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Quasi k-ideals in k-regular and intra k-regular semirings

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Abstract. A semiring S whose additive reduct is a semilattice, is called an intra k-regular semiring if for each $a \in S$ there exists $x \in S$ such that $a + xa^2x = xa^2x$ and is called a k-regular semiring if for each $a \in S$ there exists $x \in S$ such that $a + axa = axa$. Here we introduce quasi k-ideals in semirings and characterize both the k-regular and intra k-regular semirings by their quasi k-ideals.

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1 Introduction

The notion of quasi ideals in rings and semigroups was introduced and developed by Otto Steinfeld [15], [16], [17]. For an exposition of the theory we refer to the book [18] by the same author. Quasi ideals are generalizations of both left ideals and right ideals as well as a particular case of bi-ideals. The (m,n)-quasi ideal, due to S. Lajos [9] is a generalization of this notion. Lajos characterized the quasi ideals in regular semigroups [10]. Kapp [7] found that for an absorbant semigroup with 0 every $\mathcal{H}$-class together with 0 is a quasi ideal. Quasi ideals of different classes of semigroups and semirings have been characterized by many authors in [6], [8], [4].

This article is a continuation of [2] of studding semirings in SL^+ by their different types of ideals. In this article we introduce the notion of quasi k-ideals in a semiring and characterize the k-regular semirings using quasi k- ideals. Bourne [3] introduced the k-regular semirings as a generalization of regular rings. Later these semirings have been studied by Sen, Weinert, Bhuniya, Adhikari [1], [12], [13], [14]. For any semigroup F, the semiring $P(F)$ of all subsets of F is a k-regular semiring if and only if F is a regular semigroup [14]. We have introduced the intra k-regular semirings as the class of semirings to which the semiring $P(F)$ belong when F is an intra regular semiring [2]. Also a semiring S is intra k-regular if and only if every k-ideal of S is semiprime. Here we show that Q is a quasi k-ideal of the semiring $P(F)$ if and only if $Q = P(P)$ for some quasi ideal P of F. Thus it is of interest to characterize the k-regular semirings using quasi k-ideals.

Some elementary results together with prerequisites have been discussed in Section 2. Section 3 is devoted to characterize the k-regular semirings by their quasi k-ideals. In section 4, the quasi k-ideals have been used to characterize the intra k-regular semirings and the semirings which are both k-regular and intra k-regular.

2 Preliminaries

A semiring $(S, +, \cdot)$ is an algebra with two binary operations $+$ and $\cdot$ such that both the additive reduct $(S, +)$ and the multiplicative reduct $(S, \cdot)$ are semigroups and such that the following distributive laws hold:

$$x(y+z) = xy + xz \text{ and } (x+y)z = xz + yz.$$

A band is a semigroup in which every element is an idempotent. A commutative band is called a semilattice. Throughout this paper, unless otherwise stated, S is always a semiring whose additive reduct is a semilattice and the variety of all such semirings is denoted by SL^+.

A non-empty subset L of a semiring S is called a left ideal of S if $L + L \subseteq L$ and $SL \subseteq L$. The right ideals are defined dually. A subset I of S is called an ideal of S if it is both a left and a right ideal of S. A non-empty subset A is called an interior ideal of S if $A + A \subseteq A$ and $SAS \subseteq A$. A non-empty subset A of S is called semiprime if for $a \in S$, $a^2 \in A$ implies that $a \in A$.

Henriksen [5] defined an ideal (left, right) I of a semiring S to be a k-ideal (left, right) if for $a, x \in S$,

$$a, a + x \in I \ \Rightarrow \ x \in I.$$

We define interior k-ideal similarly.

Later on the notion of k-subset of a semiring evolved. A non-empty subset A of S is called a k-subset of S if for $x \in S$,

$$a \in A,\ x + a \in A \text{ implies that } x \in A.$$

The k-closure $\overline{A}$ of a non-empty subset A is given by,

$$\overline{A} = \{x \in S \mid \exists\ a, b \in A \text{ such that } x + a = b\}.$$

This is the smallest k-subset containing A. If A and B be two subsets of S such that $A \subseteq B$ then it follows that $\overline{A} \subseteq \overline{B}$. Since the additive reduct $(S, +)$ is a semilattice, it follows that an ideal(left, right) K of S is a k-ideal(left, right) if and only if $\overline{K} = K$.

