

## Symmetric convex permutominoes and involutions

FILIPPO DISANTO  
Institut für Genetik  
Universität zu Köln  
Zülpicher Str. 47a, 50674 Köln  
e-mail: disafili@yahoo.it

and

SIMONE RINALDI  
University of Siena  
Dipartimento di Scienze Matematiche e Informatiche  
Pian dei Mantellini 44, 53100, Siena, Italy  
e-mail: rinaldi@unisi.it

(Received: December 13, 2010)

**Abstract.** A permutomino of size  $n$  is a polyomino determined by particular pairs  $(\pi_1, \pi_2)$  of permutations of length  $n$ , such that  $\pi_1(i) \neq \pi_2(i)$ , for  $1 \leq i \leq n$ . In this paper we consider the class of convex permutominoes which are symmetric with respect to the diagonal  $x = y$ . We determine the number of these permutominoes according to their size and we characterize the class of permutations associated to these objects as particular involutions of length  $n$ . To do this we need to introduce a larger class of objects, called symmetric permutominides, and to study their combinatorial properties.

**Mathematics Subject Classification(2010).** 05A15.

**Keywords:** polyominoes, permutations, involutions.

### 1 Introduction and basic definitions

In the plane  $\mathbb{Z} \times \mathbb{Z}$  a *cell* is a unit square, and a *polyomino* is a finite connected union of cells having no cut point. Polyominoes are defined up to translations. A *column* (*row*) of a polyomino is the intersection between the polyomino and an infinite strip of cells lying on a vertical (horizontal) line. A polyomino is said to be *column convex* (respectively *row convex*) if all of its columns (respectively row) are connected. A polyomino is said to be *convex*, if it is both row and column convex (see Fig. 1 (a)). The *semi-perimeter* of a polyomino is just half the number of edges of cells in its boundary; thus, for any convex polyomino the semi-perimeter is the sum of the numbers of its rows and columns.

Moreover, any convex polyomino is contained in a rectangle in the square lattice which has the same semi-perimeter, called *minimal bounding rectangle*. For the main results concerning the enumeration and combinatorial properties of convex polyominoes we refer to [3, 4, 6, 8].

A polyomino is said to be *directed* when each of its cells can be reached from a distinguished cell, called the root, by a path which is contained in the polyomino and uses only north and east unitary steps. A polyomino is *directed convex* if it is both directed and convex (see Fig. 1 (b)). It is known [17]

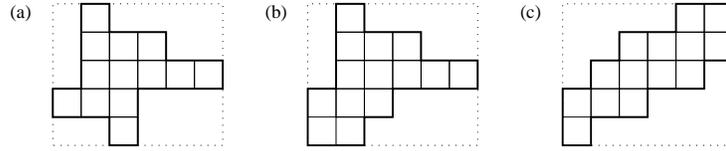


Figure 1: (a) A convex polyomino; (b) a directed convex polyomino; (c) a parallelogram polyomino.

that the number of directed convex polyominoes of semi-perimeter  $n + 2$  is equal to the  $n$ th central binomial coefficient, i.e.,  $b_n = \binom{2n}{n}$ , sequence A000984 in [16].

Finally, *parallelogram* polyominoes are a special subset of the directed convex ones, defined by two lattice paths that use north and east unit steps, and intersect only at their origin and extremity. These paths are called the *upper* and the *lower path* (see Fig. 1 (c)). It is known [17] that the number of parallelogram polyominoes having semi-perimeter  $n + 1$  is the  $n$ -th *Catalan number* (sequence A000108 in [16]),  $c_n = \frac{1}{n+1} \binom{2n}{n}$ .

**Permutominoes.** In this paper we will always deal with polyominoes having no “holes”, i.e. polyominoes where the boundary is made exactly of one component. Let us start by briefly recalling the main definitions and results concerning permutominoes. For further reading we refer to [1, 10, 9].

So, let  $P$  be a polyomino without holes, having  $n$  rows and columns,  $n \geq 1$ ; we assume without loss of generality that the south-west corner of its minimal bounding rectangle is placed in  $(1, 1)$ . Let  $\mathcal{A} = (A_1, \dots, A_{2(r+1)})$  be the list of its vertices (i.e., corners of its boundary) ordered in a clockwise sense starting from the lowest leftmost vertex, with  $A_i = (x_i, y_i)$ . We say that  $P$  is a *permutomino* if  $\mathcal{P}_1 = (A_1, A_3, \dots, A_{2r+1})$  and  $\mathcal{P}_2 = (A_2, A_4, \dots, A_{2r+2})$  represent two permutations of length  $n$ . Obviously, if  $P$  is a permutomino, then  $r = n$ , see Fig. 2.

More formally, we can define  $\pi_1(P)$  and  $\pi_2(P)$  (briefly,  $\pi_1$  and  $\pi_2$ ), as follows:

$$\pi_1(i) = y_j \quad \text{such that } x_j = 2i + 1$$

$$\pi_2(i) = y_j \quad \text{such that } x_j = 2i.$$

Now,  $P$  is a permutomino of size  $n + 1$  if and only if  $\pi_1, \pi_2 \in \mathcal{S}_{n+1}$  (as usual  $\mathcal{S}_n$  denotes the symmetric group of size  $n$ ).

From the definition any permutomino  $P$  of size  $n$  has the property that, for each abscissa (ordinate) between 1 and  $n$  there is exactly one vertical (horizontal) side in the boundary of  $P$  with that coordinate. It is simple to observe that this property is also a sufficient condition for a polyomino to be a permutomino.

Permutominoes can be viewed as special types of *permutation diagrams*, and they have been introduced by Kassel et al. [13] and then studied by F. Incitti while studying the problem of determining the  $\tilde{R}$ -polynomials associated with a pair  $(\pi_1, \pi_2)$  of permutations [12].

During the last years, a particular class of permutominoes, i.e. convex permutominoes –as the one depicted in Fig. 2– have been widely studied, and here we recall the main results:

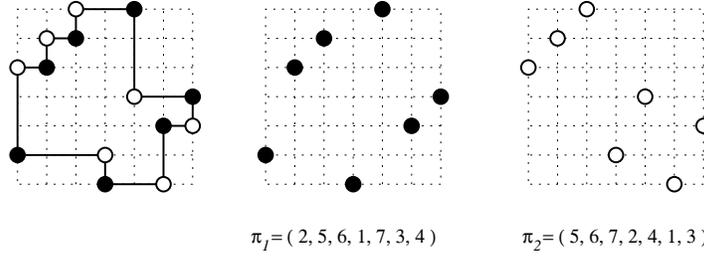


Figure 2: A convex permutomino of size 7 and the two associated permutations.

1. the number of *parallelogram permutominoes* of size  $n$  is equal to  $c_{n-1}$  [10],
2. the number of *directed convex permutominoes* of size  $n$  is equal to  $\frac{1}{2} b_{n-1}$  [10],
3. the number of *convex permutominoes* of size  $n + 1$  is [2, 9]:

$$2(n + 3)4^{n-2} - \frac{n}{2} \binom{2n}{n} \quad n \geq 1, \tag{1}$$

the first few terms being 1, 4, 18, 84, 394, 1836, 8468, ... (sequence A126020 in [16]).

**Permutations defining convex permutominoes.** Let us denote by  $\mathcal{C}_n$  the set of convex permutominoes of size  $n$ . In [1] the authors considered the following sets of permutations of length  $n$ :

$$\tilde{\mathcal{C}}_n = \{ \pi_1(P) : P \in \mathcal{C}_n \}, \quad \tilde{\mathcal{C}}'_n = \{ \pi_2(P) : P \in \mathcal{C}_n \}.$$

Easily we have that  $|\tilde{\mathcal{C}}_n| = |\tilde{\mathcal{C}}'_n|$ , and that  $\pi \in \tilde{\mathcal{C}}_n$  if and only if  $\pi^R \in \tilde{\mathcal{C}}'_n$ , where for any  $\pi = (\pi_1, \dots, \pi_n)$ , its *reversal* is  $\pi^R = (\pi_n, \dots, \pi_1)$ .

