PU. M. A. Vol. **22** (2011), No.2, pp. 173 – 208

Pattern matching in the cycle structures of permutations

MILES E. JONES
Department of Mathematics,
University of California, San Diego
La Jolla, CA 92093-0112. USA
e-mail: mej005@math.ucsd.edu

and
JEFFREY REMMEL
Department of Mathematics,
University of California, San Diego
La Jolla, CA 92093-0112. USA
e-mail: jremmel@ucsd.edu

(Received: February 24, 2011, and in revised form May 3, 2011)

Abstract. In this paper, we study pattern matching conditions in the cycle structures of permutations. We develop methods to find the joint distribution of the number of cycles and the number of cycle descents over the set of permutations in the symmetric group S_n which have no cycle occurrences or no consecutive cycle occurrences of a given pattern or set of patterns. Our methods also allow us to refine some results on the enumeration of permutations of S_n which have no occurrences or no consecutive occurrences of classical patterns in permutations in the case where the pattern starts with 1.

Mathematics Subject Classification(2010). 05A15, 05E05.

Keywords: permutation, cycle, pattern avoidance, generating function, recursion.

1 Introduction

The notion of patterns in permutations and words has proved to be a useful language in a variety of seemingly unrelated problems including the theory of Kazhdan-Lusztig polynomials, singularities of Schubert varieties, Chebyshev polynomials, rook polynomials for Ferrers boards, and various sorting algorithms including sorting stacks and permutations. The study of occurrences of patterns in words and permutations is a new, but rapidly growing, branch of combinatorics which has its roots in the works by Rotem, Rogers, and Knuth in the 1970s and early 1980s. The first systematic study of permutation patterns was not undertaken until the paper by Simion and Schmidt [26] which appeared in 1985. The field has experienced explosive growth since 1992.

First we recall the basic definitions for pattern matching in permutations. Given a sequence $\sigma = \sigma_1 \ldots \sigma_n$ of distinct integers, let $\text{red}(\sigma)$ be the permutation found by replacing the i^{th} largest integer that appears in σ by i. For example, if $\sigma = 2\ 7\ 5\ 4$, then $\text{red}(\sigma) = 1\ 4\ 3\ 2$. Given a permutation $\tau = \tau_1 \ldots \tau_j$ in the symmetric group S_j, we say a permutation $\sigma = \sigma_1 \ldots \sigma_n \in S_n$ has a *τ-match starting at position i* provided $\text{red}(\sigma_i \ldots \sigma_{i+j-1}) = \tau$. Let $\tau\text{-mch}(\sigma)$ be the number of τ-matches in the permutation σ. Similarly, we say that τ *occurs* in σ if there exist $1 \le i_1 < \cdots < i_j \le n$ such that $\mathrm{red}(\sigma_{i_1} \cdots \sigma_{i_j}) = \tau$. We say that σ *avoids* τ if there are no occurrences of τ in σ.

These definitions can naturally be extended to sets of permutations. That is, given a set of permutations Υ in the symmetric group S_j, define a permutation $\sigma = \sigma_1 \dots \sigma_n \in S_n$ to have a *Υ-match starting at position i* provided $\mathrm{red}(\sigma_i \dots \sigma_{i+j-1}) \in \Upsilon$. Let Υ-mch(σ) be the number of Υ-matches in the permutation σ. Similarly, we say that Υ *occurs* in σ if there exist $1 \le i_1 < \cdots < i_j \le n$ such that $\mathrm{red}(\sigma_{i_1} \cdots \sigma_{i_j}) \in \Upsilon$. We say that σ *avoids* Υ if there are no occurrences of Υ in σ.