Definition 2.1 A subsemiring Q is called a quasi ideal of S if $QS \cap SQ \subseteq Q$.

A quasi ideal Q is called a quasi k-ideal of S if $\overline{Q} = Q$.

For examples of quasi k-ideals of a semiring we would like to explore the following natural connection between quasi ideals of a semigroup F and quasi k-ideals of the semiring $P(F)$ of all subsets of F.

Let F be a semigroup and $P(F)$ be the set of all subsets of F. Define addition and multiplication on $P(F)$ by:

$$\begin{aligned} U + V &= U \cup V \\ \text{and } UV &= \{ab \mid a \in U, b \in V\}, \text{ for all } U, V \in P(F). \end{aligned}$$

Then $(P(F), +, \cdot)$ is a semiring whose additive reduct is a semilattice. Then we have the following result.

THEOREM 2.2 *Let F be a semigroup. Then Q is a quasi k-ideal of $P(F)$ if and only if $Q = P(P)$ for some quasi-ideal P of F.*

Proof. Let P be a quasi ideal of F and $Q = P(P)$. Let $A = \{a_1, a_2, ...a_n\} \in SQ \cap QS$ where $S = P(F)$. Then for each a_i there exist $\{s_i\}, \{t_i\} \in S$ and $\{p_i\}, \{q_i\} \in Q$ such that $a_i = s_i p_i = q_i t_i$. But $a_i \in FP \cap PF \subseteq P$ for all i. Thus $A \subseteq P$. Hence $A \in Q$. Therefore $SQ \cap QS \subseteq Q$. Thus Q is a quasi ideal of S. Now let $U \in S$ and $V_1, V_2 \in Q$ such that $U + V_1 = V_2$. Then we have $U \cup V_1 = V_2$. Which implies that $U \subseteq P$. Thus $U \in Q$. Therefore Q is a quasi k-ideal of S.
Conversely, let Q be a quasi k-ideal of $S = P(F)$. We consider $P = \bigcup_{U \in Q} U$. Then $P \subseteq F$ and $Q \subseteq P(P)$. Let$B \in P(P)$. Then $B \in Q$. Therefore $Q = P(P)$.
We now prove that P is a quasi ideal of F i.e. $FP \cap PF \subseteq P$. Let $x \in FP \cap PF$. Then there exist $p, q \in P$ and $s, t \in F$ such that $x = sp = qt$. Now $SQ \cap QS \subseteq Q \subseteq P(F) = S$. Now $\{s, t\}\{p, q\} \in SQ$ and $\{p, q\}\{s, t\} \in QS$. Also $\{sp\} = \{s\}\{p\} \in SQ$ and $\{qt\} = \{q\}\{t\} \in QS$. Thus $\{x\} \in SQ \cap QS \subseteq Q$ and so $x \in P$. Therefore $FP \cap PF \subseteq P$ and hence P is a quasi ideal of F. □

LEMMA 2.3 *Let S be a semiring. Then for all right k-ideal R and left k-ideal L of S, $R \cap L$ is a quasi k-ideal of S.*

Proof. Let R and L be a right k-ideal and left k-ideal of S respectively. Then we have

$$\begin{aligned} &(R \cap L)S \cap S(R \cap L) \\ &\subseteq RS \cap SL \text{ as } R \cap L \subseteq R \text{ and } R \cap L \subseteq L \\ &\subseteq R \cap L \text{ as } RS \subseteq R \text{ and } SR \subseteq L \end{aligned}$$

and so $R \cap L$ is a quasi ideal of S. Since intersection of two k-subsets is a k-set of a semiring, it follows that $R \cap L$ is a quasi k-ideal of S. □

Let $a \in S$. We denote $L[a] = \{\Sigma_{i=1}^{n} x_i | x_i \in \{a\} \cup Sa\}$. Since the additive reduct $(S, +)$ is a semilattice, it follows that $L[a] = \{a + sa \mid s \in S\}$. Then $L[a]$ is a subsemiring of S. Also for any $s \in S$ and $u \in L[a]$, we have $su \in Sa$ which implies that $SL[a] \subseteq L[a]$ and so $L[a]$ is a left ideal of S. As in [2], following description for the principal left k-ideal $L_k(a)$ and right k-ideal $R_k(a)$ of S can be verified easily.