It is then reasonable to study, without loss of generality, the class  $\tilde{\mathcal{C}}_n$ . In particular, if a permutation  $\pi \in \tilde{\mathcal{C}}_n$  (i.e. there is at least a permutomino  $P$  such that if  $\pi = \pi_1(P)$ ) we say that  $\pi$  is  $\pi_1$ -*associated* (briefly *associated*) with a convex permutomino.

The main problem investigated in [1] concerns the characterization of the permutations of  $\tilde{\mathcal{C}}_n$ . Let  $\pi$  be a permutation of  $\mathcal{S}_n$ , we define  $\mu(\pi)$  (briefly,  $\mu$ ) as the maximal upper unimodal sublist of  $\pi$ , and  $\sigma(\pi)$  (briefly,  $\sigma$ ) as the sublist of  $\pi$  beginning with  $\pi(1)$ , ending with  $\pi(n)$ , and containing the elements not in  $\mu$  (by convention  $\mu$  and  $\sigma$  retain the indexing of  $\pi$ ). For instance, for the convex permutomino in Fig. 2, we have  $\pi_1 = (2, 5, 6, 1, 7, 3, 4)$ , and the two subsequences  $\mu$  and  $\sigma$  are

	1	2	3	4	5	6	7
$\mu$	2	5	6	-	7	-	4
$\sigma$	2	-	-	1	-	3	4

For the sake of brevity we will represent the two sequences omitting the empty spaces, as  $\mu = (2, 5, 6, 7, 4)$ ,  $\sigma = (2, 1, 3, 4)$ . The following theorem [1] gives a characterization of the permutations of  $\mathcal{C}_n$ .

THEOREM 1.1 *Let  $\pi \in \mathcal{S}_n$  be a permutation. Then  $\pi \in \tilde{\mathcal{C}}_n$  if and only if:*

1.  $\sigma$  is lower unimodal, and
2. there are no two permutations,  $\theta \in \mathcal{S}_m$ , and  $\theta' \in \mathcal{S}_{m'}$ , such that  $m + m' = n$ , and  $\pi = \theta \oplus \theta'$ .

We recall that, given two permutations  $\theta = (\theta_1, \dots, \theta_m) \in \mathcal{S}_m$  and  $\theta' = (\theta'_1, \dots, \theta'_{m'}) \in \mathcal{S}_{m'}$ ,  $\theta \oplus \theta'$  is defined as  $(\theta_1 + m', \dots, \theta_m + m', \theta'_1, \dots, \theta'_{m'})$ .

So, for instance  $(2, 5, 6, 1, 7, 3, 4)$  satisfies conditions 1. and 2. and it defines at least one convex permutomino (as the one in Fig. 2). The reader can check that there is no convex permutomino associated with  $(2, 5, 3, 7, 4, 1, 6)$  since  $\sigma = (2, 3, 4, 1, 6)$  is not lower unimodal; there is no convex permutomino associated with  $\pi = (5, 9, 8, 7, 6, 3, 1, 2, 4)$  since  $\pi = (1, 5, 4, 3, 2) \oplus (3, 2, 1, 4)$ .

In [1] it is also proved that the cardinality of  $\tilde{\mathcal{C}}_{n+1}$  is

$$2(n+2)4^{n-2} - \frac{n}{4} \left( \frac{3-4n}{1-2n} \right) \binom{2n}{n}, \quad n \geq 2. \quad (2)$$

defining the sequence 1, 3, 13, 62, 301, 1450, ..., sequence A122122 in [16].

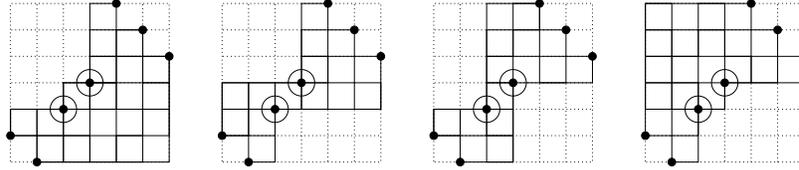


Figure 3: The four convex permutominoes associated with  $(2, 1, 3, 4, 7, 6, 5)$ . The two free fixed points are encircled.

*Remark.* The permutations  $\pi$  for which  $\sigma(\pi)$  is lower unimodal are the so called *square* permutations, introduced and enumerated by Mansour and Severini [15]. Moreover, these object have been studied also by Waton in [18], who characterized them as the set of permutations avoiding 16 patterns of length 5.

**Convex permutominoes associated with a permutation.** For any  $\pi \in \tilde{\mathcal{C}}_n$ , let us consider

$$[\pi] = \{P \in \mathcal{C}_n : \pi_1(P) = \pi\},$$

i.e., the set of convex permutominoes associated with  $\pi$ . We say that a fixed point  $i$  of  $\pi$ , with  $1 < i < n$ , is a *free fixed point* if  $\pi$  can be decomposed as the direct sum  $\pi = \sigma_1 \oplus (1) \oplus \sigma_2$ , where  $\sigma_1 \in \mathcal{S}_{i-1}$ , and  $\sigma_2 \in \mathcal{S}_{n-i}$ . Let  $\mathcal{F}(\pi)$  (briefly  $\mathcal{F}$ ) denote the set of free fixed points of  $\pi$ . By definition, the free fixed points of  $\pi$  are precisely the fixed points—different from 1 and  $n$ —lying in the ascending subsequence of  $\mu$ . In [10] it was proved the following.

THEOREM 1.2 *Let  $\pi \in \tilde{\mathcal{C}}_n$ , and let  $\mathcal{F}(\pi)$  be the set of free fixed points of  $\pi$ . Then we have  $|\pi| = 2^{|\mathcal{F}(\pi)|}$ .*

For instance, let  $\pi = (2, 1, 3, 4, 7, 6, 5)$  we have  $\mu = (2, 3, 4, 7, 6, 5)$ ,  $\sigma = (2, 1, 5)$ , and  $\pi = (2, 1) \oplus (1) \oplus (1) \oplus (3, 2, 1)$ . Hence  $\mathcal{F}(\pi) = \{3, 4\}$ , and then there are four convex permutominoes associated with  $\pi$ , as shown in Fig. 3.

**Main results.** In this paper we study the convex permutominoes which are *symmetric* with respect to the diagonal  $x = y$ . Since we will always assume that the objects we treat satisfy the convexity constrain, by abuse of notation we will often speak of *symmetric permutominoes* in place of *symmetric convex permutominoes*.

The interest in this class of permutominoes lies in the fact they are defined by a pair of involutions, and vice versa, a convex permutomino defined by a pair of involutions is necessarily symmetric. Therefore, studying these objects we can obtain interesting properties on the associated involutions.

At the beginning of Section 2 we give a simple characterization of the pairs of involutions  $(\pi_1, \pi_2)$  defining symmetric permutominoes. Moreover we prove that, if  $P$  is a symmetric permutomino, then it is uniquely determined by  $\pi_1(P)$ . Hence there is a trivial bijection between symmetric permutominoes of size  $n$  and a class of involutions of length  $n$ .

The problem of the enumeration of symmetric permutominoes according to the size turns out to be more complex. As a matter of fact, the approach used in [9], based on the ECO method, and the classical decomposition strategy (as in [3]), are here not easily applicable, since the objects we deal with must satisfy the two constraints of being permutominoes, and of being symmetric. The method we show in this paper to solve enumeration is almost entirely bijective, and resembles the method used in [4] for the enumeration of three dimensional convex polygons. It can be briefly explained in the following steps:

1. we consider a larger class of objects, namely the *symmetric permutominides*, which have the same properties of permutominoes, apart from the fact that the boundary needs not be self-avoiding. The symmetric permutominides can be of three types: proper symmetric permutominoes, *south-east oriented*, or *north-east oriented* permutominides (Section 2.1). We show that if a permutominide is of the first or of the second type, then its reentrant points form a permutation matrix. To do this, we need to apply, and then slightly extend, a result of discrete geometry recently established by Brlek et al. [5];
2. using the previous result, we are able to map the objects of the first and of the second type into a subset of 2-colored Motzkin paths, satisfying some special conditions (Section 2.2). Such paths can be easily treated with standard generating function methods. The bijection strongly uses the fact that these objects are symmetric according to the diagonal  $x = y$ ;
3. using the generating functions of the previously considered 2-colored Motzkin paths and of *south-east oriented* permutominides, by difference we determine the generating function of symmetric permutominoes (Sections 2.3 and 2.4). The main result states that the number of symmetric permutominoes of size  $n$  is

$$(n + 2) 2^{n-3} - (n - 1) \binom{n - 2}{\lfloor \frac{n-2}{2} \rfloor} - (n - 2) \binom{n - 3}{\lfloor \frac{n-3}{2} \rfloor} \quad n \geq 2.$$

Our bijective approach allows us to determine some further combinatorial properties of symmetric permutominoes. Moreover, we can prove that the number of symmetric permutominides of size  $n$  is  $n2^{n-3}$ . Such a result is quite surprising, since it states that the class of symmetric permutominides is rational, while it is obtained as the disjoint union of three algebraic (not rational) classes.

