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Pattern avoidance in flattened permutations

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Abstract. To flatten a permutation expressed as a product of disjoint cycles, we mean to form another permutation by erasing the parentheses which enclose the cycles of the original. This clearly depends on how the cycles are listed. For permutations written in the *standard cycle form*–cycles arranged in increasing order of their first entries, with the smallest element first in each cycle–we count the permutations of $[n]$ whose flattening avoids any subset of S_3. Among the sequences that arise are central binomial coefficients, Schröder numbers, and relatives of the Fibonacci numbers. In some instances, we provide combinatorial arguments of the result, while in others, our approach is more algebraic. In a couple of the cases, we define an explicit bijection between the subset of S_n in question and a restricted set of lattice paths. In another, to establish the result, we make use of the kernel method to solve a functional equation arising once a certain parameter has been considered.

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1 Introduction

We will use the following notational conventions: $\mathbb{N} = \{0, 1, 2, \ldots\}$, $\mathbb{P} = \{1, 2, \ldots\}$, $[n] = \{1, 2, \ldots, n\}$ for $n \in \mathbb{P}$, and $[m, n] = \{m, m+1, \ldots, n\}$ for $m, n \in \mathbb{P}$, with $m \le n$. By convention, we take $[0] = \varnothing$ and $[m, n] = \varnothing$ if $m > n$. A permutation σ of $[n]$ is often represented as a word $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ wherein $\sigma(i) = \sigma_i$ for each i or as a product of disjoint cycles $(a_1\cdots a_r)(b_1\cdots b_s)\cdots$ wherein the first cycle, for example, we have $\sigma(a_i) = a_{i+1}$, $1 \le i \le r-1$, and $\sigma(a_r) = a_1$. A permutation expressed as a product of disjoint cycles is said to be in *standard cycle form* if the cycles are written with the smallest element as the first entry, with cycles arranged in ascending order according to first entries.

The pattern avoidance problem for permutations has been studied extensively from various perspectives in both enumerative and algebraic combinatorics, see, e.g., [4]. The comparable problem has also been considered on other structures such as words, compositions, and set partitions. Here, we consider a notion of pattern avoidance for permutations which is analogous to the one recently introduced by Callan [2] for set partitions. Suppose σ is a permutation of $[n]$ written in standard cycle form. Define Flatten(σ) to be the permutation of $[n]$ obtained by erasing the parentheses enclosing the cycles of σ. For example, if $\sigma = 71564328 \in S_8$, then the standard form is $(172)(3546)(8)$ and Flatten$(\sigma) = 17235468$.

Here, we will say that a permutation σ *contains* an occurrence of the permutation pattern ρ if Flatten(σ) contains a subsequence isomorphic to ρ (i.e., a subsequence which, when standardized, gives ρ). Otherwise, we will say that σ *avoids* ρ. For example, the permutation $56142387 = (15263)(4)(78) \in S_8$ contains two occurrences of 231 (namely, 563 and 564) but avoids 321. Given $n \in \mathbb{P}$, let $F_n(\rho) = \{\sigma \in S_n : \text{Flatten}(\sigma) \text{ avoids } \rho\}$. In the next section, we will find the cardinality of $F_n(\rho)$ whenever ρ is a single pattern of length three. Among the sequences which arise are the central binomial coefficients and the (large) Schröder numbers (see A000984 and A006318 in [6], respectively). In the third section, results are extended to any subset of S_3.

In what follows, if $n \geq 0$ and $0 \leq m \leq n$, then let $P_{n,m}$ be the set of lattice paths from $(0,0)$ to (n,m) consisting of *up* $(1,1)$ and *down* $(1,-1)$ steps (which we will denote by u and d, respectively) and never going below the x-axis. Let $P_n = \cup_{m=0}^{n} P_{n,m}$.

2 Avoiding a pattern of length three

2.1 The cases 123 and 132

We first consider the cases of avoiding 123 and 132, both of which are easy, since requiring Flatten(σ) to avoid either of these patterns is a rather severe restriction.

PROPOSITION 2.1 *For all $n \geq 2$, we have $|F_n(123)| = 2$ and $|F_n(132)| = 2^{n-1}$.*

Proof. If $n \geq 2$, the only members of $F_n(123)$ are $(1, n, n-1, \ldots, 2)$ and $(1, n, n-1, \ldots, 3), (2)$. If $\sigma \in F_n(132)$, then Flatten$(\sigma) = 12 \cdots n$. So any subset of the $n-1$ spaces between $1, 2, \ldots, n$ can serve as dividers to form the cycles of σ, which implies the second statement. □

2.2 The cases 213 and 312

Here we consider the problem of avoiding either 213 or 312.

