i-1)\sigma(i)\sigma(i+1)\sigma(i+2)$ forms an occurrence of either the pattern* 2143 *or the pattern* 3142.

The characterization of minimal permutations with d descent in Theorem 1.11 directly leads to a *partially ordered set* (or *poset*) representation of permutations.

Consider a set of all the permutations of size n, that are minimal with d descents, and having their descents and ascents in the same positions. In all these permutations, the elements are locally ordered in the same way, even around the ascents, because of Theorem 1.11. This whole set of permutations can be represented by a partially ordered set indicating the necessary conditions on the relative order of the elements between them. For a descent, there is a link from the first and greatest element to the second and smallest one. For any ascent $\sigma(i)\sigma(i+1)$, the elements $\sigma(i-1)\sigma(i)\sigma(i+1)\sigma(i+2)$ form a diamond-shaped structure with $\sigma(i+1)$ at the top, $\sigma(i)$ at the bottom, $\sigma(i-1)$ on the left and $\sigma(i+2)$ on the right. By Theorem 1.11, any labelling of the elements of the poset respecting its ordering constraints is a minimal permutation with d descents. See Figure 3 for an example.

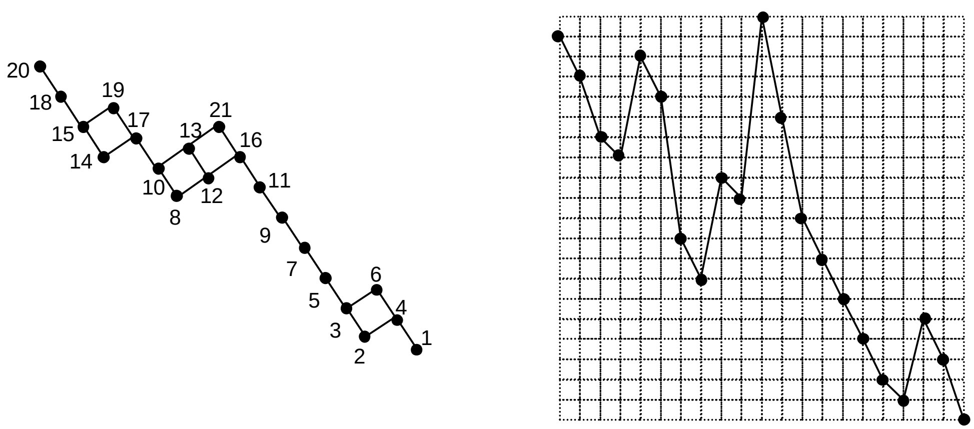

Figure 3: Permutations and authorized labelling of the posets

Recently, in [9] some interesting enumerative problems about minimal permutations are discussed.

1.3 Outline of the paper

This paper deals with some properties of *non minimal* permutations of length n with 1 ascent or, equivalently, of *non minimal* permutations of length n with $n-2$ descents. The set of *non minimal* permutations of length n with 1 ascent is denoted by $\mathcal{NM}_n^{1\uparrow}$. The work that is presented hereafter is organized as follows.

In Section 2 we give two algorithms to generate the non minimal permutations of length n with 1 ascent: the former is a constructive algorithm that, starting from an initial permutation (the decreasing sequence $n(n-1)\dots 321$) generates all the non minimal permutations of size n with 1 ascent; the latter uses an "explosion" strategy to generate the non minimal permutations of length $(n+1)$ with 1 ascent from the ones of size n.

Although in this paper we do not deal with this topic, the former algorithm given in Section 2 may be used to exhaustively generate all permutations in $\mathcal{NM}_n^{1\uparrow}$ in linear time. An algorithm for random generation of all permutations in $\mathcal{NM}_n^{1\uparrow}$ is given in [8].

In Section 3 an immediate analytical enumeration of $\mathcal{NM}_n^{1\uparrow}$ is provided. Then, in Section 4 we give a bijective enumeration of the non minimal permutations of length n with 1 ascent defining a bijection between the set $\mathcal{NM}_n^{1\uparrow}$ and a particular set of words.

In Section 5 we consider some statistics on the non minimal permutations of size n with 1 ascent, namely we count permutations according to the value of their first and last entry.

2 Generating algorithms

In this section two algorithms are provided to generate non minimal permutations of length n with 1 ascent.