LEMMA 2.4 *Let S be a semiring and $a \in S$.*

1. *Then the principal left k-ideal of S generated by a is given by $L_k(a) = \{u \in S \mid u + a + sa = a + sa,$ for some $s \in S\}$.*

2. *Then the principal right k-ideal of S generated by a is given by $R_k(a) = \{u \in S \mid u + a + as = a + as,$ for some $s \in S\}$.*

3 Quasi ideals in k-regular semirings

A semigroup S is called a regular semigroup if for each $a \in S$ there exists $x \in S$ such that $a = axa$. In [11], Von Neumann defined a ring R to be regular if the multiplicative reduct $(R, \cdot)$ is a regular semigroup. Bourne [3] defined a semiring S to be regular if for each $a \in S$ there exist $x, y \in S$ such that $a + axa = aya$. If a semiring S happens to be a ring then the Von Neumann regularity and the Bourne regularity are equivalent. This is not true in a semiring in general (For counter example we refer [12]). Adhikari, Sen and Weinert [1] renamed the Bourne regularity of a semiring as k-regularity to distinguish from the notion of Von Neumann regularity.

DEFINITION 3.1 A semiring S is called a k-regular semiring if for each $a \in S$ there exist $x, y \in S$ such that $a + axa = aya$.

Since $(S, +)$ is a semilattice, we have

$$\begin{aligned} a + axa = aya \ &\Rightarrow a + axa + (axa + aya) = aya + (axa + aya) \\ &\Rightarrow a + a(x + y)a = a(x + y)a. \end{aligned}$$

Thus, a semiring S is k-regular if and only if for all $a \in S$ there exists $x \in S$ such that

$$a + axa = axa.$$

Let S be a k-regular semiring and $a \in S$. Then there exists $x \in S$ such that $a + axa = axa$. Then we have

$$\begin{aligned} a + axa = axa \ &\Rightarrow \ a + ax(a + axa) = ax(a + axa) \\ &\Rightarrow \ a + axaxa = axaxa. \end{aligned}$$

Thus, a semiring S is k-regular if and only if for all $a \in S$ there exists $x \in S$ such that

$$a + axaxa = axaxa. \tag{1}$$

For examples and properties of k-regular semirings we refer [1], [12], [13], [14].
The following result was proved by Mukhopadhayay [] for semiring in general. We observe that the proof of this result can be made significantly simpler than the proof by Mukhopadhayay when the semiring S is taken from SL^+.

THEOREM 3.2 *Let S be a semiring. Then S is k-regular if and only if $\overline{RL} = R \cap L$ for any right k-ideal R and left k-ideal L of S.*

Proof. Let S be a k-regular semiring. Then for any right k-ideal R and left k-ideal L of S, $RL \subseteq RS \subseteq R$ and $RL \subseteq SL \subseteq L$. Then $RL \subseteq R \cap L$ implies that $\overline{RL} \subseteq R \cap L$. Also for $a \in R \cap L$ there exists $x \in S$ such that $a + axa = axa$. Then $(ax)a \in RL$ implies that $a \in \overline{RL}$ and so $R \cap L \subseteq \overline{RL}$. Thus $\overline{RL} = R \cap L$.
Conversely, let $\overline{RL} = R \cap L$ for any right k-ideal R and left k-ideal L of S. Let $a \in S$, $R = R_k(a) = \{u \in S \mid u + a + as = a + as\}$ and $L = L_k(a) = \{v \in S \mid v + a + sa = a + sa\}$. Then $a \in R \cap L = \overline{RL}$. Then there exist $u \in R$ and $v \in L$ such that $a + uv = uv$. This implies that $a + (a + as)(sa + a) = (a + as)(sa + a)$. Thus $a + aya = aya$ for some $y \in S$. Hence S is k-regular. □

Now we give several equivalent characterizations of k-regularity in terms of quasi k-ideals.