## 2 Symmetric permutominoes

Let  $Sym_n$  denote the class of symmetric permutominoes of size  $n$ , and let

$$\widetilde{Sym}_n = \{ \pi_1(P) : P \in Sym_n \}.$$

Figure 4 shows a symmetric permutomino of size 15. It is easy to prove the following fact.

**PROPOSITION 2.1** *Let  $P$  be a convex permutomino; then  $P \in Sym_n$  if and only if both  $\pi_1(P)$  and  $\pi_2(P)$  are involutions.*

Figure 3 shows that if only  $\pi_1(P)$  is an involution, then  $P$  is not symmetric.

**PROPOSITION 2.2** *If  $P$  is a symmetric permutomino, then  $\pi_1(P)$  has no free fixed points.*

**Proof.** Assume that  $i$  is a free fixed point of  $\pi_1(P)$ , then  $\pi_1(P) = \theta \oplus (1) \oplus \theta'$ . Since the boundary of  $P$  must necessarily contain the point  $I = (i, \pi_1(i))$ , and  $P$  is symmetric with respect to  $x = y$ , then the boundary of  $P$  intersects itself in  $I$ . Hence  $P$  is not a permutomino.  $\square$

As a consequence of Proposition 2.2, if  $P$  is symmetric, then  $\pi_1(P)$  has at most two fixed points.

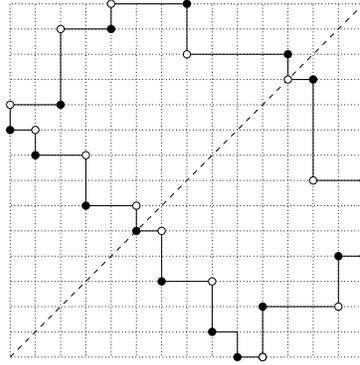


Figure 4: A symmetric permutomino of size 15 and the two associated involutions  $\pi_1 = (10, 9, 11, 7, 14, 6, 4, 15, 2, 1, 3, 13, 12, 5, 8)$ ,  $\pi_2 = (11, 10, 14, 9, 15, 7, 6, 13, 4, 2, 1, 12, 8, 3, 5)$ .

In practice, according to Proposition 2.1 and Theorem 1.2, a symmetric permutomino is uniquely determined by one of the two defining permutations, i.e. there is a trivial bijection between  $Sym_n$  and  $\widetilde{Sym}_n$ . Thus, if  $\pi \in \widetilde{Sym}_n$ , we can legitimately say that  $\pi$  defines a symmetric permutomino. The following result is straightforward.

**THEOREM 2.3** *A permutation  $\pi$  defines a symmetric permutomino if and only if  $\pi \in \widetilde{\mathcal{C}}_n$  and it is an involution without free fixed points.*

Summarizing, a permutation  $\pi$  of length  $n$  belongs to  $\widetilde{Sym}_n$  if and only if: (1)  $\sigma(\pi)$  is lower unimodal; (2)  $\pi \neq \theta_1 \ominus \theta_2$  for some  $\theta_1, \theta_2$ ; (3)  $\pi$  is an involution; (4)  $\pi$  does not contain free fixed points.

For instance, if  $\pi = (10, 9, 11, 7, 14, 6, 4, 15, 2, 1, 3, 13, 12, 5, 8)$ , we have  $\mu = (10, 11, 14, 15, 13, 12, 8)$ ,  $\sigma = (10, 9, 7, 6, 2, 1, 3, 5, 8)$ . Moreover  $\pi$  is an involution, and the unique fixed point is 6 (not a free fixed point), therefore  $\pi$  defines a symmetric permutomino, precisely the one depicted in Fig. 4. For  $n = 2, \dots, 4$ :

$$\widetilde{Sym}_2 = \{12\}, \quad \widetilde{Sym}_3 = \{132, 213\}, \quad \widetilde{Sym}_4 = \{1324, 1432, 2143, 3214\}.$$

The 4 symmetric permutominoes of size 4 are depicted in Figure 12.

### 2.1 Symmetric permutominides: a larger class of objects

Our aim in this section is to consider a larger class of objects, containing the class of symmetric permutominoes and preserving its main features. In practice, the idea is to enlarge this class including a more general class of diagrams having the basic properties of symmetric permutominoes, except for the fact that the boundary is now allowed to cross itself. These objects have nice combinatorial properties and will help us in the enumeration of symmetric permutominoes.

A set of cells is said to be *convex* if each of its rows and columns is non empty and connected (see Fig. 5 (a)). A convex set of cells  $P$  naturally defines a boundary, and the *vertices* of  $P$  are the points at the extremities of each side of maximal length in the boundary of  $P$ . To the set  $P$  we associate a closed (possibly self-intersecting) path  $p(P)$  following the boundary of  $P$  and connecting all its vertices. Such a path uses north  $N = (0, 1)$ , south  $S = (0, -1)$ , east  $E = (1, 0)$  and west  $W = (-1, 0)$  steps. It starts with a north step from the lowest leftmost vertex of  $P$ , and connects a generic vertex  $X$  to the unique vertex  $Y$ , never visited before, which can be reached from  $X$  following an horizontal or a vertical side of the boundary of  $P$  (thus  $Y$  has the same abscissa or ordinate of  $X$ ). This path naturally defines an order on the vertices of  $P$  (see Fig. 5 (b)).

So let  $\mathcal{A} = (A_1, \dots, A_{2(r+1)})$  be the list of vertices of  $P$ . Similarly to the definition of a permutomino, we say that  $P$  is a *symmetric permutominide* if  $\mathcal{P}_1 = (A_1, A_3, \dots, A_{2r+1})$  and  $\mathcal{P}_2 = (A_2, A_4, \dots, A_{2r+2})$  represent two involutions of  $\mathcal{S}_{r+1}$ , indicated as usual by  $\pi_1(P)$  and  $\pi_2(P)$ , respectively. We say that  $n + 1$  is the size of the permutominide. For instance, Figure 5 (c) presents a permutominide and the two involutions defining it.

In this paper we will study the class of convex symmetric permutominides (briefly symmetric permutominides), and we will denote with  $\mathcal{P}_n$  be the class of symmetric permutominides of size  $n$ . Clearly, concerning polyominoes, the definitions of symmetric permutominide and of symmetric permutomino coincide, whence  $Sym_n$  is strictly included in  $\mathcal{P}_n$ . Indeed permutominides retain some properties of symmetric permutominoes.

PROPOSITION 2.4 *If  $P \in \mathcal{P}_n$  then*

1.  $P$  is symmetric with respect to the diagonal  $x = y$ .
2. There is exactly one horizontal (vertical) side of  $P$  for each ordinate (abscissa) between 1 and  $n$ .

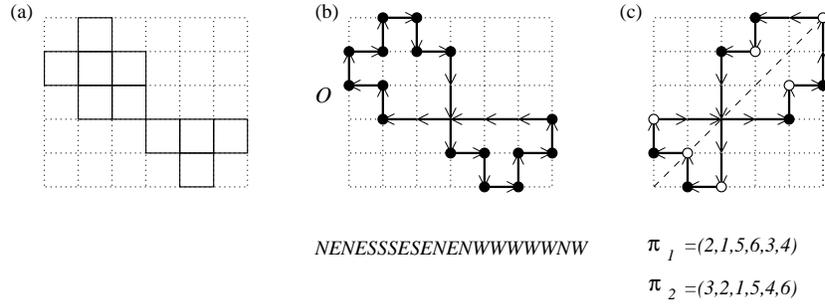


Figure 5: (a) a convex set of cells  $P$ ; (b) the path  $p(P)$ , starting from  $O$ ; (c) a permutominide of size 5, and its two defining involutions.