The former algorithm, starting from the permutation $n\,(n-1)\,\dots\,3\,2\,1$, generates all the non minimal permutations by swapping their entries.

The latter generates the non minimal permutations of length n applying a special strategy to the non minimal permutations of length $(n-1)$.

2.1 A swapping algorithm

The following algorithm generates the minimal permutations of length n.

For $1 \leq i \leq (n-1)$, the *non minimal* permutations of length n with the unique ascent in position i are generated by the following procedure.

Algorithm A

1. Set $\pi = n(n-1)\dots(n-i+1)(n-i)\dots 321$;

2. the first permutation π_0 is obtained swapping $\pi(i)$ for $\pi(i+1)$;

3. π_0 generates $(n-i-1)$ permutations π_j such that π_j is obtained by π_{j-1} swapping $\pi_{j-1}(i)$ for $\pi_{j-1}(i+j+1)$, $(1 \leq j \leq n-i-1)$:

$$\begin{aligned}
\pi_0 &= n(n-1)\dots(n-i+2)(n-i)(n-i+1)(n-i-1)\dots 1 \\
\pi_1 &= n(n-1)\dots(n-i+2)(n-i-1)(n-i+1)(n-i)\dots 1 \\
&\vdots \\
\pi_{n-i-1} &= n(n-1)\dots(n-i+2)1(=\pi(n))(n-i+1)(n-i)\dots 2
\end{aligned}$$

4. π_0 generates $(i-1)$ more permutations $\bar{\pi}_k$ such that $\bar{\pi}_k$ is obtained by $\bar{\pi}_{k-1}$ swapping $\bar{\pi}_{k-1}(i-k)$ for $\bar{\pi}_{k-1}(i+1)$ $(\bar{\pi}_0 = \pi_0)$:

$$\begin{aligned}
\bar{\pi}_1 &= n(n-1)\dots(n-i+3)(n-i+1)(n-i)(n-i+2)(n-i-1)\dots 1 \\
\bar{\pi}_2 &= n(n-1)\dots(n-i+2)(n-i+1)(n-i)(n-i+3)(n-i-1)\dots 1 \\
&\vdots \\
\bar{\pi}_{i-1} &= (n-1)\dots(n-i+2)(n-i+1)(n-i)n(n-i-1)\dots 1
\end{aligned}$$

□

Then, for any i such that $1 \leq i \leq (n-1)$, $1+(n-i-1)+(i-1) = n-1$ permutations are generated.

EXAMPLE 2.1 Let be $n = 6$. The $(n-1)^2 = 25$ permutations obtained by applying the above procedure are listed in Figure 4.

PROPOSITION 2.2 *Algorithm A generates all and only the non minimal permutations of size n with 1 ascent.*

Proof. In order to prove that the obtained permutations are non minimal, it is necessary to show that they belong to the set of avoiding permutations $S_n(2143, 3142)$. The four positions that could form a pattern of type 2143 or 3142 are

$$(i-1)(i)(i+1)(i+2).$$

$$i=1 \begin{cases} \pi_0 &= 564321 \\ \pi_1 &= 465321 \\ \pi_2 &= 365421 \\ \pi_3 &= 265431 \\ \pi_4 &= 165432 \end{cases} \quad i=2 \begin{cases} \pi_0 &= 645321 \\ \pi_1 &= 635421 \\ \pi_2 &= 625431 \\ \pi_3 &= 615432 \\ \bar{\pi}_1 &= 546321 \end{cases} \quad i=3 \begin{cases} \pi_0 &= 653421 \\ \pi_1 &= 652431 \\ \pi_2 &= 651432 \\ \bar{\pi}_1 &= 643521 \\ \bar{\pi}_2 &= 543621 \end{cases}$$

$$i=4 \begin{cases} \pi_0 &= 654231 \\ \pi_1 &= 654132 \\ \bar{\pi}_1 &= 653241 \\ \bar{\pi}_2 &= 643251 \\ \bar{\pi}_3 &= 543261 \end{cases} \quad i=5 \begin{cases} \pi_0 &= 654312 \\ \bar{\pi}_1 &= 654213 \\ \bar{\pi}_2 &= 653214 \\ \bar{\pi}_3 &= 643215 \\ \bar{\pi}_4 &= 543216 \end{cases}$$

