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Almost avoiding pairs of permutations

William Griffiths
Department of Mathematics,
Southern Polytechnic State University,
e-mail: wgriffit@spsu.edu

and
Rebecca Smith
Department of Mathematics,
SUNY Brockport,
e-mail: rnsmith@brockport.edu

and
Daniel Warren
Department of Mathematics,
Ohio State University,
e-mail: warren@math.ohio-state.edu

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Abstract. In 2009, Brignall et al. defined almost avoidance of a permutation pattern, and the permutations of length n which almost avoid each pattern of length three were enumerated. We generalize their definition to sets of permutations and find generating functions and closed formulas for the number of permutations of length n which almost avoid each set of two patterns of length three.

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1 Introduction

The permutation $p = p_1p_2 \dots p_n$ is said to *contain* the pattern $q = q_1q_2 \dots q_k$ if there exists a sequence $1 \leq \alpha_1 < \alpha_2 < \dots < \alpha_k \leq n$ such that $p_{\alpha_i} < p_{\alpha_j}$ if and only if $q_i < q_j$. If no such sequence exists, p is said to *avoid* q.

Much of the founding work on pattern avoidance in permutations was by Knuth [11, 12], who considered the permutations that could be obtained by running the identity permutation $123 \dots n$ through a stack; these turn out to be precisely the permutations that avoid 312. The set of permutations that can be sorted (rearranged into the identity permutation) by a stack are then the permutations that avoid 231.

From there, other instances of pattern avoidance were investigated, some with clear applications and others from a more theoretical vantage. In particular, Simion and Schmidt [15] enumerated the permutations of length n in each class of permutations defined by the avoidance of two distinct permutations of length three. The class of permutations avoiding 312 and 231 are precisely the permutations that can be obtained from a pop stack (starting with the identity) and also the class of permutations that can be sorted by a pop stack.

In 2009, Brignall, Ekhad, Smith, and Vatter [6] considered a variation on pattern avoidance called *almost avoidance.*

DEFINITION 1.1 A permutation p *almost avoids* a pattern q if there is an element of p that can be removed to obtain a q-avoiding permutation.

Generating functions enumerating the permutations of length n for both distinct Wilf classes for almost avoiding a single permutation of length three (non-monotonic/monotonic) were discovered. The generating function enumerating the permutations of length n that almost avoid 231 thus gives the permutations that could be sorted by a single stack if one was allowed to remove one "bad" element. Similarly, the same generating function enumerates the permutations of length n that almost avoid 312 and thus gives permutations that could be obtained from a stack less one element.

The definition of almost avoidance of a single permutation can easily be extended to sets of permutations.

DEFINITION 1.2 To say that the permutation *p almost avoids a set S of permutations* is to say that there is a way to remove a single element of p to get a permutation that avoids all elements in S.

In this paper we study the permutations that almost avoid each pair of distinct permutations of length three.

EXAMPLE 1.3 The permutation $p = 1347256$ almost avoids the pair $\{231, 213\}$ since removing the 2 from p results in the $\{231, 213\}$-avoiding permutation $p\prime = 123645$.

We will use terminology (and methods) from [6, 15] in some of our results. In particular, we use the concept of an *essential element* as defined below.

DEFINITION 1.4 If a permutation p almost avoids S, an element p_i is *essential* if removing p_i from p results in an S-avoiding permutation.

EXAMPLE 1.5 If $p = 6175423$, and $S = \{213, 231\}$, then 6 and 7 are essential since if either is removed, the resulting permutation avoids both 213 and 231.

Throughout this paper we shall speak of insertion of elements in the permutation. An *insertion* of an element x into the i^{th} a permutation will increase the value of elements greater than or equal to x by one and increase the position of elements in the ith or greater position by one. All other values and positions will remain the same.

EXAMPLE 1.6 If we insert 5 into the fourth position of $p = 153264$, then the resulting permutation is $p' = 1635274$.

For a set S of permutations, let $L_n(S)$ be the set of permutations of length n that almost avoid S.