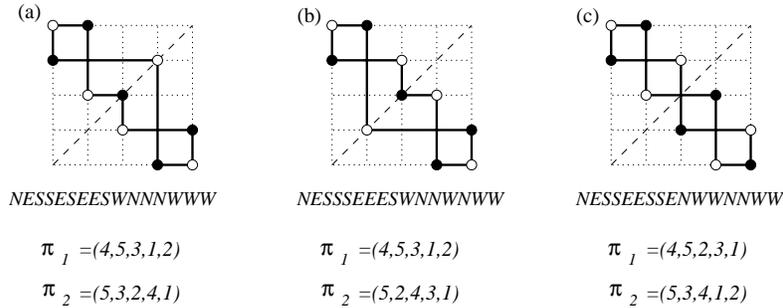


Figure 6: (a),(b) Two symmetric permutominides, and the associated paths and involutions; observe that they have the same  $\pi_1$ ; (c) a permutominide which is symmetric with respect to  $x = y$ , but is not a symmetric permutominide.

In Figure 6 (c) we see that conditions 1. and 2. are not sufficient for a permutominide  $P$  to be a symmetric permutominide. In contrast to what happened for symmetric permutominides, a symmetric permutominide  $P$  is not uniquely determined by simply  $\pi_1(P)$ . For instance, Figure 6 (a) and (b) show two distinct symmetric permutominides  $P_1$  and  $P_2$  such that  $\pi_1(P_1) = \pi_1(P_2)$ . In this section we give the main properties of symmetric permutominides and of the associated permutations.

### Three classes of symmetric permutominides.

The class  $\mathcal{P}_n$  can be partitioned into three disjoint subsets:

1. the set  $Sym_n$  of the symmetric permutominides,
2. the set  $NE_n$  of the symmetric permutominides of size  $n$ , which are not permutominides and such that  $\pi_1(1) < \pi_1(n)$ ; these objects are called, for obvious graphical reasons, *north-east oriented* permutominides (briefly *ne*-permutominides), (see Fig. 5 (c));
3. the set  $SE_n$  of the symmetric permutominides of size  $n$ , which are not permutominides and such that  $\pi_1(1) > \pi_1(n)$ ; these objects are called *south-east oriented* permutominides (briefly

*se*-permutominides), as the one in Fig. 7.

The *ne*-permutominides and the *se*-permutominides have a simple and rather predictable characterization, which relates the problem of their enumeration to the problem of the enumeration of some special subclasses of symmetric permutominoes.

PROPOSITION 2.5 *Let  $P$  be a symmetric permutominide of size  $n$ . We have:*

(i)  *$P$  is a *ne*-permutominide if and only if  $P$  can be uniquely decomposed into a sequence  $V_1 \dots V_h$  of symmetric permutominoes concatenated along the diagonal  $x = y$ , with  $h \geq 2$ , starting from the point  $(1, 1)$ , and where*

1.  $V_1$  is the reflection of a directed convex permutomino with respect to  $x + y = 0$ ,
2.  $V_2, \dots, V_{h-1}$  are parallelogram permutominoes,
3.  $V_h$  is a directed convex permutomino (see for instance Fig. 7 (a)).

(ii)  *$P$  is a *se*-permutominide if and only if  $P$  can be uniquely decomposed into a sequence of  $2h + 1$  permutominoes  $V_1 \dots V_h W V'_h \dots V'_1$ , with  $h \geq 1$ , concatenated along the diagonal  $x + y = n$ , starting from the point  $(1, n)$ , and where,*

1.  $V_1$  is the reflection of a directed convex permutomino with respect to the  $y$ -axis,
2.  $V_2, \dots, V_h$  are the reflection of parallelogram permutominoes with respect to the  $y$ -axis,
3.  $W$  is symmetric with respect to  $x = y$  and is the reflection with respect to the  $y$ -axis of a parallelogram permutomino,
4. each  $V'_i$  is the reflection of  $V_i$  with respect to  $x = y$  (the decomposition of a south-east oriented permutominide is shown in Fig. 7 (b)).

## 2.2 A path representation for symmetric permutominides

Our aim in this section is to enumerate permutominides, and in particular symmetric permutominoes, according to their size. To do this we will encode permutominides in terms of paths in the plane.

**Reentrant points in a symmetric permutominide.** For any  $P \in \mathcal{P}_n$  we consider the path  $p(P)$ , already defined in the previous section, represented by a word in the alphabet  $\{N, E, S, W\}$ .

Any occurrence of a sequence  $NE$ ,  $ES$ ,  $SW$ , or  $WN$  in the word  $p(P)$  defines a *salient point* of  $P$ , while any occurrence of a sequence  $EN$ ,  $SE$ ,  $WS$ , or  $NW$  defines a *reentrant point* of  $P$  (see for instance, Figure 7). Reentrant and salient points were considered in [7], and successively in [5], in a more general context, and it was proved that in any closed path (hence in any permutominide) the difference between the number of salient and reentrant points is congruous to  $0 \pmod 4$ . More specifically, Brlek et al. [5] proved that in any polyomino (hence in any permutomino) the difference between the number of salient and reentrant points is equal to 4.

Hence, a permutomino of size  $n \geq 2$  has exactly  $n - 2$  reentrant and  $n + 2$  salient points. Moreover if  $P$  is a convex permutomino, then there is exactly one reentrant point for each abscissa and ordinate

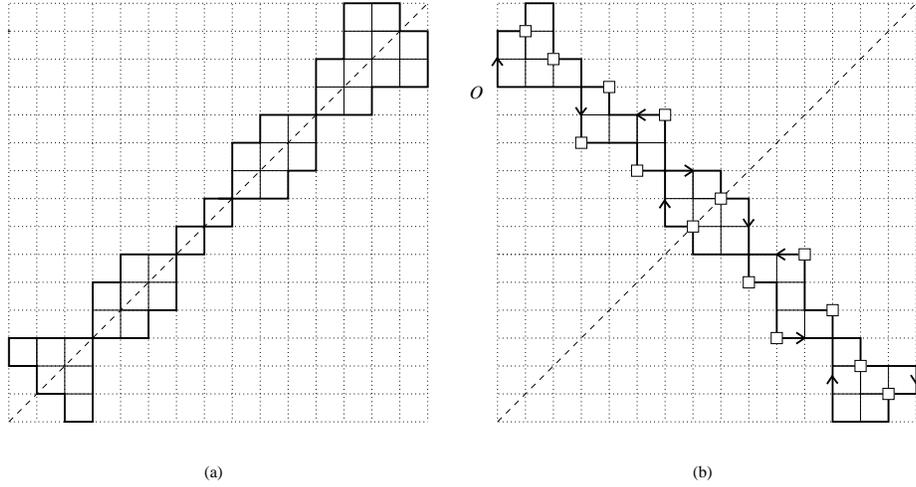


Figure 7: (a) A north-east oriented permutominide; (b) A south-east oriented permutominide and its reentrant points. The path determined by its boundary, starting from  $O$ , has been put in evidence by means of arrows.

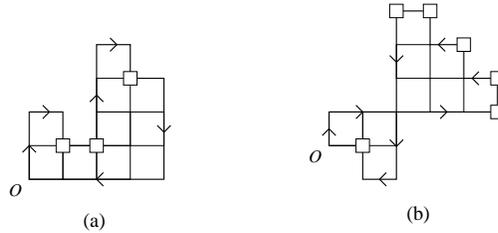


Figure 8: (a) A column-convex permutominide and its reentrant points; (b) a north-east oriented permutominide and its reentrant points.

between 2 and  $n$  (see also [9]). Equivalently, the set of reentrant points of  $P$  defines a permutation matrix of dimension  $n - 2$ .