In this paper, we shall study matching conditions within the cycle structure of a permutation. Suppose that $\tau = \tau_1 \dots \tau_j$ is a permutation in S_j and σ is a permutation in S_n with k cycles $C_1 \dots C_k$. We shall always write cycles in the form $C_i = (c_{0,i}, \dots, c_{p_i-1,i})$ where $c_{0,i}$ is the smallest element in C_i and p_i is the length of C_i and assume that we have arranged the cycles by decreasing smallest elements. That is, we arrange the cycles of σ so that $c_{0,1} > \cdots > c_{0,k}$. Then we say that σ has a *cycle τ-match* (c-τ-match) if there is an i such that $C_i = (c_{0,i}, \dots, c_{p_i-1,i})$ where $p_i \ge j$ and an r such that $\mathrm{red}(c_{r,i}c_{r+1,i} \dots c_{r+j-1,i}) = \tau$ where we take indices of the form $r+s \bmod p_i$. Let c-τ-mch(σ) be the number of cycle τ-matches in the permutation σ. For example, if $\tau = 2\ 1\ 3$ and $\sigma = (1, 10, 9)(2, 3)(4, 7, 5, 8, 6)$, then 9 1 10 is a cycle τ-match in the first cycle and 7 5 8 and 6 4 7 are cycle τ-matches in the third cycle so that c-τ-mch$(\sigma) = 3$. Similarly, we say that τ *cycle occurs* in σ if there exists an i such that $C_i = (c_{0,i}, \dots, c_{p_i-1,i})$ where $p_i \ge j$ and there is an r with $0 \le r \le p_i - 1$ and indices $0 \le i_1 < \cdots < i_{j-1} \le p_i - 1$ such that $\mathrm{red}(c_{r,i}c_{r+i_1,i} \dots c_{r+i_{j-1},i}) = \tau$ where the indices $r + i_s$ are taken mod p_i. We say that σ *cycle avoids* τ if there are no cycle occurrences of τ in σ. For example, if $\tau = 1\ 2\ 3$ and $\sigma = (1, 10, 9)(2, 3)(4, 8, 5, 7, 6)$, then 4 5 7, 4 5 6, 5 6 8, and 5 7 8 are cycle occurrences of τ in the third cycle.

We can extend the notion of cycle matches and cycle occurrences to sets of permutations in the obvious fashion. That is, suppose that Υ is a set of permutations in S_j and σ is a permutation in S_n with k cycles $C_1 \dots C_k$. Then we say that σ has a *cycle Υ-match* (c-Υ-match) if there is an i such that $C_i = (c_{0,i}, \dots, c_{p_i-1,i})$ where $p_i \ge j$ and an r such that $\mathrm{red}(c_{r,i} \dots c_{r+j-1,i}) \in \Upsilon$ where we take indices of the form $r + s$ modulo p_i. Let c-Υ-mch(σ) be the number of cycle Υ-matches in the permutation σ. Similarly, we say that Υ *cycle occurs* in σ if there exists an i such that $C_i = (c_{0,i}, \dots, c_{p_i-1,i})$ where $p_i \ge j$ and there is an r with $0 \le r \le p_i - 1$ and indices $0 \le i_1 < \cdots < i_{j-1} \le p_i - 1$ such that $\mathrm{red}(c_{r,i}c_{r+i_1,i} \dots c_{r+i_{j-1},i}) \in \Upsilon$ where the indices $r + i_s$ are taken mod p_i. We say that σ *cycle avoids* Υ if there are no cycle occurrences of Υ in σ.

The study of patterns in cycle structures in not entirely new. That is, Callan [7] and Vella [29] studied cycle pattern avoidance in n-cycles in S_n. For example, they independently proved that the number of n-cycles in S_n which cycle avoid 1324 is the Fibonacci number F_{2n-3}, the number of n-cycles in S_n which cycle avoid 1342 is $2^{n-1} - (n-1)$, and the number of n-cycles in S_n which cycle avoid 1234 is $2^n + 1 - 2n - \binom{n}{3}$. However, neither Callan or Vella considered the more general problem of cycle avoidance in general cycle structures of permutations. We shall see below that one can use the theory of exponential structures to reduce the problem of finding certain generating functions on the cycle of structures of permutations in S_n to finding certain corresponding generating functions on n-cycles in S_n. Thus it is not difficult to extend the results of Callan and Vella to cycle structure of permutations in S_n. This idea was used by Deutsch and Elizalde [8] to study various generating functions for the analogue of up-down permutations relative to cycle structures of permutations in S_n. We shall see below that for even cycles, their definition of up-down cycles is equivalent to having no cycle $\{123,321\}$-matches.