THEOREM 2.2 *If $n \geq 1$, then there are $\binom{2n-2}{n-1}$ members of $F_n(213)$.*

Proof. Suppose $\sigma \in F_n(213)$, $n \geq 1$, and that the letters of the first cycle of σ form the word 1α. If σ has only one cycle, then α is itself a 213-avoiding permutation on the letters $\{2, 3, \ldots, n\}$ and thus there are c_{n-1} possibilities in this case, where c_m denotes the m-th Catalan number. Suppose then $\sigma = C_1 C_2 \cdots C_r$ for some $r \geq 2$, where $C_j = (i_j \cdots)$ denotes the j-th cycle. Let $b = i_2 - 1$. Then the set of letters comprising the first cycle of σ must be $[b] \cup [t+1, n]$ for some t, where $b+1 \leq t \leq n$, since if $x > i_2$ belongs to the first cycle of σ, then so do all of the members of $[n]$ greater than x in order to avoid an occurrence of 213 in Flatten(σ).

Note further that the members of $[b]$ must occur in increasing order going from left to right; i.e., the letters of the first cycle of σ can be written as

$$1\alpha = 1w_1 2w_2 \cdots bw_b, \tag{1}$$

where (i) each w_i has letters in $[t+1, n]$ or is empty, (ii) if $i < j$, then all of the letters in w_i are greater than all of those in w_j, and (iii) each w_i constitutes a 213-avoiding permutation of its letters.

We now define a bijection f between words of the form in (1) having length m and the lattice paths in $P_{2m,2}$. To do so, first map each w_i in (1) above, $1 \leq i \leq b$, to a Catalan path p_i having semilength $|w_i|$ using any choice of bijection from the set of 213-avoiding permutations to the set of Catalan paths (see, e.g., [5]). Define the mapping f by

$$f(1w_1 2w_2 \cdots bw_b) = up_1dup_2d \cdots up_{b-1}dup_bu.$$

Note that the number of returns to the x-axis in the produced path is $b-1$; hence, f is seen to be a bijection.

Applying the same reasoning to the second, to the third, and, subsequently, to all but the last cycle shows further that any cycle except the last can be expressed using its letters as a word whose form is analogous to (1) above. Thus, we may apply the bijection f to the k-th cycle $C_k = (i_k \cdots)$ to obtain a lattice path $f(C_k) \in P_{2m_k,2}$, where m_k denotes the number of elements in the k-th cycle, $1 \leq k < r$. For the last cycle C_r, we express it as a word $i_r\beta$, where β is 213-avoiding. Let $g(C_r) = \beta'$, where β' denotes the Catalan path obtained by applying one of the known bijections; note that the semilength of β' is one less than the length of C_r. Now define $h(\sigma)$, where $\sigma = C_1C_2 \cdots C_r \in F_n(213)$, by

$$h(\sigma) = f(C_1)f(C_2) \cdots f(C_{r-1})g(C_r),$$

where the lattice paths are concatenated. Note that h is also defined when $r = 1$, in this case we simply apply g to the cycle.

Then h is a mapping from the subset of $F_n(213)$ consisting of those members having exactly r cycles to $P_{2n-2,2r-2}$. Upon decomposing such paths according to the first and last returns to height i for each i, the mapping h is seen to be bijection. Combining the mappings h for all r, $1 \leq r \leq n$, we obtain a bijection between $F_n(213)$ and P_{2n-2}. The cardinality of P_{2n-2} is known to be the central binomial coefficient $\binom{2n-2}{n-1}$; see Theorem 1 on page 74 of Feller [3]. This completes the proof. Below we illustrate the bijection h with the permutation $\sigma = (1, 15, 2, 12, 14, 13, 3), (4, 11, 10, 5, 9), (6, 8, 7) \in F_{15}(213)$.