Figure 4: Non minimal permutations of length 6 with 1 ascent

In π_0 we have

$$\pi_0(i+2) < \pi_0(i) < \pi_0(i+1) < \pi_0(i-1)$$

so π_0 contains the pattern of type 4231. In the permutations π_j the entry $\pi_j(i)$ is swapped for an element smaller than $\pi_j(i+2)$, so $\pi_j(i) < \pi_j(i+2) < \pi_j(i+1) < \pi_j(i-1)$ is a pattern of type 4132. The first $(i-1)$ entries of π_0 are in decreasing order and they are greater than both $(n-i)$ and $(n-i+1)$. Therefore, in the permutations $\bar{\pi}_k$ the entry $\bar{\pi}_k(i+1)$ is swapped for an element greater than $\bar{\pi}_k(i-1)$; so $\pi_j(i+2) < \pi_j(i) < \pi_j(i-1) < \pi_j(i+1)$ is a pattern of type 3241. Therefore, the permutations generated by algorithm A are non minimal.

Now we prove that Algorithm A generates *all* the non minimal permutations of length n with 1 ascent, that is to say, if σ is a non minimal permutation of size n with 1 ascent, then σ can be generated applying Algorithm A. Let σ be a non minimal permutation of size n with 1 ascent in position i. We know that $\sigma(i) < \sigma(i-1)$, $\sigma(i) < \sigma(i+1)$ and $\sigma(i+2) < \sigma(i+1)$; so there are five possible configurations:

1. $\sigma(i) < \sigma(i-1) < \sigma(i+2) < \sigma(i+1)$
2. $\sigma(i) < \sigma(i+2) < \sigma(i-1) < \sigma(i+1)$
3. $\sigma(i+2) < \sigma(i) < \sigma(i+1) < \sigma(i-1)$
4. $\sigma(i) < \sigma(i+2) < \sigma(i+1) < \sigma(i-1)$
5. $\sigma(i+2) < \sigma(i) < \sigma(i-1) < \sigma(i+1)$

The first two configurations are not admissible since they correspond to patterns of type 2134 and 3142, respectively, and we assumed that σ is non minimal. Configuration 3. is a pattern of type 4231, so σ is π_0. Configuration 4. is a pattern of type 4132, so $\sigma = \pi_j$, for a $j = 1, \ldots, n-i-1$. Finally, configuration 5. is a pattern of type 3241, so $\sigma = \bar{\pi}_k$, for a $k = 1, \ldots, i-1$. □

LEMMA 2.3 *In the permutations π_j, $0 \leq j \leq (n-i-1)$, the ascent in position i, $1 \leq i \leq (n-1)$, is $\pi_j(i) = (n-i-j)$ and the first $(i-1)$ elements are the decreasing sequence $n(n-1)\ldots(n-i+2)$. Therefore, there are not consecutive elements greater than $\pi_j(i)$ on the left of $\pi_j(i)$.*

LEMMA 2.4 *In the permutations $\bar{\pi}_k$, $1 \leq k \leq (i-1)$ the ascent in position i, $1 \leq i \leq (n-1)$, is $\bar{\pi}_k(i) = (n-i)$ and there are k consecutive elements greater than $\bar{\pi}_k(i)$ on the left of $\bar{\pi}_k(i)$.*

LEMMA 2.5 *In a non minimal permutation σ of length n with the unique ascent in position i, $1 \leq i \leq (n-1)$, and such that $\sigma(i) < (n-i)$, there are not consecutive elements greater than $\sigma(i)$ on the left of $\sigma(i)$.*

Proof. Let $\sigma(i) = n-i-q$, $q > 0$, and let us assume that there are $k > 0$ consecutive elements greater than $\sigma(i)$ on the left of $\sigma(i)$. Then, $\sigma(i-1) = (n-i-q+1)$, $\sigma(i+1) = (n-i+k+1)$ and $\sigma(i+2) = (n-i-q+k+1)$. Therefore σ contains the forbidden pattern 2143. □

2.2 An explosion algorithm

An algorithm is defined so that it generates the non minimal permutation of length $(n+1)$ with 1 ascent from the ones of length n.