This property is true *a fortiori* if  $P$  is a symmetric convex permutomino. Observe that the statement does not hold if the permutomino is not convex (see Fig. 8 (a)). We will prove that instead this property holds for *se*-permutominides.

**PROPOSITION 2.6** *If  $P \in SE_n$  then there is exactly one reentrant point for each abscissa and ordinate between 2 and  $n$ . Equivalently,  $R(P)$  defines a permutation matrix of dimension  $n - 2$ .*

**Proof.** By Proposition 2.5,  $P$  can be uniquely decomposed in the concatenation of  $2h+1$  permutominoes  $V_1 \dots V_h V_{h+1} V_{h+2} \dots V_{2h+1}$ , with  $h \geq 1$ , concatenated along the diagonal  $x + y = n$ . By the results we quoted above, for each of these components the statement of the Proposition holds.

Let  $R(P)$  and  $S(P)$  denote, respectively, the sets of reentrant and salient points of  $P$ . Moreover, with  $1 \leq i \leq 2h + 1$ , let  $R(V_i)$  and  $S(V_i)$  denote, respectively, the sets of reentrant and salient points

of the component  $V_i$ , seen as a single permutomino. By the decomposition given in Proposition 2.5, each of the components  $V_2, \dots, V_{2h}$  is the reflection of a parallelogram permutomino according to the  $y$ -axis. For each of these objects  $V_i$ , we denote with  $\hat{S}(V_i)$  the set of the salient points of  $V_i$  except the two points in the leftmost upper corner and in the rightmost lower corner of  $V_i$ .

Looking at Figures 8 (b), and 9, one easily sees that  $R(P)$  can be obtained as

$$R(P) = R(V_1) \cup \dots \cup \hat{S}(V_i) \cup R(V_{i+1}) \cup \dots \cup R(V_{2h+1}), \quad 2 \leq i \leq 2h.$$

This simply brings to the proof of the proposition. □

Observe that the result of Proposition 2.6 does not hold for  $ne$ -permutominides (see Fig. 8 (b)). From the previous characterization we have the following.

**COROLLARY 2.7** *If  $P \in Sym_n \cup SE_n$  then  $P$  is uniquely determined by the set  $R(P)$  of its reentrant points.*

Finally, one can easily prove that if  $P \in Sym_n \cup SE_n$ , then its reentrant points are regularly distributed in the path  $p(P)$ . Let us consider the following points on  $p(P)$  (see Fig. 9):

- $A$  is the vertex of  $P$  with maximal ordinate among the two with abscissa 1;
- $B$  is the vertex of  $P$  with minimal abscissa among the two with ordinate  $n$ ;
- $C$  is the vertex of  $P$  with minimal ordinate among the two with abscissa  $n$ ;
- $D$  is the vertex of  $P$  with maximal abscissa among the two with ordinate 1.

The following proposition describes the positions of the reentrant points of  $P$  within the path  $p(P)$ :

**COROLLARY 2.8** *Let  $P \in Sym_n \cup SE_n$ . Then*

- *the reentrant points of type  $EN$  lie in the subpath of  $p(P)$  from  $A$  to  $B$ ;*
- *the reentrant points of type  $SE$  lie in the subpath of  $p(P)$  from  $B$  to  $C$ ;*
- *the reentrant points of type  $WS$  lie in the subpath of  $p(P)$  from  $C$  to  $D$ ;*
- *the reentrant points of type  $NW$  lie in the subpath of  $p(P)$  from  $D$  to  $A$ .*

**A path encoding for symmetric permutominides.** Let us consider paths in the plane using *up* steps  $U = (1, 1)$ , *down* steps  $D = (1, -1)$ , and two types of *horizontal* steps,  $(1, 0)$ , namely  $\alpha$  and  $\beta$  steps, graphically represented by *solid* and *dotted* horizontal steps, respectively. Let  $M_n$  be the set of these paths running from  $(0, 0)$  to  $(n, 0)$ , remaining weakly above the  $x$ -axis and with the further requirements that:

- (1) an  $\alpha$  (solid) step can never occur after the first occurrence of a down step;
- (2) a  $\beta$  (dotted) step can never occur before the last occurrence of an up step.

Examples of paths of this class are given in Fig. 11, 13, 16.

Now we define the mapping  $\varphi : Sym_n \cup SE_n \rightarrow M_{n-2}$ . Let  $P$  be a permutominide in  $Sym_n \cup SE_n$ . The function  $\varphi$  maps each type of reentrant point of  $P$  into a different type of step (as sketched in Fig. 10):

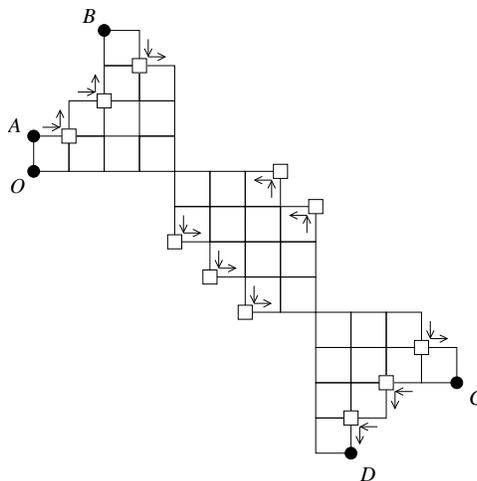


Figure 9: The distribution of the reentrant points of a *se*-permutomino.

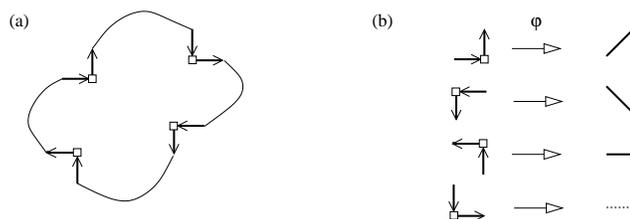


Figure 10: (a) The four types of reentrant points of a permutomino; (b) the coding of each reentrant point through the mapping  $\varphi$ .

- a reentrant point  $EN$  is mapped to an up step  $U$ ;
- a reentrant point  $SE$  is mapped to a horizontal dotted step  $\beta$ ;
- a reentrant point  $WS$  is mapped to a down step  $D$ ;
- a reentrant point  $NW$  is mapped to an up a horizontal solid step  $\alpha$ .

The path  $\varphi(P)$  is obtained as the concatenation, starting from  $(0,0)$  of the steps encoding the reentrant points of  $P$  read from left to right. This is possible since, by Proposition 2.6 there is exactly one reentrant point for each abscissa of  $P$ . For instance, the permutomino in Fig. 4 is mapped to the path in Fig. 11 (a), while the south-east oriented crossing permutomino in Fig. 7 is mapped to the path in Fig. 11 (b).

The images through  $\varphi$  of elements of  $Sym_4 \cup SE_4$  are depicted in Fig. 12.

We first ensure the reader that, if  $P \in Sym_n \cup SE_n$ , then  $\varphi(P) \in M_{n-2}$ :

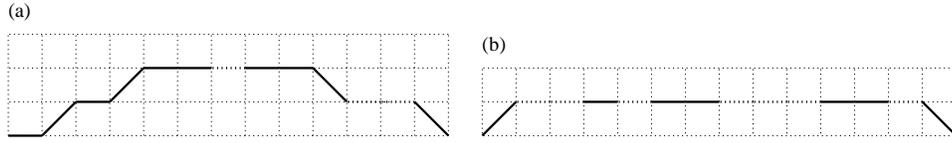


Figure 11: The image, through  $\varphi$ , of the permutominoes of Fig. 4 (a), and Fig. 7 (b).