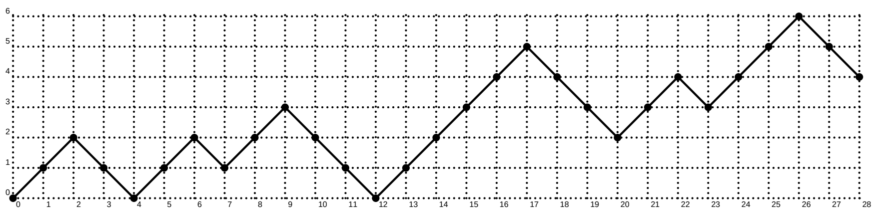

Figure 1: The lattice path $h(\sigma)$.

□

The previous theorem may be refined by fixing the number of cycles.

COROLLARY 2.3 *If $n \geq 1$ and $0 \leq i \leq n$, then there are $\frac{2i+1}{n+i+1}\binom{2n}{n-i}$ members of $F_{n+1}(213)$ having exactly $i+1$ cycles.*

Proof. From the proof of Theorem 2.2 above, we count, equivalently, the members of $P_{2n,2i}$. To do so, we will use generating functions. Given $j \geq 0$, let $F_j(x)$ denote the generating function for the number of lattice paths with u and d steps starting from the origin which have final height j and never go below the x-axis. Let

$$C(x) = \sum_{n\geq 0} c_n x^n = \frac{1-\sqrt{1-4x}}{2x},$$

where c_n denotes the n-th Catalan number. Considering whether or not a lattice path enumerated by $F_j(x)$ returns to the x-axis yields

$$F_j(x) = xF_{j-1}(x) + x^2C(x^2)F_j(x), \qquad j \geq 1,$$

with $F_0(x) = C(x^2)$. Thus, we have

$$F_j(x) = \frac{x}{1-x^2C(x^2)}F_{j-1}(x) = xC(x^2)F_{j-1}(x), \qquad j \geq 1,$$

which we iterate to obtain

$$F_j(x) = x^jC^{j+1}(x^2), \qquad j \geq 0.$$

Taking $j = 2i$ gives

$$F_{2i}(x) = x^{2i}C^{2i+1}(x^2) = \sum_{m\geq 0} \frac{(2i+1)(2m+2i)!}{m!(m+2i+1)!}x^{2m+2i},$$

by (2.5.16) in [7]. Letting $n = m + i$, we see that there are

$$\frac{(2i+1)(2n)!}{(n-i)!(n+i+1)!} = \frac{2i+1}{n+i+1}\binom{2n}{n-i}$$

lattice paths, as required. □

Remark. Summing the expression in Corollary 2.3 above over i yields the following binomial coefficient identity which we have not previously seen:

$$\sum_{i=0}^{n} \frac{2i+1}{n+i+1}\binom{2n}{n-i} = \binom{2n}{n}, \qquad n \geq 0. \tag{2}$$

Our proof of (2) is combinatorial since the summand on the left-hand side was shown to count the lattice paths enumerated by the right-hand side according to the final height $2i$, $0 \leq i \leq n$ (an equivalent interpretation may of course be given in terms of permutations). Dividing both sides of (2) by $\binom{2n}{n}$ shows further that a certain sum involving the quotients of binomial coefficients is independent of n:

$$\sum_{i=0}^{n} \frac{2i+1}{i+1}\frac{\binom{n}{i}}{\binom{n+i+1}{i+1}} = 1, \qquad n \geq 0. \tag{3}$$

We now consider the case of avoiding 312.

Theorem 2.4 *If $n \geq 1$, then there are $\binom{2n-2}{n-1}$ members of $F_n(312)$.*

Proof. Suppose $\sigma \in F_n(312)$, $n \geq 1$, and that $\sigma = C_1C_2 \cdots C_r$ for some $r \geq 2$, where $C_j = (i_j \cdots)$ denotes the j-th cycle (we will assume for now that σ has at least two cycles, the case for a single cycle being easier). Let $b = i_2 - 1$. Then the set of letters comprising the first cycle C_1 of σ must be $[b] \cup [b+2, t]$ for some t, where $b+1 \leq t \leq n$. For if $x > b+1$ belongs to the first cycle of σ, then so do all of the members of $[n]$ less than x and greater than $b+1$ in order to avoid an occurrence of 312 in Flatten(σ).