Algorithm B

1. From every permutation $\pi_j \in \mathcal{NM}_n^{1\uparrow}$ generate a permutation $\sigma_i \in \mathcal{NM}_{n+1}^{1\uparrow}$ inserting $(n+1)$ on the left of $\pi_j(1)$;
2. let $\pi_j \in \mathcal{NM}_n^{1\uparrow}$ be a permutation such that $\pi_j(1) < n$; from π_j generate a permutation $\sigma_i \in \mathcal{NM}_{n+1}^{1\uparrow}$ increasing every element of π_j by 1 and adding 1 on its right;
3. insert in the set $\mathcal{NM}_{n+1}^{1\uparrow}$ the permutation $n\,(n-1)\,\ldots\,2\,1\,(n+1)$, that is, the permutation ending with $(n+1)$ and with the ascent in position n;

4. insert in the set $\mathcal{NM}_{n+1}^{1\uparrow}$ the permutation $1\,(n+1)\,n\ldots 2$, that is, the permutation having 1 as first entry and with the ascent in position 1.

□

Figure 5 shows the permutations generated by applying Algorithm B for $n = 2, 3, 4, 5$; the permutations generated at step 1. are written in **bold**, the permutations inserted in $\mathcal{NM}_{n+1}^{1\uparrow}$ at steps 3. and 4. are in *italics* and the regular font is used for the permutations generated at step 2.

						45321	35421	25431	*15432*
			3421	2431	*1432*	**53421**	**52431**	**51432**	*43215*
	231	*132*	**4231**	**4132**	*3214*	**54231**	**54132**	**53214**	43251
12	**312**	*213*	**4312**	**4213**	3241	**54312**	**54213**	**53241**	43521
$n=2$	$n=3$		$n=4$			$n=5$			

Figure 5: Explosion generation of $\mathcal{NM}_n^{1\uparrow}$

PROPOSITION 2.6 *Algorithm B generates all and only non minimal permutations of size $(n+1)$ with 1 ascent.*

Proof. The proof is by induction on n. By hypothesis the permutations in $\mathcal{NM}_n^{1\uparrow}$ are non minimal with the ascent in position i, $1 \leq i \leq (n-1)$, that is, they can be broken down into two (possibly empty) sequences of descents separated by the ascent.

Each permutation σ_i generated at step 1. keeps the same ascent as π_j and it is non minimal because the new entry $(n+1)$ on the left of $\pi_j(1)$ increases the length of the first sequence of descents merely by 1.

The first n elements of each permutation σ_i generated at step 2. are order-isomorphic to π_j; so, if the ascent in π_j is in position k, then $\sigma_i(k)$ is the ascent, too. Therefore, σ_i is non minimal because the entry 1 on the right of $\pi_j(n)$ increases the length of the second sequence of descents merely by 1.

The permutations in $\mathcal{NM}_n^{1\uparrow}$ with value 1 in position $(n-1)$ end with the pair $1\,j$, $2 \leq j \leq n$. None of the permutations generated at steps 1. and 2. in Algorithm B ends with the pair $1\,(n+1)$, so such a permutation is inserted in $\mathcal{NM}_{n+1}^{1\uparrow}$ at step 3. Remark that all the other permutations in $\mathcal{NM}_{n+1}^{1\uparrow}$ ending with $1\,j$, $2 \leq j \leq n$, are generated at step 1.

None of the permutations generated at steps 1., 2. and 3. has value 1 as first element; on the other hand, for any length m of the permutations, only one permutation has value 1 as first element, that is the permutation

$$1\, m\, (m-1)\, \ldots\, 2$$

which is added to $\mathcal{NM}_{n+1}^{1\uparrow}$ at step 4. of Algorithm B. □

3 Analytical enumeration of $\mathcal{NM}_n^{1\uparrow}$

Let $\mathcal{S}_n^{(n-2)\downarrow}$ be the set of permutations of length n with $n-2$ descents. From (1) with $k = n-2$ we can prove that the cardinality of $\mathcal{S}_n^{(n-2)\downarrow}$, is

$$|\mathcal{S}_n^{(n-2)\downarrow}| = 2^{\,n} - n - 1$$

In [3] it is proved that the number of minimal permutations of length $n(= d+2)$ with $n-2$ descents is $2^{\,n} - n(n-1) - 2$. So, the number $|\mathcal{NM}_n^{(n-2)\downarrow}|$ of *non minimal* permutations of length n with $n-2$ descents is:

$$\begin{aligned} |\mathcal{NM}_n^{(n-2)\downarrow}| &= |\mathcal{S}_n^{(n-2)\downarrow}| - (2^{\,n} - n(n-1) - 2) \\ &= n^2 - 2n + 1 \end{aligned}$$

that is

$$|\mathcal{NM}_n^{(n-2)\downarrow}| = (n-1)^{\,2}$$

Clearly a permutation of length n with $n-2$ descents has a unique ascent, and vice versa. So, $|\mathcal{NM}_n^{(n-2)\downarrow}|$ is the number of non minimal permutations of length n with 1 ascent, that is, $|\mathcal{NM}_n^{(n-2)\downarrow}| = |\mathcal{NM}_n^{1\uparrow}|$.