We first note that $L_n(123, 321)$ is 0 for $n \geq 6$, as a consequence of the 1935 result of Erdös and Szekers [9] governing the avoidance of such pairs. The values for $n = 4$ and 5 are 22 and 52, respectively.

The main result of this paper is to give the number of permutations of length n which almost avoid each nontrivial pair of patterns of length three as categorized in the following theorem.

THEOREM 1.7 *Since almost avoidance of a set of permutations of the same length is invariant within symmetry classes, the characterization of permutations which almost avoid each nontrivial pair of patterns of length three fall into four equivalence classes. In particular,*

$$\begin{aligned} &L_n(213,231) = L_n(132,231) = L_n(132,312) = L_n(213,312) \\ &L_n(312,321) = L_n(123,132) = L_n(123,213) = L_n(231,321) \\ &L_n(213,132) = L_n(231,312) \\ &L_n(132,321) = L_n(123,231) = L_n(123,312) = L_n(213,321) \end{aligned} \tag{1}$$

The number of permutations of length n which almost avoid these pairs are as follows:

$$L_n(213,231) = 2^{n-2} \cdot (n-2)^2 + 2^{n-1} \text{ for } n \geq 2 \tag{2}$$

$$L_n(312,321) = (n-2)(n-3)2^{n-2} + 2^n - 2 \text{ for } n \geq 3 \tag{3}$$

$$L_n(213,132) = 2^{n-4} \cdot (4n^2 - 16n + 23) \text{ for } n \geq 4 \tag{4}$$

$$L_n(132,321) = \frac{1}{2}n^4 - \frac{5}{2}n^3 + 21n - 30 \text{ for } n \geq 3. \tag{5}$$

The proof of this result comprises the next four sections. As mentioned earlier, the instance of complete avoidance for each pair has already been resolved. Simion and Schmidt looked at many combinations of patterns to be avoided including all possible combinations of two patterns of length three. In most cases we utilize the construction given in the avoidance result to construct the set of permutations which almost avoid each pair.

2 Almost avoiding the pair of patterns $\{231, 213\}$

We begin with the problem of complete avoidance of the pair of patterns $\{231, 213\}$.

THEOREM 2.1 *(Simion and Schmidt) The number of n-permutations that avoid both 231 and 213 is $t_0 = 1$ for $n = 0$ and $t_n = 2^{n-1}$ for $n \geq 1$.*

Proof. The first element of any such permutation must be either 1 or n. Any other initial element would serve as a '2' in either a 231 or 213 with 1 and n as the '1' and '3' respectively. For the same reason, there are two choices for each element of the permutation in positions $1, 2, 3, \ldots n-2$. Then there are two ways to finish the permutation, giving us a total of 2^{n-1} permutations. □

This result also gives us a nice decomposition of a permutation that avoids 213 and 231 into an increasing sequence $1, 2, \ldots r$ and a decreasing sequence $n, n-1, \ldots r$.

We now prove part 1 of Theorem 1.7,

$$a_n = L_n(213,231) = 2^{n-2} \cdot (n-2)^2 + 2^{n-1} \text{ for } n \geq 2$$

Proof. We first notice that such a permutation has one of the three following properties:

1. The permutation begins with a 1 or an n and is followed by an almost avoiding permutation of length $n-1$.

2. The permutation begins with $3, \dots, n-2$ and is followed by a permutation that avoids both 213 and 231.

3. The permutation has 1 or n inserted into a permutation of length $n-1$ that avoids both 213 and 231 in any place but the first position and now begins with a 2 or an $n-1$ respectively.

Thus a_n has initial conditions $a_0 = 1, a_1 = 1, a_2 = 2$ and for $n \geq 3$ satisfies the following recurrence relation:

$$\begin{aligned} a_n &= 2a_{n-1} + (n-4)2^{n-2} + (n-1)2^{n-2} \\ &= 2a_{n-1} + (2n-5)2^{n-2} \end{aligned}$$

Proceeding by induction we find $a_n = 2^{n-2} \cdot (n-2)^2 + 2^{n-1}$ for $n \geq 2$. □

3 Almost avoiding the pair of patterns $\{321, 312\}$

We will attack this problem by considering how a permutation that almost avoids the pair $\{321, 312\}$ begins. First we mention the result of Simion and Schmidt [15] on complete avoidance in this instance.