Note further that all of the members of $[b]$ must occur prior to all of those in $[b+2, t]$ within C_1 so as to avoid 312. Thus, the letters of C_1 may be expressed as a word $1v_1v_2$, where v_1 and v_2, respectively, are permutations of the sets $[2, b]$ and $[b+2, t]$ that avoid 312. Let $a(C_1) = q_1uq_2u$, where q_i, $i = 1, 2$, is the Catalan path of semilength $|v_i|$ obtained by applying to v_i any choice of bijection from the set of 312-avoiding permutations to the set of Catalan paths (see, e.g., [5]); note that $a(C_1)$ is a lattice path starting from $(0, 0)$ having final height 2 that never dips below the x-axis. Similarly, one can define $a(C_i)$, $1 \leq i < r-1$, since cycles other than the last one have analogous structures on their respective sets of letters. For the last cycle C_r, we express it as a word $i_r\beta$, where β is 312-avoiding. Let $b(C_r) = \beta'$, where β' denotes the Catalan path obtained by applying any of the known bijections; note that the semilength of β' is one less than the length of C_r. Now define $c(\sigma)$, where $\sigma = C_1C_2 \cdots C_r \in F_n(312)$, by

$$c(\sigma) = a(C_1)a(C_2) \cdots a(C_{r-1})b(C_r),$$

where the lattice paths are concatenated; note that c is also defined in the case $r = 1$.

Then c is seen to be a bijection and combining the mappings c corresponding to various r, we obtain a bijection from $F_n(312)$ to P_{2n-2}, which has cardinality $\binom{2n-2}{n-1}$. Note that the statistic recording the number of cycles has the same distribution on $F_n(312)$ as it does on $F_n(213)$. Below we illustrate the bijection c with the permutation $\sigma = (1, 4, 3, 2, 7, 6, 8), (5, 10, 9, 12), (11), (13, 15, 14) \in F_{15}(312)$.

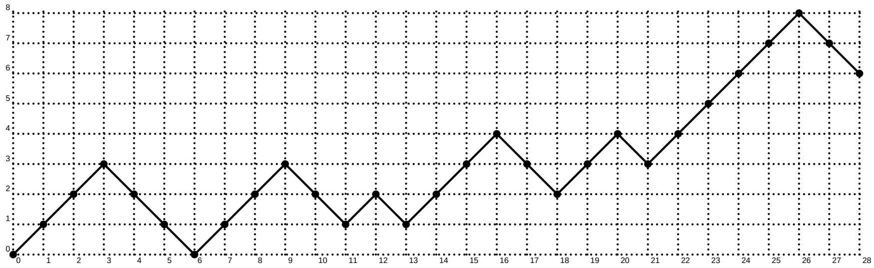

Figure 2: The lattice path $c(\sigma)$.

□

2.3 The cases 231 and 321

We first consider the case of avoiding 231. Recall that a *Schröder path* of semilength n is a lattice path from $(0, 0)$ to $(2n, 0)$ having $(1, 1)$, $(1, -1)$, and $(2, 0)$ steps which never goes below the x-axis. Such paths are enumerated by the n-th Schröder number, which we will denote by $\mathcal{S}_n$.

Theorem 2.5 *The generating function for the number of members of $F_n(231)$, $n \geq 1$, according to the number of cycles is given by*

$$\frac{y}{2}\left(1 - xy - \sqrt{(1-xy)^2 - 4x}\right).$$

Hence, there are $\frac{1}{n}\binom{n}{k}\binom{2n-k}{n-1}$ members of $F_{n+1}(231)$ having exactly $k+1$ cycles if $n \geq 1$ and $0 \leq k \leq n$, which is the number of Schröder paths of semilength n having k peaks. In particular, we have $|F_{n+1}(231)| = \mathcal{S}_n$ for all $n \geq 0$.

Proof. We prove only the first statement; the rest follows from known facts concerning Schröder paths (see, e.g., A060693 in [6]). Let $b_n(y)$ denote the distribution polynomial on $F_n(231)$ for the statistic recording the number of cycles; i.e.,

$$b_n(y) = \sum_{\sigma \in F_n(231)} y^{\alpha(\sigma)}, \qquad n \geq 1,$$

where $\alpha(\sigma)$ denotes the number of cycles of the permutation σ. Considering the position of n within $\sigma \in F_n(231)$ yields the recurrence

$$b_n(y) = yb_{n-1}(y) + \frac{1}{y}\sum_{k=1}^{n-1} b_k(y)b_{n-k}(y), \qquad n \geq 2, \tag{4}$$