THEOREM 3.3 *Let S be a k-regular semiring and A be a non-empty subset of S. Then A is a quasi k-ideal of S if and only if $A = \overline{RL}$, where R is a right k-ideal and L is a left k-ideal of S.*

Proof. Let A be a quasi k-ideal of k-regular semiring S and $a \in A$. Then $R = R_k(a)$ and $L = L_k(a)$ are right k-ideal and left k-ideal of S respectively. Since S is k-regular and $a \in A \subseteq S$, there is $x \in S$ such that $a + axaxa = axaxa$. Now $a \in R \Rightarrow ax \in R$ and $a \in L \Rightarrow (ax)a \in L$. Then $axaxa \in RL$. Thus $a \in \overline{RL}$. Therefore $A \subseteq \overline{RL}$. Now consider $u \in R$ and $v \in L$. Then, by Lemma 2.4, there are $s, t \in S$ such that $u + as + a = as + a$ and $v + ta + a = ta + a$. Then $uv + (as + a)v = (as + a)v$ implies that $uv + (as + a)(ta + a) = (as + a)(ta + a) \Rightarrow uv + a(st + s + t)a + a^2 = a(st + s + t)a + a^2$. Again $a(st + s + t)a + a^2 \in AS \cap SA \subseteq A$ shows that $uv \in \overline{A} \subseteq A$. Then $RL \subseteq A$ and so $\overline{RL} \subseteq A$. Hence $A = \overline{RL}$.
Conversely, let for a non-empty subset A of S, $A = \overline{RL}$, where R is a right k- ideal and L is a left k-ideal of S. But by the above theorem, for a k-regular semiring S, $\overline{RL} = R \cap L$. Then $A = R \cap L$. But $R \cap L$ is quasi k-ideal of S[Lemma 2.3]. Thus A is a quasi k-ideal of S. □

THEOREM 3.4 *For a semiring S the following conditions are equivalent:*

1. *S is k-regular.*
2. *$Q = \overline{QSQ}$ for every quasi k-ideal Q of S.*

Proof. **(1) ⇒ (2):** Let Q be a quasi k-ideal of S. Then $QSQ \subseteq QS \cap SQ \subseteq Q$ implies that $\overline{QSQ} \subseteq Q$. Let $a \in Q$. Since S is k-regular, there is $x \in S$ such that $a + axa = axa$. Then $axa \in QSQ$ implies that $a \in \overline{QSQ}$, whence $Q \subseteq \overline{QSQ}$. Thus $Q = \overline{QSQ}$.
(2) ⇒ (1) : Let $a \in S$. Then $Q = L_k(a) \cap R_k(a)$ is a quasi k-ideal of S, by Lemma 2.3 and 2.4. Then $a \in Q = \overline{QSQ}$ and this implies that there exist $q_1,\ q_2,\ q_3,\ q_4 \in Q$ and $s_1,\ s_2 \in S$ such that

$$\begin{aligned}
&a + q_1 s_1 q_2 = q_3 s_2 q_4 \\
\Rightarrow\ & a + (q_1 + q_2 + q_3 + q_4)(s_1 + s_2)(q_1 + q_2 + q_3 + q_4) = (q_1 + q_2 + q_3 + q_4)(s_1 + s_2)(q_1 + q_2 + q_3 + q_4) \\
\Rightarrow\ & a + qsq = qsq,
\end{aligned}$$

where $q = (q_1 + q_2 + q_3 + q_4) \in Q = L_k(a) \cap R_k(a)$ and $s = (s_1 + s_2) \in S$. Then there exist $x, y \in S$ such that $q + xa + a = xa + a$ and $q + ay + a = ay + a$[Lemma 2.4]. Thus we have

$$\begin{aligned}
a + qsq = qsq\ &\Rightarrow a + (q + ay + a)s(q + xa + a) = (q + ay + a)s(q + xa + a) \\
&\Rightarrow a + (ay + a)s(xa + a) = (ay + a)s(xa + a) \\
&\Rightarrow a + a(ysx + ys + sx + s)a = a(ysx + ys + sx + s)a \\
&\Rightarrow a + ata = ata,
\end{aligned}$$

where $t = ysx + ys + sx + s \in S$, and so S is a k-regular semiring. □

THEOREM 3.5 *For a semiring S the following conditions are equivalent:*

1. *S is k-regular.*
2. *$Q \cap J = \overline{QJQ}$ for every quasi k-ideal Q and every k-ideal J of S.*
3. *$Q \cap I = \overline{QIQ}$ for every quasi k-ideal Q and every interior k-ideal I of S.*