By Theorem 1.11, in a non minimal permutation σ of length n with the ascent in position i, the four elements $\sigma(i-1)$, $\sigma(i)$, $\sigma(i+1)$ and $\sigma(i+2)$ form a pattern of type 4231 or 4132 or 3241.

4 Bijective enumeration of $\mathcal{NM}_n^{1\uparrow}$

In this section a bijective proof of the following theorem is given.

THEOREM 4.1 *The number of non minimal permutations of length n with 1 ascent is $(n-1)^2$.*

The proof is given by defining a bijection ϕ between the set $\mathcal{NM}_n^{1\uparrow}$ of non minimal permutations of size n with 1 ascent and the following set $\Sigma_{2(n-1)}$ of words:

$$\Sigma_{2(n-1)} = \{\mathrm{B}^l \mathrm{A}\mathrm{B}^{n-l-2} \mathrm{A}^m \mathrm{B}\mathrm{A}^{n-m-2} \,:\, 0 \leq l, m \leq n-2\}$$

that is to say $\Sigma_{2(n-1)}$ is the set of words of 2 distinct letters, A and B, each of them with $n-1$ copies and with 2 fixed points. A *fixed point* is the occurrence of the letter A in a position $i \in [1\,..\,n-1]$ or the occurrence of the letter B in a position $j \in [n\,..\,2n-2]$. The cardinality of $\Sigma_{2(n-1)}$ is $(n-1)^2$ (see sequence A000290 of [12]).

Let $\sigma \in \mathcal{NM}_n^{1\uparrow}$ be a permutation with the ascent in position i; then $\omega = \phi(\sigma)$ is obtained in the following way:

1. place a letter B in position $n-1+i$ (the position of the ascent in σ determines the position of the last B in ω);
2. let k be the number of consecutive elements greater than $\sigma(i)$ that are on the left of $\sigma(i)$; place the first A in position $j = \sigma(i) + k$;
3. complete ω so as to respect the properties of the words in $\Sigma_{2(n-1)}$

Example 4.2 Let $\sigma = 6\,4\,3\,2\,5\,1$ be a permutation of $\mathcal{NM}_6^{1\uparrow}$. The ascent of σ is in position 4, then the last B in the word ω of length 10 is in position 9. $\sigma(4) = 2$ and there are two consecutive elements greater than 2 on the left of $\sigma(4)$, then the first A is in position 4. Finally, we have

$$\omega = \phi(\sigma) = \text{BBBABAAABA}$$

Vice versa, given a word $\omega \in \Sigma_{2(n-1)}$ with the last B in position $(n-1+i)$, $1 \le i \le (n-1)$, and the first A in position ℓ, $1 \le \ell \le (n-1)$, the permutation $\sigma = \phi^{-1}(\omega)$ has the ascent in position i and

- if $\ell < (n-i)$, then $\sigma(i) = \ell$, the first $(i-1)$ of σ are the decreasing sequence beginning from n, while the last $(n-i)$ are the decreasing sequence of the remaining elements starting with $(n-i+1)$;
- if $\ell \ge (n-i)$, the $\sigma(i) = (n-i)$ and there are $k = \ell - (n-i)$ consecutive elements greater than $\sigma(i)$ on the left of $\sigma(i)$. Therefore, the first $(i-k-1)$ elements of σ are the decreasing sequence beginning from n, followed by the values $(\ell+k)\,(\ell+k-1)\ldots(\ell+1)\,\ell$ and, finally, the last $(n-i)$ elements are the decreasing sequence of the remaining elements starting with $(n-i+k+1)$.