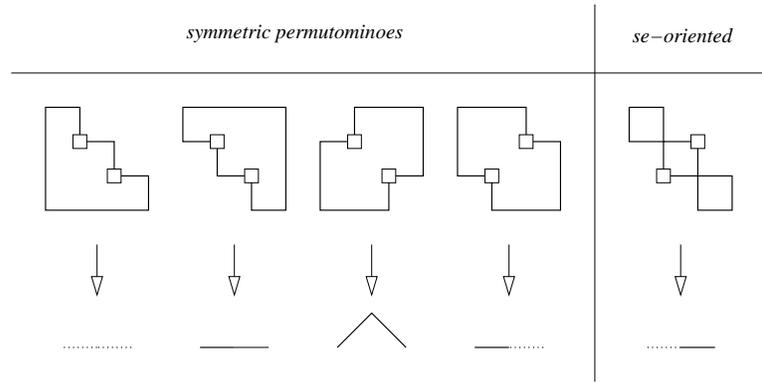
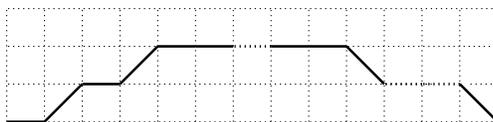


Figure 12: The image of the symmetric permutominides in  $Sym_4 \cup SE_4$  through  $\varphi$ .

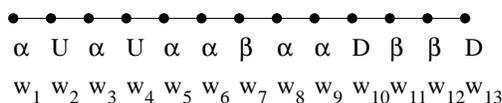
- i. it is clear that  $\varphi(P)$  is a path made of  $n - 2$  steps; moreover, since  $P$  is symmetric,  $\varphi(P)$  has the same number of up and down steps;
- ii. we know from Corollary 2.8 that the reentrant points of type  $EN$ —giving up steps—are located in the subpath from  $A$  to  $B$ , while those of type  $WS$ —giving down steps—are located in the subpath from  $C$  to  $D$ . Hence, if  $P$  is a permutomino, then clearly  $\varphi(P)$  cannot pass below the  $x$ -axis. On the other side, if  $P \in SE_n$ , the decomposition of Proposition 2.5 suggests that all the up steps in  $\varphi(P)$  must come before its down steps, so  $\varphi(P)$  remains strictly above the  $x$ -axis;
- iii. using the same arguments, we see that, from left to right, the reentrant points of type  $EN$  must all come before the first reentrant point of type  $SE$ , and similarly, the reentrant points of type  $NW$  must all come before the first reentrant point of type  $WS$ . Therefore an  $\alpha$  step in  $\varphi(P)$  can never occur after the leftmost occurrence of a down step, and a  $\beta$  step in  $\varphi(P)$  can never occur before the rightmost occurrence of an up step.

PROPOSITION 2.9 For any  $n \geq 2$ , the function  $\varphi$  is a bijection between the sets  $Sym_n \cup SE_n$  and  $M_{n-2}$ .

Proof. We prove the statement defining a function  $\xi : M_{n-2} \rightarrow Sym_n \cup SE_n$  such that, for all  $P \in Sym_n \cup SE_n$ , we have  $\xi(\varphi(P)) = P$ . Let  $w = w_1 \dots w_{n-2} \in M_{n-2}$ , and let  $r$  (respectively  $q$ ) be the number of the horizontal solid  $\alpha$  steps (respectively horizontal dotted steps  $\beta$ ) in  $w$ . The first step to build  $\xi(P)$  is to place its reentrant points, then we will collect them in the unique possible way in order to obtain the associated permutominide.

Figure 13: A path  $w \in M_{13}$ .

For simplicity, we represent the path  $w = w_1 \dots w_{n-2}$  as a sequence of points on a line as, for example, it is shown below



Now we build a matching of the points  $\{1, \dots, n-2\}$  following the rules:

- connect the  $k$ -th up step to the  $k$ -th down step,
- connect the  $k$ -th solid step ( $\alpha$ ) to the  $(r-k+1)$ -th solid step;
- connect the  $k$ -th dotted step ( $\beta$ ) to the  $(q-k+1)$ -th dotted step.

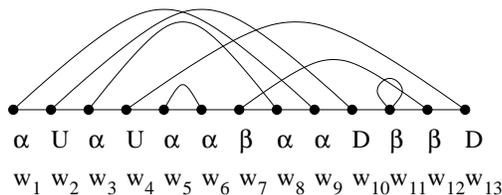


Figure 14: The matching obtained from the path in Fig.13.

Now everything is set to obtain the symmetric permutominide  $P = \xi(w)$  such that  $\varphi(P) = w$ :

- For each up-down edge, starting in  $w_i$  and ending in  $w_l$ , we place a reentrant point of type  $EN$  in the position  $(i+1, l+1)$  and a reentrant point of type  $WS$  in  $(l+1, i+1)$ .
- For each dotted-dotted edge, starting in  $w_i$  and ending in  $w_l$ , we place a reentrant point of type  $SE$  in the positions  $(i+1, l+1)$  and  $(l+1, i+1)$ .
- For each solid-solid edge, starting in  $w_i$  and ending in  $w_l$ , we place a reentrant point of type  $NW$  in the positions  $(i+1, l+1)$  and  $(l+1, i+1)$ .

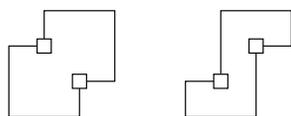
The example in Fig. 15 shows the path  $w$  and the reentrant points it determines. Once the  $n-2$  reentrant points have been placed, by connecting them we can easily build the unique permutominide

of size  $n$  having these points as reentrant points. By the construction of  $\xi$  the reader can easily prove that  $\xi = \varphi^{-1}$ .  $\square$

Definitively, Proposition 2.9 ensures that

$$|Sym_n| + |SE_n| = |Sym_n \cup SE_n| = |M_{n-2}|. \tag{3}$$

*Remark.* We would like to point out that we may naturally define the function  $\varphi$  also on the class of convex permutominoes of size  $n$ . The image through  $\varphi$  of a convex permutomino is still a path in  $M_{n-2}$ , but in this case  $\varphi$  is not injective. For instance the two convex permutominoes below



are both mapped into the path  $UD$ .

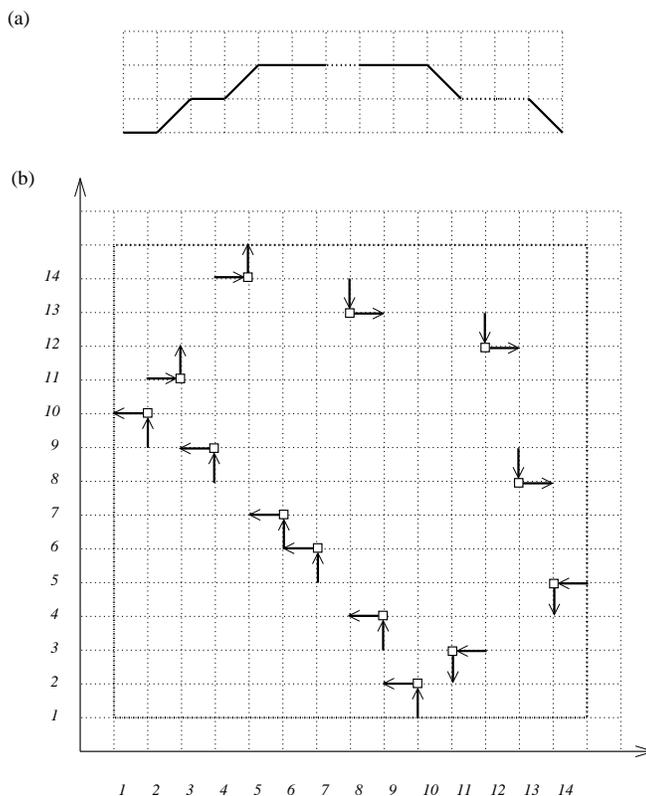


Figure 15: (a) The path  $w \in M_{13}$ ; (b) the set of the reentrant points of  $\varphi^{-1}(w)$ ; joining these points in the natural way we obtain the permutomino of size 15 depicted in Fig. 7.

### 2.3 Enumeration of various classes of symmetric permutominoes

Here we consider some subclasses of symmetric permutominoes and enumerate them using the bijection  $\varphi$ .

PROPOSITION 2.10 *Let  $P \in \text{Sym}_n$ . Then:*

1.  $P$  is directed convex if and only if  $\varphi(P)$  does not contain  $\alpha$  steps (see Fig. 16 (a));
2.  $P$  is parallelogram if and only if  $\varphi(P)$  does not contain  $\alpha$  or  $\beta$  steps, i.e. it is a Dyck path (see Fig. 16 (b)).