Proof. Since each k-ideal is an interior k-ideal, it is clear that (3) $\Rightarrow$ (2). Hence we are to prove (1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (1) only.
(1) $\Rightarrow$ (3): Let Q be a quasi k-ideal and I be an interior k-ideal of S. Then $QIQ \subseteq QSQ \subseteq QS \cap SQ \subseteq Q$ and $QIQ \subseteq SIS \subseteq I$ implies that $QIQ \subseteq Q \cap I$ and so $\overline{QIQ} \subseteq Q \cap I$. Let $a \in Q \cap I$. Since S is k-regular, there is $x \in S$ such that $a + axaxa = axaxa$, by (1). Now $a(xax)a \in Q(SIS)Q \subseteq QIQ$ implies that $a \in \overline{QIQ}$ and so $Q \cap I \subseteq \overline{QIQ}$. Thus $Q \cap I = \overline{QIQ}$.
(2) $\Rightarrow$ (1): Let Q be a quasi k-ideal of S. Since S is a k-ideal of S, $Q \cap S = \overline{QSQ}$ i.e. $Q = \overline{QSQ}$. Hence S is a k-regular semiring, by Theorem 3.4. □

THEOREM 3.6 *For a semiring S the following conditions are equivalent:*

1. *S is k-regular.*
2. *$R \cap L \subseteq \overline{RL}$ for every right k-ideal R and every left k-ideal L of S.*
3. *$Q \cap L \subseteq \overline{QL}$ for every quasi k-ideal Q and every left k-ideal L of S.*

Proof. $(\mathbf{1}) \Rightarrow (\mathbf{3})$: Let Q be a quasi k-ideal and L be a left k-ideal of S respectively. Let $a \in Q \cap L$. Since S is k-regular, there is $x \in S$ such that $a + axa = axa$. Now $a(xa) \in Q(SL) \subseteq QL$. Then $a \in \overline{QL}$. Thus $Q \cap L \subseteq \overline{QL}$.
$(\mathbf{3}) \Rightarrow (\mathbf{2})$: Since every right k-ideal is a quasi k-ideal of S, it follows that $R \cap L \subseteq \overline{RL}$.
$(\mathbf{2}) \Rightarrow (\mathbf{1})$: Let $a \in S$. Consider $L = L_k(a)$, $R = R_k(a)$. Then $a \in R \cap L$ implies that there exist $r \in R$ and $l \in L$ such that $a + rl = rl$. Again there exist $s, t \in S$ such that $r + as + a = as + a$ and $l + ta + a = ta + a$. Thus we have

$$\begin{aligned} a + rl = rl &\Rightarrow a + (r + as + a)(l + ta + a) = (r + as + a)(l + ta + a) \\ &\Rightarrow a + a^2 + aua = a^2 + aua, \text{ for some } u \in S \\ &\Rightarrow a + a(a + a^2 + aua) + aua = a(a + a^2 + aua) + aua \\ &\Rightarrow a + ava = ava, \end{aligned}$$

for some $v = a + au + u \in S$, whence S is k-regular semiring. □

The left-right dual of this theorem is as follows:

THEOREM 3.7 *For a semiring S the following conditions are equivalent:*

1. *S is k-regular.*
2. *$Q \cap R \subseteq \overline{RQ}$ for every quasi k-ideal Q and every right k-ideal R of S.*

THEOREM 3.8 *For a semiring S, the following conditions are equivalent:*

1. *S is k-regular.*
2. *$R \cap Q \cap L \subseteq \overline{RQL}$ for every right k-ideal R, every quasi k-ideal Q and every left k-ideal L of S.*

