A note on divided ideals

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Abstract. Let R be an integral domain with quotient field K and let I be a nonzero ideal of R which is comparable to every ideal of R. We investigate when I^{-1} is a ring. We show that if I is not invertible, then I^{-1} is a ring if and only if I is prime.

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1 Introduction

In this paper, we consider an integral domain R where has a nonzero ideal I which is comparable to every principal ideal of R, such ideals are called divided ideal. Suppose that K is the quotient field of R, the R-submodule J of K is called fractional ideal if there exists an element $a \in R$ such that $aJ \subseteq R$. For a nonzero fractional ideal J of R, the fractional ideal $(R : J) = \{ x \in K \mid xJ \subseteq R \}$ is called the dual of J and we show with J^{-1}. In [7], Huckaba and Papick studied the question of when I^{-1} is a ring, and this question has received further attention in [1], [2], [3], [4], [5] and [6].

We note that while (I : I) is always an overring of R, I^{-1} need not be a ring at all. In fact, $(I : I)$ is the largest overring of R in which I is still an ideal. Clearly $(I : I) \subseteq I^{-1}$, and if we have equality, then I^{-1} is a ring. Example 3.1 of [1] shows that I^{-1} may be a ring but $I^{-1} \neq (I : I)$. Our purpose here is to determine when I^{-1} is a ring, where I is a divided ideal of R.

We start by recalling the following results proved in [7].

Proposition 1.1 *(See [7; Lemma 2.0]) If I is a proper invertible ideal of R, then I^{-1} is not a subring of K.*

Proposition 1.2 *(See [7; Proposition 3.5]) Let I be an ideal of a valuation domain R. Then I^{-1} is a subring of K if and only if I is a noninvertible prime ideal.*

2 Divided ideals

Definition 2.1 Let R be an integral domain and I be a proper ideal of R. If I is comparable to every principal ideal of R, then I is called divided ideal.

Proposition 2.2 *Let R be an integral domain and P be a nonzero ideal of R. The following statements are equivalent:*

1. P is a divided prime ideal.

2. For every element $x \in R \setminus P$, $P = xP$.

Proof. **1 ⇒ 2.** Let $x \in R \setminus P$. Then $P \subseteq Rx$. Thus for each element $a \in P$, there exists $r \in R$ such that $rx = a \in P$ and so $r \in P$. Hence $P \subseteq xP \subseteq P$. Therefore $P = xP$.
2 ⇒ 1. Let $x \in R$. If $x \in P$, then $Rx \subseteq P$ and if $x \notin P$, then $P = xP \subseteq Rx$. Therefore P is divided ideal. We now assume that $xy \in P$, for $x, y \in R$. If $x \notin P$, then $P = xP$ and so $xy \in P = xP$. Hence there exists an element $a \in P$ such that $xy = xa$. Thus $y = a \in P$ and consequently P is a prime ideal. □

We can now prove a result which shows that an invertible divided prime ideal is a principal maximal ideal.

PROPOSITION 2.3 *Let R be an integral domain and P be a nonzero divided prime ideal of R. If P is invertible, then R is a local ring with maximal ideal P. Furthermore, P is a principal ideal.*

Proof. Since P is invertible, $PP^{-1} = R$ and so $1 = \sum_{i=1}^{n} a_i b_i$, for some elements $a_1, a_2, \cdots, a_n$ of P and $b_1, b_2, \cdots, b_n$ of P^{-1}. For every element $a \in R \setminus P$, we have $P = aP$, by Proposition 2.2. Thus, for each i, $(1 \leq i \leq n)$, there exists an element $s_i \in P$ such that $as_i = a_i$. Hence,

$$1 = \sum_{i=1}^{n} a_i b_i = \sum_{i=1}^{n} a s_i b_i = a(\sum_{i=1}^{n} s_i b_i).$$

Since $s_i b_i \in PP^{-1} = R$, for all i, a is an unit of R. Therefore P is the unique maximal ideal of R and consequently R is a local ring. On the other hand, since P is an invertible ideal of a local ring, P is principal ideal. □

In sequel, we obtain for a divided ideal I, the following result on when I^{-1} is a ring. This result is a generalization of Proposition 1.2.

THEOREM 2.4 *Let R be an integral domain and I be a nonzero divided ideal of R. Then I^{-1} is a overring of R if and only if I is a noninvertible prime ideal. Moreover, in this case $I^{-1} = (I : I)$.*

Proof. Suppose that I^{-1} is a ring. Then I is not invertible, by Proposition 1.1. Let $a, b \in R$. If $a, b \notin I$, then $I \subset Ra$ and $I \subset Rb$. Thus $\frac{1}{a}, \frac{1}{b} \in I^{-1}$. Since I^{-1} is a ring, so $\frac{1}{ab} = \frac{1}{a}\frac{1}{b} \in I^{-1}$. Hence $\frac{1}{ab} I \subseteq R$ and consequently $I \subseteq Rab$. If $I = Rab$, then I is invertible, a contradiction. Therefore $I \subset Rab$ and so $ab \notin I$. Hence I is a prime ideal.

Now, suppose that I is a prime ideal. If $II^{-1} \subseteq I$, then $I^{-1} = (I : I)$ is a ring. In contrary, we assume that $II^{-1} \nsubseteq I$. Hence there exist elements $x \in I^{-1}$ and $a \in I$ such that $b = ax \notin I$. Since I is divided and $b \notin I$, then $I \subset Rb$ and so $a = rb$ for some $r \in I$, because I is prime. Thus $b = xrb$. Hence $xr = 1$. Therefore $II^{-1} = R$ which is a contradiction. □

COROLLARY 2.5 *For every divided prime ideal P of the integral domain R, either P is maximal and so R is a local ring or P^{-1} is a ring.*

PROPOSITION 2.6 *Let R be an integral domain and I be a proper ideal of R. If I^n is a divided ideal for every positive integer n, then*

1. *$P = \bigcap_{n=1}^{\infty} I^n$ is a divided prime ideal.*
2. *For every prime ideal Q which $Q \subset I$, we have $Q \subseteq P$.*
3. *If $I \neq I^2$, then P^{-1} is a ring.*

Proof. **1.** Let $x, y \in R$ and $xy \in P$. If $x \notin P$, then $x \notin I^n$ for some integer n. Thus $I^n \subset Rx$. Hence for each integer m, we have

$$xy \in Rxy \subseteq P \subseteq I^{m+n} = I^m I^n \subseteq I^m x$$

whence $y \in I^m$. Therefore $y \in P$ and so P is a prime ideal. Obviously P is divided.
2. If $Q \nsubseteq P$, then there exists an element $x \in Q$ such that $x \notin P$. Thus $x \notin I^n$ for some n. Since I^n is divided, then $I^n \subseteq Rx \subseteq Q$. Hence $I \subseteq Q$, because Q is prime. This is a contradiction. Therefore $Q \subseteq P$.
3. Since $I \neq I^2$, $P \subset I$ and so P is not maximal. Therefore P^{-1} is a ring, by Corollary 2.5. □

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