THEOREM 3.1 *(Simion and Schmidt) The number of n-permutations that avoid both 312 and 321 is $t_0 = 1$ for $n = 0$ and $t_n = 2^{n-1}$ for $n \geq 1$.*

We now prove part 2 of Theorem 1.7,

$$b_n = L_n(312, 321) = (n-2)(n-3)2^{n-2} + 2^n - 2 \text{ for } n \geq 3$$

Proof. First we consider how a permutation that almost avoids $\{312, 321\}$ begins. There are three different cases.

1. The first entry of the permutation is a 1 or 2. In this case, the first entry cannot be part of a 312 or a 321 pattern. Therefore, there are b_{n-1} ways to complete a $\{312, 321\}$ almost avoiding permutation that begins with a 1 or 2.

2. The first entry of the permutation is one of $4, 5, \dots, n$. In this case, the first entry must be essential and so the remaining portion of the permutation must avoid both 312 and 321. Thus there 2^{n-2} ways to complete a $\{312, 321\}$ almost avoiding permutation that begins with one of $4, 5, \dots, n$.

3. The first entry of the permutation is 3. This entry must be part of exactly one 312 or one 321 pattern. As such, the 3 is only essential if this is the only "bad" pattern in the permutation. There are now three ways this permutation can continue.

   (a) The second entry is 1. Then 2 is an essential element and so there 2^{n-4} ways to place the rest of the elements and then $n-2$ ways to insert the 2.

   (b) The second entry is 2. This case is the same as the first and so there are also $(n-4)2^{n-4}$ to complete this permutation.

(c) The second entry is 4. Now the last $n-1$ entries are also a $\{312, 321\}$ almost avoiding permutation beginning with a 3. Therefore if there are h_n $\{312, 321\}$ almost avoiding permutations of length n beginning with a 3, we have h_{n-1} ways to complete the permutation for this case.

Therefore if the permutation begins with a 3, it satisfies the recurrence $h_n = (n-2)2^{n-3} + h_{n-1}$ for $n \geq 4$ where $h_3 = 2$. Hence the closed form can be seen to be $h_n = 2 + (n-3)2^{n-2}$.

We now have $b_n = 2 + (n-3) \cdot 2^{n-1} + 2b_{n-1}$. Solving this recurrence, we obtain the desired result $b_n = (n-2)(n-3)2^{n-2} + 2^n - 2$ for $n \geq 4$. □

4 Almost avoiding the pair of patterns $\{213, 132\}$

It will be useful to note the form of a permutation that avoids both 213 and 132. Since each inversion in such a permutation can be neither followed by a greater element (to avoid 213) nor preceeded by a lesser element (to avoid 132), such a permutation consists of consecutively increasing sequences, in a decreasing order, such as 562341. There are 2^{n-1} such completely avoiding permutations, as there is a choice of 2 elements to insert to the front of such a permutation on $[n-1]$, either n or the element which will lengthen the beginning consecutively increasing sequence. Further, 2^{n-2} of the avoiding permutations on $[n]$ begin with n, 2^{n-2} begin with a consecutively increasing sequence of length at least 2, 2^{n-3} begin with a consecutively increasing sequence of length at least 3, and so forth.

We now prove part 3 of Theorem 1.7,

$$c_n = L_n(213, 132) = 2^{n-4} \cdot (4n^2 - 16n + 23) \text{ for } n \geq 4$$

Proof. We consider the following cases:

1. The permutation begins with an essential and is not completely avoiding. After deletion, we are left with one of the 2^{n-2} avoiding permutations on $[n-1]$. There are $n-2$ choices for an essential to be inserted on the front of such a permutation: One may not choose n nor the element which would lengthen the initial consecutively increasing sequence of the permutation. Hence there are $(n-2)2^{n-2}$ such almost avoiding permutations.