Proof. **(1) ⇒ (2):** Let R, Q and L be any right k-ideal, any quasi k-ideal and any left k-ideal of S respectively. Let $a \in R \cap Q \cap L$. Since S is k-regular there exists $x \in S$ such that $a + axaxa = axaxa$. However $(ax)a(xa) \in RQL$, whence $a \in \overline{RQL}$. Thus $R \cap Q \cap L \subseteq \overline{RQL}$.
(2) ⇒ (1): Let R and L be any right k-ideal and any left k-ideal of S respectively. Then $R \cap L$ is quasi k-ideal of S. Then we have

$$R \cap (R \cap L) \cap L \subseteq \overline{R(R \cap L)L} \Rightarrow R \cap L \subseteq \overline{RL}$$

and so S is a k-regular semiring, by Theorem 3.6. □

4 Quasi ideals in intra k-regular semirings

In this section we characterize intra-k-regular semirings using quasi-k-ideals. For a semigroup F, the semiring $P(F)$ is intra k-regular if and only if F is an intra regular semigroup [2]. Again the Theorem 2.2 shows that the quasi k-ideals are natural analogue in semirings of the notion of quasi ideals of a semigroup. Thus it is natural to extend the results, characterising the intra regular semigroups by quasi ideals to semirings.

Definition 4.1 A semiring S is called an intra k-regular semiring if for each $a \in S$, $a \in \overline{Sa^2S}$.

It is easy to check that a semiring S is intra k-regular if and only if for each $a \in S$ there exists $x \in S$ such that

$$a + xa^2x = xa^2x. \tag{2}$$

In the following theorem we characterized the intra k-regular semirings by their left and right k-ideals.

Theorem 4.2 ([2]) *Let S be a semiring. Then the following conditions are equivalent:*

1. *S is intra k-regular.*
2. *$L \cap R \subseteq \overline{LR}$ for every left k-ideal L and every right k-ideal R of S.*

We left it to check to the readers that a semiring S is both k-regular and intra k-regular if and only if for each $a \in S$ there exists $z \in S$ such that

$$a + aza^2za = aza^2za. \tag{3}$$

Theorem 4.3 *For a semiring S, the following conditions are equivalent:*

1. *S is both k-regular and intra k-regular.*
2. *$Q = \overline{Q^2}$ for every quasi k-ideal Q of S.*

Proof. $(\mathbf{1}) \Rightarrow (\mathbf{2})$: Let Q be a quasi k-ideal of S and $a \in Q$. Since S is both k-regular and intra k-regular there exists $x \in S$ such that $a + axa^2xa = axa^2xa$. However $axa \in QSQ \subseteq QS \cap SQ \subseteq Q$ implies that $a \in \overline{Q^2}$. Hence $Q \subseteq \overline{Q^2}$. Again, since Q is a k-subsemiring, it follows that $Q^2 \subseteq Q$, whence $\overline{Q^2} \subseteq Q$. Thus $Q = \overline{Q^2}$.
$(\mathbf{2}) \Rightarrow (\mathbf{1})$: Let L and R be a left k-ideal and a right k-ideal of S respectively. Then $Q = R \cap L$ is a quasi k-ideal of S. Then $R \cap L = \overline{(R \cap L)^2} = \overline{(R \cap L)(R \cap L)} \subseteq \overline{RL} \cap \overline{LR}$. Thus S is both k-regular and intra k-regular, by Theorem 3.6 and Theorem 4.2. □

THEOREM 4.4 *For a semiring S, the following conditions are equivalent:*

1. *S is k-regular and intra k-regular.*
2. *$P \cap Q \subseteq \overline{PQ}$ for every quasi k-ideals P and Q of S.*
3. *$B \cap Q \subseteq \overline{BQ}$ for every k-bi-ideal B and every quasi k-ideal Q of S.*
4. *$P \cap B \subseteq \overline{PB}$ for every quasi k-ideal P and every k-bi-ideal B of S.*
5. *$G \cap Q \subseteq \overline{GQ}$ for every generalized k-bi-ideal G and every quasi k-ideal Q of S.*
6. *$P \cap G \subseteq \overline{PG}$ for every quasi k-ideal P and every generalized k-bi-ideal G of S.*