The enumeration of these two classes is then straightforward, using standard combinatorial techniques. The generating function of directed convex symmetric permutominoes is

$$\text{Dir}(x) = \frac{x(1 - 2x - \sqrt{1 - 4x^2})}{2(2x - 1)}. \quad (4)$$

The number of symmetric directed convex permutominoes of size  $n$  is therefore

$$\binom{n-2}{\lfloor \frac{n-2}{2} \rfloor}.$$

The generating function of parallelogram symmetric permutominoes is

$$\text{Par}(x) = \frac{1 - \sqrt{1 - 4x^2}}{2}, \quad (5)$$

and the number of parallelogram symmetric permutominoes of size  $2n$  is equal to the Catalan number  $c_{n-2}$ .

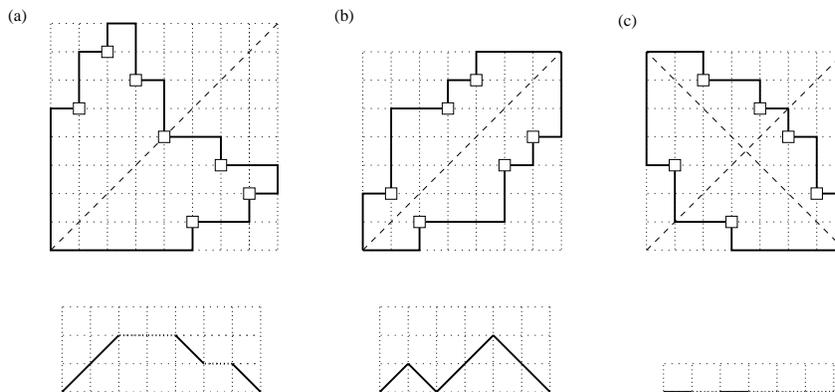


Figure 16: (a) a symmetric directed convex permutomino; (b) A symmetric parallelogram permutomino; (c) a symmetric reflected parallelogram permutomino, and their encoding path.

In this section we are also interested in the class of symmetric permutominoes which are the reflection of a parallelogram permutomino with respect to the  $x$ -axis, which we will call *reflected parallelogram symmetric permutominoes*. Clearly, such permutominoes can only have reentrant points of type  $SE$  or  $NW$ , and  $\varphi$  maps them into paths without up or down steps (see Fig. 16 (c)).

PROPOSITION 2.11 *The number of reflected parallelogram symmetric permutominoes having size  $n$  is equal to  $\binom{n-1}{\lfloor \frac{n-1}{2} \rfloor}$ .*

*Proof.* A reflected parallelogram symmetric permutomino of size  $n$  is uniquely determined by the sub-path  $p$  of its boundary running from  $(1, n)$  to the diagonal  $x = y$ , and remaining weakly above the diagonal  $x + y = n + 1$ , as shown in Fig. 17. Such a path  $p$  is (up to a rotation) a prefix of a Dyck path of length  $n - 1$ . Since the number of prefixes of Dyck paths of length  $n - 1$  is equal to  $\binom{n-1}{\lfloor \frac{n-1}{2} \rfloor}$  [17], we obtain the thesis.  $\square$

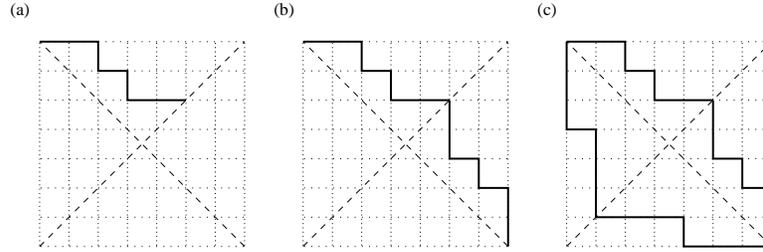


Figure 17: (a) The path  $p$ ; (b) the symmetric of  $p$  with respect to the diagonal  $x = y$ ; (c) the unique permutomino obtained from  $p$ .

### 2.4 Enumeration results

To obtain the enumeration of symmetric permutominoes, we plan to use identity (3), and then we need to count the sets  $SE_n$  and  $M_n$ . Using the decomposition given in Proposition 2.5, and the generating functions of the various classes involved in such a decomposition – determined in the previous section – it is a simple exercise to compute the generating function of  $SE_n$ . We also need to recall from [10] that the generating functions of directed convex permutominoes and of parallelogram permutominoes are, respectively,

$$D(x) = \frac{x(1 - \sqrt{1 - 4x})}{2\sqrt{1 - 4x}} \quad P(x) = \frac{1 - 2x - \sqrt{1 - 4x}}{2}.$$

PROPOSITION 2.12 *The generating function of south-east oriented permutominides according to the size is*

$$SE(x) = \frac{x^2}{2(1 - 2x)} \left( \frac{1}{\sqrt{1 - 4x^2}} - 1 \right). \tag{6}$$

*Proof.* By the decomposition in Proposition 2.5, the generating function is

$$\text{Ref}(x) \frac{D(x^2)}{x^2} \left( 1 + \frac{P(x^2)}{x^2} + \frac{P(x^2)^2}{x^4} + \dots + \frac{P(x^2)^k}{x^{2k}} + \dots \right) = \frac{\text{Ref}(x) D(x^2)}{x^2 - P(x^2)},$$

where  $\text{Ref}(x)$  is the generating function of reflected parallelogram symmetric permutominoes. Hence the assertion of the proposition follows.  $\square$

Consequently, the number of south east oriented permutominides of size  $n$  is:

$$|SE_n| = \frac{n-1}{2} \binom{n-2}{\lfloor \frac{n-2}{2} \rfloor} - 2^{n-3},$$

with  $n \geq 4$ . Using standard combinatorial techniques we can then determine the generating function of the class  $M_n$ .

PROPOSITION 2.13 *The generating function of the class  $M_n$  is*

$$M(x) = \frac{3 - 4x - \sqrt{1 - 4x^2}}{2(1 - 2x)^2}. \quad (7)$$

Now we have all ingredients to count symmetric permutominoes.

THEOREM 2.14 *The generating function of the class  $Sym_n$  is*

$$Sym(x) = \frac{x(1-x)^2}{(1-2x)^2} - \frac{x^2(1+x)}{(1-2x)\sqrt{1-4x^2}} - x, \quad (8)$$

and therefore the number of symmetric permutominoes of size  $n$  is equal to:

$$|Sym_n| = (n+2)2^{n-3} - (n-1) \binom{n-2}{\lfloor \frac{n-2}{2} \rfloor} - (n-2) \binom{n-3}{\lfloor \frac{n-3}{2} \rfloor} \quad n \geq 2. \quad (9)$$

The sequence begins with 1, 2, 4, 10, 22, 54, 120, 284, ... and it is not in [16]. Due to the decomposition given in Proposition 2.5, the generating function of north-east oriented permutominides is given by  $\frac{\text{Dir}^2(x)}{x - \text{Par}(x)}$ , hence it is equal to

$$NE(x) = \frac{x^2(2x - 1 + \sqrt{1 - 4x^2})}{2(1 - 2x)^2}.$$

Finally, recalling that  $|\mathcal{P}_n| = |Sym_n| + |SE_n| + |NE_n|$ , we have

THEOREM 2.15 *The generating function of the class  $\mathcal{P}_n$  is*

$$\mathcal{P}(x) = \frac{x^2(1-x)}{(1-2x)^2}, \quad (10)$$

therefore the number of symmetric permutominides of size  $n$  is  $n2^{n-3}$ ,  $n \geq 2$ .

The result in Theorem 2.15 is quite surprising, since it states that symmetric permutominides are a rational class of objects, while symmetric permutominoes are an algebraic class. The table below lists the first terms of the main sequences treated in the paper, and their identifying number in the Sloane database [16]. We point out that diagonally symmetric polyominoes were first studied in [14], where the sequence  $Ref_n$  was also determined.