with $b_1(y) = y$. To see this, first note that $yb_{n-1}(y)$ counts the members of $F_n(231)$ in which the 1-cycle (n) occurs. Suppose now that σ is a member of $F_n(231)$ not containing the 1-cycle (n). Note that all of the letters in σ to the left of n are smaller than all of those to the right of n. Then the set of letters to the left of n is $[k]$ for some k, $1 \leq k \leq n-1$. Let σ' denote the permutation of $[k]$ obtained by writing the letters of $[k]$ in cycles as they occur within σ; note that $\sigma' \in F_k(231)$. Similarly, the remaining letters $[k+1, n]$ constitute a member σ'' of $F_{n-k}(231)$ (where here n is treated as the "smallest letter", being the first letter in the leftmost cycle). Note that $\alpha(\sigma) = \alpha(\sigma') + \alpha(\sigma'') - 1$ since the cycle containing n is counted in both $\alpha(\sigma')$ and $\alpha(\sigma'')$. Thus, the sum on the right-hand side of (4) above counts the remaining members of $F_n(231)$ according to the number, k, of letters occurring to the left of n.

Let $B(x,y) = \sum_{n\geq 1} b_n(y)x^n$. Multiplying (4) by x^n and summing over $n \geq 2$ yields

$$B(x,y) - xy = xyB(x,y) + \frac{1}{y}B^2(x,y),$$

which gives

$$B(x,y) = \frac{y}{2}\left(1 - xy - \sqrt{(1-xy)^2 - 4x}\right),$$

as required. □

We now consider the problem of enumerating the members of $F_n(321)$. Given $n \geq 2$ and $2 \leq t \leq n$, we let $a_n(t)$ denote the number of members σ in $\mathcal{F}_n = F_n(321)$ such that the second element in Flatten(σ) is t, where t belongs to the same cycle as 1 in σ. Let $\mathcal{F}_{n,t}$ denote the subset of $\mathcal{F}_n$ counted by $a_n(t)$. Let $a_n(y;t)$ denote the distribution polynomial on $\mathcal{F}_{n,t}$ for the statistic recording the number of cycles; i.e.,

$$a_n(y;t) = \sum_{\sigma \in \mathcal{F}_{n,t}} y^{\alpha(\sigma)}, \qquad n \geq 2 \text{ and } 2 \leq t \leq n,$$

where $\alpha(\sigma)$ denotes the number of cycles of the permutation σ. Let $a_n(y)$ denote the distribution polynomial on $\mathcal{F}_n$ for the number of cycles, i.e.,

$$a_n(y) = \sum_{\sigma\in\mathcal{F}_n} y^{\alpha(\sigma)}, \qquad n \geq 1.$$

Note that

$$a_n(y) = ya_{n-1}(y) + \sum_{t=2}^{n} a_n(y;t), \qquad n \geq 2, \tag{5}$$

with $a_1(y) = y$, for the sum gives the distribution of the statistic recording the number of cycles over all the members of $\mathcal{F}_n$ for which the cycle (1) does not occur, while $ya_{n-1}(y)$ is the distribution over the members of $\mathcal{F}_n$ for which it does.

We can write an explicit recurrence satisfied by the $a_n(y;t)$.

LEMMA 2.6 *If $n \geq 3$, then*

$$a_n(y;t) = ya_{n-1}(y;t-1) + \sum_{j=t-1}^{n-1} a_{n-1}(y;j), \qquad 3 \leq t \leq n, \tag{6}$$

with $a_n(y;2) = a_{n-1}(y)$ for all $n \geq 2$.

Proof. That $a_n(y;2) = a_{n-1}(y)$ is clear since one may simply delete the 2 from the cycle containing 1 and consider the resulting permutation of length $n-1$. Assume then $n \geq 3$ and $3 \leq t \leq n$. Let $\sigma \in \mathcal{F}_{n,t}$ and let j denote the third letter of Flatten(σ). Then either $j = 2$ or $j \in [t+1, n]$, in order to avoid an occurrence of 321. First suppose $j = 2$. Then σ is of either the form (i) $\sigma = (1t)(2\cdots)(i_3\cdots)\cdots$ or (ii) $\sigma = (1t2\cdots)(i_2\cdots)(i_3\cdots)\cdots$. Note that members of $\mathcal{F}_n$ in (i) are synonymous with members of $\mathcal{F}_{n-1}$ on the letters $[n]-\{1\}$ of the form $(2t\cdots)(i_3\cdots)\cdots$, which are counted (according to the number of cycles) by $ya_{n-1}(y;t-1)$. Members of $\mathcal{F}_n$ in (ii) are synonymous with members of $\mathcal{F}_{n-1}$ on the letters $[n]-\{2\}$ of the form $(1t\cdots)(i_2\cdots)(i_3\cdots)\cdots$, which are counted by $a_{n-1}(y;t-1)$.