Proof. It is clear that $(6) \Rightarrow (4) \Rightarrow (2)$ and $(5) \Rightarrow (3) \Rightarrow (2)$. So we have to prove $(1) \Rightarrow (6)$, $(1) \Rightarrow (5)$ and $(2) \Rightarrow (1)$.
$(\mathbf{1}) \Rightarrow (\mathbf{6})$: Let P and G be a quasi k-ideal and a generalized k-bi-ideal of S respectively. Let $a \in P \cap G$. Since S is both k-regular and intra k-regular, there exists $x \in S$ such that $a + axa^2xa = axa^2xa$, by (3). Then $axa \in PS \cap SP \subseteq P$ and $axa \in GSG \subseteq G$ implies that $axa^2xa \in PG$ and so $a \in \overline{PG}$. Thus $P \cap G \subseteq \overline{PG}$.
$(\mathbf{1}) \Rightarrow (\mathbf{5})$: Similar to $(1) \Rightarrow (6)$.
$(\mathbf{2}) \Rightarrow (\mathbf{1})$: Let Q be a quasi k-ideal. Then $Q \cap Q \subseteq \overline{QQ}$. Also $\overline{QQ} \subseteq Q$, since Q is a subsemiring and k-set. Thus $Q = \overline{Q^2}$ and so S is both k-regular and intra k-regular, by Theorem 4.3. □

THEOREM 4.5 *For a semiring S, the following conditions are equivalent:*

1. *S is k-regular and intra k-regular.*
2. *$L \cap R \subseteq \overline{LR} \cap \overline{RL}$ for every left k-ideal L and every right k-ideal R of S.*
3. *$L \cap Q \subseteq \overline{LQ} \cap \overline{RL}$ for every left k-ideal L and every quasi k-ideal Q of S.*
4. *$R \cap Q \subseteq \overline{RQ} \cap \overline{QR}$ for every right k-ideal R and every quasi k-ideal Q of S.*
5. *$P \cap Q \subseteq \overline{PQ} \cap \overline{QP}$ for all quasi k-ideals P and Q of S.*
6. *$Q \cap B \subseteq \overline{QB} \cap \overline{BQ}$ for every quasi k-ideal Q and every k-bi-ideal B of S.*
7. *$Q \cap G \subseteq \overline{QG} \cap \overline{GQ}$ for every quasi k-ideal Q and every generalized k-bi-ideal G of S.*

Proof. It is clear that$(7) \Rightarrow (6) \Rightarrow (5) \Rightarrow (4) \Rightarrow (2)$and $(5) \Rightarrow (3) \Rightarrow (2)$. So we are to prove $(1) \Rightarrow (7)$ and $(2) \Rightarrow (1)$ only.
$(\mathbf{1}) \Rightarrow (\mathbf{7})$: Let G be a two generalized k-bi-ideal and Q be a quasi k -ideal of S. Let $a \in G \cap Q$. Since S is both k-regular and intra k-regular, there exists $x \in S$ such that $a + axa^2xa = axa^2xa$. This can be written as$a + (axa)(axa) = (axa)(axa)$. Now $axa \in QSQ \subseteq QS \cap SQ \subseteq Q$ and $axa \in G$. Then $(axa)(axa) \in QG$ and $(axa)(axa) \in GQ$. Thus $a \in \overline{QG}$ and $a \in \overline{GQ}$. Therefore $a \in \overline{QG} \cap \overline{GQ}$. Hence $G \cap Q \subseteq \overline{GQ} \cap \overline{QG}$.
$(\mathbf{2}) \Rightarrow (\mathbf{1})$: Let $L \cap R \subseteq \overline{LR} \cap \overline{RL}$. Then $L \cap R \subseteq \overline{LR}$ and $L \cap R \subseteq \overline{RL}$. Thus S is both k-regular and intra-k-regular by Theorem 3.6 and Theorem 4.2. □

THEOREM 4.6 *For a semiring S, the following conditions are equivalent:*

1. *S is k-regular and intra k-regular.*
2. *$Q \cap L \subseteq \overline{QLQ}$ for every quasi k-ideal Q and every left k-ideal L of S.*
3. *$Q \cap R \subseteq \overline{QRQ}$ for every quasi k-ideal Q and every right k-ideal R of S.*
4. *$Q \cap P \subseteq \overline{QPQ}$ for all quasi k-ideals Q and P of S.*
5. *$Q \cap B \subseteq \overline{QBQ}$ for every quasi k-ideal Q and every k-bi-ideal B of S.*
6. *$Q \cap G \subseteq \overline{QGQ}$ for every quasi k-ideal Q and every generalized k-bi-ideal G of S.*
7. *$B \cap Q \subseteq \overline{BQB}$ for every k-bi-ideal B and every quasi k-ideal Q of S.*
8. *$G \cap Q \subseteq \overline{GQG}$ for every generalized k-bi-ideal G and every quasi k-ideal Q of S.*