2. The permutation is an avoiding permutation that does not begin with n, an almost avoiding permutation not beginning with n in which either the second element is essential, or the permutation begins with a consecutively increasing sequence of length at least 2 and neither of the first two elements are essential nor will the removal of the first element result in an almost avoiding permutation in which the new first element is essential. These permutations can be obtained from the following procedure: Begin with an almost avoiding permutation on $[n-1]$. If the first element of the permutation is not essential, or if all elements of the permutation are essential, and if the permutation begins with element k, then insert k on the beginning of the permutation. Otherwise, let the second element be k, and insert k as the first element of the permutation. The permutation is still almost avoiding, as after removal of an essential from the original permutation, the new element fits the pattern of a completely avoiding permutation. Further, the new element is not essential unless all elements of the permutation are essential, as after its removal,

the resulting permutation retains any patterns. There are c_{n-1} permutations obtainable from this procedure.

3. The permutation begins with element n, which cannot be essential unless all elements are essential. In this case, one can also begin with any almost avoiding permutation on $[n-1]$, and insert element n at the beginning. The resulting permutation will be almost avoiding, as after deletion of any previous essential, the avoiding permutation obtained may simply have n added to the front. There are c_{n-1} such permutations.

4. Finally, we collect all almost avoiding permutations that contain a pattern, do not begin with n, neither of the first two elements are essential, but either deletion of both the first two elements results in an avoiding permutation, or the first two elements are not a consecutively increasing sequence. There are a number of ways to obtain such a permutation:

    (a) The permutation begins with $n-1$. It follows that element n is essential in the permutation. Thus, after deletion of n, we obtain an avoiding permutation beginning with its largest element. We begin with any avoiding permutation beginning with a consecutively increasing sequence of length at least 2 on $[n-2]$ in 2^{n-4} ways, inserting $n-1$ as the first element. n will be essential on insertion if we do not insert it at the beginning of the permutation, nor immediately following $(n-1)$, so that we do not obtain an avoiding permutation. Further, we must not insert n at the end of the first consecutively increasing subsequence of the initial avoiding permutation, as doing so would make $n-1$ essential. Inserting n as the third element of the permutation would make the second element essential, so with $(n-4)$ choices for insertion of n, there are $(n-4)2^{n-4}$ such permutations. Otherwise, begin with a permutation on $[n-2]$ that begins with $(n-2)$ in 2^{n-4} ways, and follow the above argument, excepting that the third element of the permutation is the end of the first consecutively increasing subsequence, so that there are $(n-3)2^{n-4}$ such permutations.

    (b) The permutation begins with a consecutively increasing sequence of length exactly 2, say k and $k+1$, with the third element being $k+3$, with $1 \leq k \leq n-3$. In this case, k+2 is essential, but as deletion of k and $k+1$ will result in an avoiding permutation, $k+2$ is necessarily either immediately following the first consecutively increasing sequence in the resultant avoiding permutation, or at the end of the second consecutively increasing sequence (should one exist.) If $k+2$ is moved to the third entry of the original permutation, we would have an avoiding permutation beginning with a consecutively increasing sequence of length at least 4. In all but one of these 2^{n-4} such avoiding permutations, there are 2 places to move the third element. The identity permutation is the exception, as it is but one consecutively increasing sequence, and hence there are $2^{n-3}-1$ almost avoiding permutations in this case. In addition, if $k=n-2$ with no requirement on the third element, we can still move n to the third entry, giving an avoiding permutation beginning with a consecutively increasing sequence of length exactly 3. Of these there are 2^{n-4}, and n must have been the end of the following consecutively increasing sequence, obtaining another 2^{n-4} almost avoiding permutations.

    (c) The permutation begins with a consecutively increasing sequence of length exactly 2, say k and $k+1$, $k+3$ is not the third element, with $k+2$ being the fourth element, so that it is not essential. Further, $k < n-3$, to avoid overlap with the previous case. The third element of the permutation must be essential. Since deletion of both k and $k+1$ results in an avoiding permutation, this essential third element must be either n or $k-1$, and as $k < n-3$, deletion of the third element will give an avoiding permutation beginning with a consecutively increasing sequence of length at least 4. As there are 2^{n-5} such avoiding permutations, and with 2 possible insertions for the essential third element, we obtain 2^{n-4} almost avoiding permutations.