Example 4.3 Let $\omega = \text{BBBBAAAABA}$ a word of Σ_{10}, that is $n = 6$. The last B is in position 9 and the first A is in position $\ell = 5$. The permutation $\sigma = \phi^{-1}(\omega)$ has the ascent in position $i = 4$ and, since $\ell > (n-i)$, $\sigma(4) = 2$ and there are 3 consecutive elements greater than 2 on the left of $\sigma(4)$. So we have $\sigma = \phi^{-1}(\omega) = 5\,4\,3\,2\,6\,1$.

Now we can state the following theorem.

THEOREM 4.4 *The function* ϕ *is a bijection between the set* $\mathcal{NM}_n^{1\uparrow}$ *of non minimal permutations of size* n *with 1 ascent and the set of words* $\Sigma_{2(n-1)}$.

Proof. It follows from Lemmas 2.3, 2.4 and 2.5. □

5 Some statistics on $\mathcal{NM}_n^{1\uparrow}$

Table 1.a) shows the number of non minimal permutations of length n with 1 ascent and with the *first* element equal to $1, 2, \ldots, n$, for some values of n.

n / $\pi(1)$	3	4	5	6	7	8	n / $\pi(n)$	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1	4	9	16	25	36
2	2	1	1	1	1	1	2	2	3	4	5	6	7
3	1	3	1	1	1	1	3	1	1	1	1	1	1
4		4	4	1	1	1	4		1	1	1	1	1
5			9	5	1	1	5			1	1	1	1
6				16	6	1	6				1	1	1
7					25	7	7					1	1
8						36	8						1

a) **b)**

Table 1: **a)**: number of non minimal permutations of length $n = 3, 4 \ldots, 8$ with 1 ascent and with *first* entry $1, 2 \ldots, 8$; **b)**: number of non minimal permutations of length $n = 3, 4 \ldots, 8$ with 1 ascent and with *last* entry $1, 2 \ldots, 8$

PROPOSITION 5.1 *The following properties hold for non minimal permutations of length* n *with 1 ascent:*

1. *the number of permutations with the first entry equal to* $1, 2, \ldots, (n-2)$ *is one, for each value;*

2. *the number of permutations with the first entry equal to* $(n-1)$ *is* $(n-1)$*;*

3. *the number of permutations with the first entry equal to* n *is* $(n-2)^2$.

Proof. We refer to the sequence of non minimal permutations generated by Algorithm A.

1. The permutations σ with the first element between 1 and $(n-2)$ have the ascent in position $i = 1$. Indeed, since $\sigma(i+1)$ is necessarily equal to n and $\sigma(i+2) = (n-1)$, if the ascent is in position $i > 1$, the substring $\sigma(i-1)\,\sigma(i)\,\sigma(i+1)\,\sigma(i+2)$ is an occurrence of the pattern 2143, contrary to the assumption that σ is a non minimal permutation. So the permutations having a value $j = 1, 2, \ldots, (n-2)$ as first entry are the permutations π_{n-j-1} with the ascent in position $i = 1$;

2. when $i = 1$ there is only one permutation, that is π_0, with the first entry equal to $(n-1)$; for each i such that $2 \leq i \leq (n-1)$ all the permutations have the first entry equal to n except the permutation $\bar{\pi}_{i-1}$ which has the value n in position $(i+1)$;

3. when $i = 1$, no permutation has the first entry equal to n, while, as we said before, for each i such that $2 \leq i \leq (n-1)$ there are $(n-2)$ permutations with the first entry equal to n.

□

Table 1.b) shows the number of non minimal permutations of length n with 1 ascent and with the *last* element equal to $1, 2, \ldots, n$, for some values of n.

PROPOSITION 5.2 *Every column in Table 1.b) is opposite versus the corresponding column in Table 1.a), i.e., it is read from bottom to top.*

Proof. The permutations which have the last element equal to a value j between 3 and n have the ascent in position $i = (n-1)$, that is, they are the permutations $\bar{\pi}_{j-2}$. Indeed, if the ascent is in position $i < (n-1)$, since the element at position i must be 1, the substring defined by the positions $(i-1)$, (i), $(i+1)$ and $(i+2)$ is an occurrence of either the pattern 2143 or 3142, in opposition to the assumption that the permutations are non minimal.
For each i such that $1 \leq i \leq (n-2)$, all permutations end with the value 1 except permutation π_{n-i-1}, whose last element is 2; when $i = (n-1)$ only permutation π_0 has 2 as last element.

□

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