Proof. It is clear that $(8) \Rightarrow (7) \Rightarrow (4) \Rightarrow (3)and(6) \Rightarrow (5) \Rightarrow (4) \Rightarrow (2)$. So it is sufficient to prove $(1) \Rightarrow (8), (1) \Rightarrow (6), (3) \Rightarrow (1)$ and $(2) \Rightarrow (1)$.
$(\mathbf{1}) \Rightarrow (\mathbf{8})$: Let G be a generalized k-bi-ideal and Q be a quasi k-ideal of S. Let $a \in G \cap Q \subseteq S$. Since S is both k-regular and intra k-regular, there exists $x \in S$ such that
$a + axa^2xa = axa^2xa$
$\Rightarrow\ a + axa^2x(a + axa^2xa) = axa^2xa(a + axa^2xa)$
$\Rightarrow\ a + axa^2xaxa^2xa = axa^2xaxa^2xa$
$\Rightarrow\ a + (axa)(axaxa)(axa) = (axa)(axaxa)(axa)$.
Now $axa \in G$ and $axaxa \in QSQ \subseteq QS \cap SQ \subseteq Q$. Then $(axa)(axaxa)(axa) \in GQG$. Thus $a \in \overline{GQG}$. Hence $G \cap Q \subseteq \overline{GQG}$.
$(\mathbf{1}) \Rightarrow (\mathbf{6})$: Proceeding as above we can similarly prove that $Q \cap G \subseteq \overline{QGQ}$
$(\mathbf{2}) \Rightarrow (\mathbf{1})$: Let L and R be a left k-ideal and a right k-ideal of S respectively. Then $Q = R \cap L$ is a quasi k-ideal of S. Therefore $Q \cap L \subseteq \overline{QLQ}$ implies that $R \cap L = R \cap L \cap L = \overline{(R \cap L)L(R \cap L)} \subseteq \overline{RL(R \cap L)} \subseteq \overline{R(LL)} \subseteq \overline{RL}$. Similarly $R \cap L = R \cap L \cap L = \overline{(R \cap L)L(R \cap L)} \subseteq \overline{(R \cap L)LR} \subseteq \overline{RLR} \subseteq \overline{LR}$. Thus $L \cap R \subseteq \overline{LR} \cap \overline{RL}$. Hence S is both k-regular and intra k-regular, by Theorem 4.5.
$(\mathbf{3}) \Rightarrow (\mathbf{1})$: Similar to $(\mathbf{2}) \Rightarrow (\mathbf{1})$. □

THEOREM 4.7 *For a semiring S, the following conditions are equivalent:*

1. *S is k-regular and intra k-regular.*
2. *$Q \cap R \cap L = \overline{QRL}$ for every quasi k-ideal Q,every right k-ideal R and every left k-ideal L of S.*

Proof. $(\mathbf{1}) \Rightarrow (\mathbf{2})$: Assume that S is a k-regular and intra k -regular semiring. Let R, Q and L be a right k-ideal, a quasi k-ideal and a left k-ideal of S respectively. Let $a \in Q \cap R \cap L$. Since S is k-regular there exist $x \in S$ such that $a + axa^2xa = axa^2xa$. This can be written as $a + (axa)(ax)a = (axa)(ax)a$. Since R is a right k-ideal and Q is a quasi k-ideal of S, so $ax \in R$ and $axa \in QSQ \subseteq QS \cap SQ \subseteq Q$. Thus $(axa)(ax)a \in QRL$. Then $a \in \overline{QRL}$. Hence $Q \cap R \cap L \subseteq \overline{QRL}$.
$(\mathbf{2}) \Rightarrow (\mathbf{1})$: Let L and R be a left k-ideal and a right k-ideal of S respectively. As L and R are also quasi k-ideal of S, we have $L \cap R = R \cap R \cap L \subseteq \overline{RRL} \subseteq \overline{RL}$ and $L \cap R = L \cap R \cap L \subseteq \overline{LRL} \subseteq \overline{LR}$. Thus $L \cap R \subseteq \overline{RL} \cap \overline{LR}$. Hence S is both k-regular and intra k-regular, by Theorem 4.5. □

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