(d) The permutation begins with an increase of value of exactly 2, say k and $k+2$, with $1 \leq k \leq n-2$. In this case, $k+1$ is an essential element of the permutation. After deletion of $k+1$, the resulting avoiding permutation must begin with a consecutively increasing sequence of length $n-k \geq 2$, and so if $k+1$ is placed as the second entry of this avoiding permutation, it begins with a consecutively increasing sequence of length at least 3. If $k < n-2$, this consecutively increasing sequence is of length at least 4, and there are 2^{n-4} such avoiding permutations. $k+1$ can be made essential by moving it to one of $n-5$ positions in the permutation, which excludes the following: It cannot be placed first, as the almost avoiding permutation begins with k; it cannot be placed second or the permutation is completely avoiding; it cannot be placed third, as this would make $k+2$ essential; it cannot be placed after n, as then k is essential; it cannot be placed at the end of the consecutively increasing sequence following n, again to prevent k being essential. Note that there will be another exception for the identity permutation, which does not have a consecutively increasing sequence following n. Hence there are $(n-5)2^{n-4}+1$ such almost avoiding permutations. If $k = n-2$, then begin with an avoiding permutation with an initial consecutively increasing sequence of length exactly 3 in 2^{n-4} ways, and move $n-1$ to one of the $n-4$ positions as described above, as $n = k+2$, obtaining another $(n-4)2^{n-4}$ almost avoiding permutations.

Combining the preceeding cases, we obtain $(n-3)2^{n-2}$ almost avoiding permutations.

We now have $c_n = (n-2)2^{n-2} + 2c_{n-1} + (n-3)2^{n-2}$. Solving this recurrence, we obtain the desired result. □

5 Almost avoiding the pair of patterns $\{132, 321\}$

We now prove part 4 of Theorem 1.7,

$$d_n = L_n(132, 321) = \frac{1}{2}n^4 - \frac{5}{2}n^3 + 21n - 30 \text{ for } n \geq 3$$

To avoid some of the hassle of dealing with overcounts that show up in a direct count, we will count the permutations which almost avoid $\{132, 321\}$ in stages.

Firstly, the number of permutations of length n which strictly avoid the pair was proven by Simion and Schmidt [15] to be $\binom{n}{2}+1$. A consequence of that proof is that every permutation which totally avoids the pair must consist of a sequence of one, two, or three blocks, each of which is an increasing sequence of consecutive entries, as shown in figure 1.

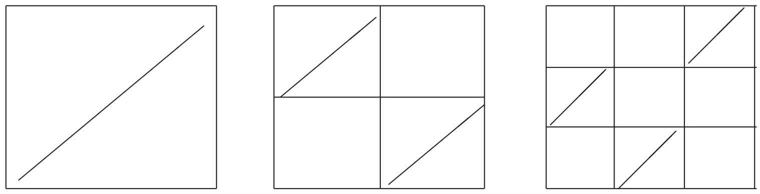

Figure 1: Structure of the three types of $\{132, 321\}$-avoiders.

As the issue of complete avoidance is resolved, we need only count the permutations which almost avoid the pair but do not completely avoid it – we will say these permutations *nearly* avoid the pair. Let g_n be the number of permutations of length n which nearly avoid $\{132, 321\}$. Furthermore, the first four values of g_n are easy enough to compute by hand, so we will assume hereafter that $n \geq 5$.