Now suppose $j \in [t+1, n]$. Then j must go in the first cycle after t since $t \geq 3$. Then $\sigma = (1tj\cdots)(i_2\cdots)\cdots$, which is synonymous with a member of $\mathcal{F}_{n-1}$ on the letters $[n]-\{t\}$ of the form $(1j\cdots)(i_2\cdots)\cdots$, of which there are $a_{n-1}(y;j-1)$. Summing over j, we see that there are $\sum_{j=t+1}^{n} a_{n-1}(y;j-1)$ possibilities in this case. Combining all of the cases yields (6). □

If $n \geq 2$, then let

$$A_n(y;q) = ya_{n-1}(y) + \sum_{t=2}^{n} a_n(y;t)q^{t-2},$$

with $A_1(y;q) = y$; note that $A_n(y;1) = a_n(y)$, by (5). The polynomials $A_n(y;q)$ satisfy the following relation.

LEMMA 2.7 *If $n \geq 3$, then*

$$A_n(y;q) = \frac{q(y-q-yq)}{1-q}A_{n-1}(y;q) + \frac{1+y-yq}{1-q}A_{n-1}(y;1) - qy(1+y)A_{n-2}(y;1), \tag{7}$$

with $A_1(y;q) = y$ and $A_2(y;q) = y + y^2$.

Proof. Multiplying (6) by q^{t-2} and summing on $3 \le t \le n$, we obtain

$$\sum_{t=3}^{n} a_n(y;t)q^{t-2} = y\sum_{t=3}^{n} a_{n-1}(y;t-1)q^{t-2} + \sum_{j=2}^{n-1} a_{n-1}(y;j)\sum_{t=3}^{j+1} q^{t-2},$$

which implies

$$\begin{aligned} A_n(y;q) - (1+y)A_{n-1}(y;1) &= yq(A_{n-1}(y;q) - yA_{n-2}(y;1)) + \sum_{j=2}^{n-1} \frac{q-q^j}{1-q} a_{n-1}(y;j) \\ &= yq(A_{n-1}(y;q) - yA_{n-2}(y;1)) + \frac{q}{1-q}(A_{n-1}(y;1) - yA_{n-2}(y;1)) \\ &\quad - \frac{q^2}{1-q}(A_{n-1}(y;q) - yA_{n-2}(y;1)), \end{aligned}$$

since $a_{m+1}(y;2) = a_m(y) = A_m(y;1)$ for all m. The final equality rearranges to give (7), as required. □

The following theorem implies that the statistic recording the number of cycles has the same distribution on $F_n(321)$ as it does on $F_n(231)$ for all n.

THEOREM 2.8 *The generating function for the number of members of $F_n(321)$, $n \ge 1$, according to the number of cycles is given by*

$$\frac{y}{2}\left(1 - xy - \sqrt{(1-xy)^2 - 4x}\right).$$

Proof. Let $A(x,y;q) = \sum_{n\ge1} A_n(y;q)x^n$; we seek $A(x,y;1)$. Multiplying (7) by x^n and summing over all $n \ge 3$ yields

$$\begin{aligned} A(x,y;q) - xy - x^2(y+y^2) &= \frac{xq(y-q-yq)}{1-q}(A(x,y;q) - xy) + \frac{x(1+y-yq)}{1-q}(A(x,y;1) - xy) \\ &\quad - x^2yq(1+y)A(x,y;1), \end{aligned}$$

which may be rewritten as

$$\begin{aligned} \frac{1-(1+xy)q + x(1+y)q^2}{1-q}A(x,y;q) &= \frac{x(1+y-yq-xyq(1+y)(1-q))}{1-q}A(x,y;1) \\ &\quad + xy(1-xq-xyq). \end{aligned} \tag{8}$$

To solve (8), we use the *kernel method* (see [1]). Setting the coefficient of $A(x,y;q)$ on the left-hand side of (8) equal to zero and solving for $q = q_0$ in terms of x and y, we get

$$q_0 = \frac{1+xy-\sqrt{(1-xy)^2-4x}}{2x(1+y)}.$$

(Of the two roots, only this one yields q as a power series in x and y.) Letting $q = q_0$ in (8) then gives

$$\begin{aligned} A(x,y;1) &= \frac{y(1-q_0)(xyq_0 + xq_0 - 1)}{1+y-yq_0 - xyq_0(1+y)(1-q_0)} = \frac{xyq_0}{1-xyq_0} \\ &= \frac{y}{2}\left(1-xy-\sqrt{(1-xy)^2-4x}\right), \end{aligned}$$

as required. □

Note that one may substitute the expression for $A(x, y; 1)$ back into (8) and recover $A(x, y; q)$. It would be interesting to find a direct bijection between $F_{n+1}(321)$ and the set of Schröder paths of semilength n.