$n$	2	3	4	5	6	7	8	9	10	...	Sequence in [16]
$\mathcal{P}_n$	1	3	8	20	48	112	256	576	1280	...	A001792
$M_{n-2}$	1	2	5	12	29	68	158	360	813	...	---
$Sym_n$	1	2	4	10	22	54	120	284	626	...	---
$SE_n$	0	0	1	2	7	14	38	76	187	...	A107373
$NE_n$	0	1	3	8	19	44	98	216	467	...	---
$Dir_n$	1	1	2	3	6	10	20	35	70	...	A001405
$Par_n$	1	0	1	0	2	0	5	0	14	...	A000108
$Ref_n$	1	2	3	6	10	20	35	70	126	...	A001405

### 3 Further work and open problems

We start presenting some minor studies related to the arguments treated in the paper. Since the results are quite simple, we will not provide detailed proofs to the following statements.

**Involutions associated with symmetric permutominides.** We have considered the problem of establishing if a given involution of length  $n$  belongs to the set

$$\tilde{P}_n = \{ \pi_1(P) : P \text{ is a symmetric permutominide} \}.$$

It is convenient to partition this set into three subsets:  $\widetilde{Sym}_n$ , which has already been studied, and

1.  $\widetilde{NE}_n = \{ \pi_1(P) : P \in NE_n \}$ ,
2.  $\widetilde{SE}_n = \{ \pi_1(P) : P \in SE_n \}$ .

1. If  $\pi \in \widetilde{NE}_n$  then we can easily build a symmetric permutomino  $P$  such that  $\pi_1(P) = \pi$ , hence  $\pi \in Sym_n$ ; however there are permutations in  $\widetilde{Sym} \setminus \widetilde{NE}$  (the simplest one is  $(2, 1, 3)$ ). Then  $\widetilde{NE}_n \subset \widetilde{Sym}$ .
2. If  $\pi \in \widetilde{SE}_n$  then  $\pi$  is necessarily a square involution which can be decomposed as  $\pi = \theta \ominus \theta'$ . Moreover, for every square involution which can be decomposed as the direct difference of two permutations, unless  $\pi = (1) \ominus \theta$  (which is the same as  $\pi(1) = n$ ), it is possible to construct at least a south-east oriented symmetric permutominide  $P$  such that  $\pi_1(P) = \pi$ .

Hence, referring to the conditions (1), (2), (3), (4) characterizing the permutations of  $\widetilde{Sym}_n$ , and presented in the Section 2, we can easily give a characterization of the set  $\tilde{P}_n$ .

**PROPOSITION 3.1** *A permutation  $\pi \in \tilde{P}_n$  if and only if  $\pi$  satisfies (1), (3), (4), and (2) is replaced by the weaker*

$$(2') \quad \pi \neq (1) \ominus \pi'.$$

Clearly, since  $\pi$  is an involution, (2') implies that  $\pi \neq \pi' \ominus (1)$ . For instance, the permutation

$$(9, 11, 12, 10, 7, 8, 5, 6, 1, 3, 4, 2) = (1, 3, 4, 2) \ominus (1, 2) \ominus (1, 2) \ominus (1, 3, 4, 2)$$

is in  $\widetilde{P}_{11}$  but not in  $\widetilde{Sym}_{11}$ .

**Permutominides symmetric according to the main diagonal.** Using the results in the last section it is now simple to determine the number of permutominides which are symmetric according to  $x = y$ . In practice, to symmetric permutominides we must add those permutominides which can be decomposed—along the diagonal  $x+y = n+1$ —into a sequence of  $2h$  permutominoes  $V_1 \dots V_h V'_h \dots V'_1$ , with properties given in Proposition 2.5 (ii) (such as for instance, the permutominide 6 (c)). In terms of generating functions, to  $\mathcal{P}(x)$ , we must add the generating function  $\frac{D(x^2)}{x^2 - P(x^2)} = \frac{x^2}{1 - 4x^2}$ .

Then the number of permutominides symmetric according to the main diagonal, and having size  $n$  is  $n2^{n-3}$ , if  $n$  is odd, and  $n2^{n-3} + 4^{\frac{n}{2}-1}$ , if  $n$  is even.

**Some open problems.** There are some problems related to the study of symmetric permutominoes, which we plan to consider in some further research. Below we give a brief list of the problems in which we are more interested:

1. It would be interesting and useful to prove in a bijective way that the number of symmetric permutominides of size  $n$  is  $n2^{n-3}$ . This proof should be related to some bijective proof that the number of convex permutominides (not necessarily symmetric) of size  $n$  is  $2n4^{n-3}$ . We believe that such proofs should give us many information on the properties of symmetric permutominoes, and of convex permutominoes.
2. Involutions of the symmetric group have been studied by several authors and by various points of view. We would like to study some algebraic properties of the involutions defining symmetric permutominoes, and particularly, to see if these involutions inherit some well-known properties of involutions. For instance we could consider the poset defined on these involutions by the Bruhat order (also referring to the poset defined by involutions [11]), or some classical statistics such as number of descents, excedences, fixed points, and many others.
3. In this paper we have considered the class of *convex* symmetric permutominoes. It would be interesting to determine similar results for some larger classes of symmetric permutominoes, for instance the *directed symmetric permutominoes*.

## References

- [1] A. BERNINI, F. DISANTO, R. PINZANI AND S. RINALDI *Permutations defining convex permutominoes*, J. Integer Seq., 10 (2007) Article 07.9.7.
- [2] P. BOLDI, V. LONATI, R. RADICIONI, AND M. SANTINI, *The number of convex permutominoes*, Proceedings of LATA 2007, International Conference on Language and Automata Theory and Applications, Tarragona, Spain.

- 
- [3] M. BOUSQUET-MÉLOU, *A method for the enumeration of various classes of column convex polygons*, Discrete Math., 154 (1996) 1–25.
- [4] M. BOUSQUET-MÉLOU AND A. J. GUTTMANN, *Enumeration of three dimensional convex polygons*, Ann. Comb., 1 (1997) 27–53.
- [5] S. BRLEK, G. LABELLE AND A. LACASSE, *Properties of the countour path of discrete sets*, Internat. J. Found. Comput. Sci., 17 (2006) 543–556.
- [6] S. J. CHANG AND K. Y. LIN, *Rigorous results for the number of convex polygons on the square and honeycomb lattices*, J. Phys. A, 21 (1988) 2635–2642.
- [7] A. DAURAT AND M. NIVAT, *Salient and reentrant points of discrete sets*, Discrete Appl. Math., 151 (2005) 106–121.
- [8] M. DELEST AND X. G. VIENNOT, *Algebraic languages and polyominoes enumeration*, Theoret. Comput. Sci., 34 (1984) 169–206.
- [9] F. DISANTO, A. FROSINI, R. PINZANI AND S. RINALDI, *A closed formula for the number of convex permutominoes*, Electron. J. Combin., 14 (2007) Article #R57.
- [10] I. FANTI, A. FROSINI, E. GRAZZINI, R. PINZANI AND S. RINALDI, *Characterization and enumeration of some classes of convex permutominoes*, Pure Math. Appl. (P.U.M.A.), 18 (2007) 265–290.
- [11] F. INCITTI, *The Bruhat Order on the Involutions of the Symmetric Group*, J. Algebraic Combin., 20 (2004) 243–261.
- [12] F. INCITTI, *Permutation diagrams, fixed points and Kazhdan-Lusztig R-polynomials*, Ann. Comb., 10 (2006) 369–387.
- [13] C. KASSEL, A. LASCoux AND C. REUTENAUER, *The singular locus of a Schubert variety*, J. Algebra, 269 (2003) 74–108.
- [14] P. LEROUX, E. RASSART AND A. ROBITAILLE, *Enumeration of Symmetry Classes of Convex Polyominoes in the Square Lattice*, Adv. in Appl. Math., 21 (1998) 343–380.
- [15] T. MANSOUR AND S. SEVERINI, *Grid polygons from permutations and their enumeration by the kernel method*, FPSAC 2007, Tianjin, China, 2007.
- [16] N. J. A. SLOANE, *The On-Line Encyclopedia of Integer Sequences*, at <http://oeis.org/>
- [17] R. P. STANLEY, *Enumerative Combinatorics 2*, Cambridge University Press, Cambridge, 1999.
- [18] S. WATON, *On Permutation Classes Generated by Token Passing Networks, Gridding Matrices and Pictures: Three Flavours of Involvement*, PhD Thesis, University of St. Andrews, 2007.