Some of the easiest near-avoiders to count are those in which n is an essential element; we need only count the ways to add n to a $\{132, 321\}$-avoider of length $n-1$ such that the resulting permutation no longer avoids the pair. If we start with the identity or a two-block avoider, n can be placed in $n-2$ different positions without creating a larger avoider – anywhere except at the end or directly before 1. If we start with a three-block avoider, n can be inserted into any of the $n-1$ positions except the last. Hence, the number of permutations of length n which nearly avoid $\{132, 321\}$, where n is an essential element, is

$$(1 + (n-2))(n-2) + \binom{n-2}{2}(n-1).$$

Next, notice that adding n to the end of a permutation of length $n-1$ does not introduce any new instances of 132 or 321. It follows that we can get all of the near-avoiders where n is the last element, but not an essential one by starting with with any almost-avoider of length $n-1$ and inserting n in the last position. Hence the number of permutations of length n which nearly avoid $\{132, 321\}$ where n is last but non-essential is g_{n-1}, the number of near-avoiders of length $n-1$.

It remains to count the permutations which nearly avoid $\{132, 321\}$ for which n is non-essential and not last. Let p be a permutation on $[n-1]$ which avoids $\{132, 321\}$. We must count the number of ways to insert an element which is less than n into p so that the resulting permutation does not avoid $\{132, 321\}$, n is not last, and n is not essential. There are three cases, depending on the structure of p.

Since the identity permutation and all three-block avoiders end in their largest element, any element less than n that we insert must be put into the last position to prevent n from being last in the result.

If p is the identity permutation or a three-block avoider where the third block is a single element, any element can be inserted into the last position except $n-1$ and the first element of the permutation, both of which would make n essential. If p is a three-block avoider where the last block has at least two elements, any element in $\{1, \ldots, n-2\}$ may be inserted into the last position without making n essential. It follows that there are

$$(n-3)(1 + (n-3)) + (n-2)\binom{n-3}{2}$$

permutations that can be built this way from the identity and three-block avoiders.

Suppose now that p is a two-block avoider. Since these permutations don't end in their largest element, we must consider all $n(n-1)$ insertions of elements less than n into all positions and rule out the ones that produce larger avoiders, permutations where n is essential or last, or duplicates. First, consider the case where both blocks have at least two elements. We can ignore all $(n-1)$ insertions of any element directly before itself, as those produce larger avoiders. Since inserting i directly after i produces the same permutation as inserting $i+1$ directly before i, we may ignore all $(n-1)$ insertions of an element directly after itself to avoid counting duplicates. Finally, to avoid creating a permutation where n is essential, neither $n-1$ nor the first element of p can be inserted in the last position, and 1 cannot be inserted directly before $n-1$. Hence, there are

$$n(n-1) - 2(n-1) - 3$$

permutations which can be created from each such permutation, of which there are $(n-4)$.

If p has a singleton block, we must be a bit more restrictive. Suppose that $p = (n-1)12\ldots(n-2)$. As before, we must rule out all $2(n-1)$ insertions of any element before or after itself, and the insertion of $n-1$ into the last position, as those all create either larger avoiding permutations or duplicates. Furthermore, insertion of any element in the first or second positions or insertion of 1 into any other position creates a permutation where n is essential, so we must throw out all $(n-2)+2(n-3)$ of those insertions that have not already been counted. It follows that there are

$$n(n-1) - 2(n-1) - 1 - (n-2) - 2(n-3)$$

legal permutations that may be produced from $(n-1)12\ldots(n-2)$. A symmetric argument shows that the same number of permutations are created from $23\ldots(n-1)1$.

There are still a few permutations that we have counted twice, which arise from adjacent cases. For example, notice that inserting 3 into the first position of 45123 and inserting 4 into the last position of 45123 give us the same permutation, 356124. Each permutation of this type, which is a two-block avoider with the first and last elements swapped, is created exactly twice, once from each pair of two-block avoiders of length $n-1$ wh ose first block sizes are consecutive but both bigger than 1. There are $(n-3)$ such permutations and thus $(n-4)$ consecutive pairs, so we have overcounted by $(n-4)$.

Adding up the preceding arguments, we gain the recurrence

$$g_n = 2n^3 - \frac{21}{2}n^2 + \frac{17}{2} + 19 + g_{n-1}.$$

Solving this recurrence and adding in the $\binom{n}{2}+1$ $\{132, 321\}$-avoiders of length n gives us (5).

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