3 Further results

In this section, we consider the further problem of avoiding two or more patterns. Given the classical patterns $\rho_1, \rho_2, \ldots \rho_r$, we let $F_n(\rho_1, \rho_2, \ldots, \rho_r)$ denote the set of permutations σ of $[n]$ such that Flatten(σ) avoids all of the patterns and let $f_n(\rho_1, \rho_2, \ldots, \rho_r) = |F_n(\rho_1, \rho_2, \ldots, \rho_r)|$. The following two propositions yield a complete solution to the problem of finding $f_n(T)$ when T is any subset of S_3. By Proposition 2.1 above, we need not consider the case when T contains either 123 or 132.

Proposition 3.1 *We have*

1. $f_n(213, 231) = f_n(231, 312) = f_n(312, 321) = 2 \cdot 3^{n-2}, \quad n \geq 2;$
2. $f_n(213, 312) = n \cdot 2^{n-2}, \quad n \geq 2;$
3. $f_n(213, 321) = (n-2)2^{n-1} + 2, \quad n \geq 1;$
4. $f_n(231, 321) = a_n, \quad n \geq 1,$ *where* $a_n = 4a_{n-1} - 2a_{n-2}$ *if* $n \geq 3$, *with the initial conditions* $a_1 = 1$ *and* $a_2 = 2$ *(see sequence A006012 in [6]).*

Proof. We do not provide all the details of the proof here. The equalities in the first statement follow from inductive arguments, upon considering the position of the letter n. We supply proofs of 3 and 4; the proof of 2 is similar to that of 3.

To show 3, suppose $\sigma \in F_n(213, 321)$, where $n \geq 3$. If n is the final letter of Flatten(σ), then there are $2^{n-2} + 2^{n-2} = 2^{n-1}$ possibilities in this case, since the letters in $[n-1]$ must occur in increasing order in Flatten(σ), with n either belonging to a cycle by itself or occurring as the last letter in the last cycle. So suppose n is *not* the final letter of Flatten(σ). We will consider the two subcases: (i) n belongs to the first cycle of σ; (ii) n does not belong to the first cycle. In the first case, we have Flatten(σ) of the form $1\alpha' n\alpha''$, where α'' is an *interval* in $[2, n-1]$ of length k, $1 \leq k \leq n-2$ (otherwise there would be an occurrence of either 213 or 321; note that numbers both to the left and to the right of n must increase). Furthermore, the numbers to the right of n may be partitioned in any of $2 \cdot 2^{k-1} = 2^k$ ways, as n may or may not be the last letter in the first cycle. Thus, the number of possibilities in this case is

$$\sum_{k=1}^{n-2} (n-k-1)2^k = 2^n - 2n.$$

For subcase (ii), let $t+1$ denote the first letter of the cycle containing n, where $1 \leq t \leq n-3$. Note that the members of $[t]$ must occur in cycles in increasing order in 2^{t-1} ways. Applying the same reasoning as in (i) above to the remaining numbers $[t+1, n]$, we see that the total number of possibilities in this case is $2^{t-1}(2^{n-t} - 2(n-t))$. Summing over all t, we get

$$\sum_{t=1}^{n-3} 2^{t-1}(2^{n-t} - 2(n-t)) = (n-5)2^{n-1} + 2n + 2$$

possibilities in this case. Combining all of the cases, we have $2^{n-1}+(2^n-2n)+((n-5)2^{n-1}+2n+2) = (n-2)2^{n-1}+2$ members of $F_n(213,321)$ in all.

To show 4, first note that all letters to the left of n in $\sigma \in F_n(231,321)$ are smaller than all of those to the right, with those to the right in ascending order. If $n \geq 3$, then members of $F_n(231,321)$ may be formed, recursively, from members of $F_{n-1}(231,321)$ by (i) adding n as the 1-cycle (n), (ii) adding n to the end of the last cycle, (iii) replacing $n-1$ by n and adding $n-1$ to the end of the last cycle, or (iv) replacing $n-1$ by n and adding the 1-cycle $(n-1)$ to the end. Note that (iii) and (iv) cannot be implemented if $n-1$ occurs by itself in a cycle, for this would violate the ordering within and among cycles. Thus, the sequence $f_n(231,321)$ satisfies the same recurrence as the sequence a_n for all $n \geq 3$, with the two agreeing on the initial values. □

PROPOSITION 3.2 *We have*

1. $f_n(213,231,312) = f_n(213,312,321) = 2^n - 2, \quad n \geq 2;$
2. $f_n(213,231,321) = n \cdot 2^{n-2}, \quad n \geq 2;$
3. $f_n(231,312,321) = b_n, \quad n \geq 1$, *where* $b_n = 2b_{n-1}+2b_{n-2}$ *if* $n \geq 3$, *with the initial conditions* $b_1 = 1$ *and* $b_2 = 2$ *(see sequence A002605 in [6]);*
4. $f_n(213,231,312,321) = 3 \cdot 2^{n-2}, \quad n \geq 3.$

Proof. We prove 1 and leave the others as exercises for the reader. If $\sigma \in F_n(213,231,312)$ and n is the final letter of Flatten(σ), then there are 2^{n-1} possibilities since the members of $[n-1]$ must occur in their natural order within the cycles. If n is not the final letter of Flatten(σ), then the letters to the left of n must comprise $[t]$ for some t, $1 \leq t \leq n-2$, and occur in increasing order. Those letters to the right of n must decrease and thus occur in the same cycle as n, except possibly the last letter, which may also occur in its own cycle. Hence, there are $\sum_{t=1}^{n-2} 2 \cdot 2^{t-1} = 2^{n-1} - 2$ possibilities in this case and $2^{n-1} + (2^{n-1}-2) = 2^n - 2$ members of $F_n(213,231,312)$ in all.

If $\sigma \in F_n(213,312,321)$, then n is either the last or next-to-last letter of Flatten(σ), in the former case there being 2^{n-1} possibilities, so assume the latter. Suppose further that n belongs to a cycle other than the first and let $t+1$ be the smallest element of that cycle, where $1 \leq t \leq n-3$. For each t, there are $(n-t-2)2^t$ possible permutations, as there are $n-t-2$ choices regarding the element that follows n (which may go in its own cycle or at the end of the last cycle) and 2^{t-1} ways to divide up the members of $[t]$ in cycles. Thus, there are

$$\sum_{t=1}^{n-3}(n-t-2)2^t = 2^{n-1} - 2(n-1)$$

possibilities in this case. By similar reasoning, there are $2(n-2)$ possible σ in which n belongs to the first cycle and is the next-to-last letter of Flatten(σ). In all, we have

$$2^{n-1} + (2^{n-1} - 2(n-1)) + 2(n-2) = 2^n - 2$$

members of $F_n(213,312,321)$. □

One might consider the further problem of enumerating $F_n(\rho)$, where ρ is a pattern of length four or more. We have considered several cases when ρ has length four, and while it is sometimes possible to find the generating function, or a relation satisfied by it, a simple closed formula is usually not apparent. One might also consider the further problem of counting the members of a subset of S_n which avoid a given pattern ρ such as the alternating group or the set of derangements.

References

[1] C. Banderier, M. Bousquet-Mélou, A. Denise, P. Flajolet, D. Gardy and D. Gouyou-Beauchamps, *Generating functions for generating trees*, Discrete Math., 246 (2002) 29–55.

[2] D. Callan, *Pattern avoidance in “flattened” partitions*, Discrete Math., 309 (2009) 4187–4191.

[3] W. Feller, An Introduction to Probability Theory and its Applications, Vol. 1, John Wiley & Sons, Inc., 1957.

[4] S. Kitaev, Patterns in Permutations and Words, Springer, 2011.

[5] T. Mansour, E. Deng and R. Du, *Dyck paths and restricted permutations*, Discrete Appl. Math., 154 (2006) 1593–1605.

[6] N. J. Sloane, *The On-Line Encyclopedia of Integer Sequences*, at http://oeis.org.

[7] H. S. Wilf, generatingfunctionology, A K Peters, 2006.