In this paper, we shall give generating functions for the joint distributions of cycles and cycle descents over the set of permutations of S_n which either have no cycle occurrences or no cycle matches for certain patterns or sets of patterns. That is, given a cycle $C=(c_0,\ldots,c_{p-1})$ where c_0 is the smallest element in the cycle, we let $\mathrm{cdes}(C)=1+\mathrm{des}(c_0\ldots c_{p-1})$. Thus $\mathrm{cdes}(C)$ counts the number of descent pairs as we traverse once around the cycle because the extra factor of 1 counts the descent pair $c_{p-1}>c_0$. For example if $C=(1,5,3,7,2)$, then $\mathrm{cdes}(C)=3$ which counts the descent pairs 53, 72, and 21 as we traverse once around C. By convention, if $C=(c_0)$ is one-cycle, we let $\mathrm{cdes}(C)=1$. If σ is a permutation in S_n with k cycles $C_1\ldots C_k$, then we define $\mathrm{cdes}(\sigma)=\sum_{i=1}^k \mathrm{cdes}(C_i)$. We let $\mathrm{cyc}(\sigma)$ denote the number of cycles of σ. Our generating functions for such joint distributions are new. However, in the case where the pattern τ starts with 1, then our generating functions can be viewed as refinements of generating functions for the number of permutations of S_n which have no classical matches of τ.

Given $\Upsilon\subseteq S_j$, we let $\mathcal{AV}_n(\Upsilon)$ (resp. $\mathcal{CAV}_n(\Upsilon)$) denote the set of permutations of S_n which avoid (resp. cycle avoid) Υ and $AV_n(\Upsilon)=|\mathcal{AV}_n(\Upsilon)|$ (resp. $CAV_n(\Upsilon)=|\mathcal{CAV}_n(\Upsilon)|$). Similarly, we let $\mathcal{NM}_n(\Upsilon)$ (resp. $\mathcal{NCM}_n(\Upsilon)$) denote the set of permutations of S_n which have no Υ-matches (resp. no cycle Υ-matches) and $NM_n(\Upsilon)=|\mathcal{NM}_n(\Upsilon)|$ (resp. $NCM_n(\Upsilon)=|\mathcal{NCM}_n(\Upsilon)|$). Throughout this paper, when $\Upsilon=\{\tau\}$ is a singleton, we shall just write the τ rather than $\{\tau\}$. Thus for example, we shall write $\mathcal{AV}_n(\tau)$ for $\mathcal{AV}_n(\Upsilon)$ when $\Upsilon=\{\tau\}$.

Given α and β in S_j, we say that α and β are *Wilf equivalent* if $AV_n(\alpha)=AV_n(\beta)$ for all n. We say that α and β are *matching Wilf equivalent* (m-Wilf equivalent) if $NM_n(\alpha)=NM_n(\beta)$ for all n. For any permutation $\sigma=\sigma_1\ldots\sigma_n$, we let σ^r be the reverse of σ and σ^c be the complement of σ. That is, $\sigma^r=\sigma_n\ldots\sigma_1$ and $\sigma^c=(n+1-\sigma_1)\ldots(n+1-\sigma_n)$. It is well known that Wilf equivalence classes and m-Wilf equivalence classes are closed under reverse and complementation. We say that α and β are *cycle avoidance Wilf equivalent* (ca-Wilf equivalent) if $CAV_n(\alpha)=CAV_n(\beta)$ for all n and we say that α and β are *cycle matching Wilf equivalent* (cm-Wilf equivalent) if $NCM_n(\alpha)=NCM_n(\beta)$ for all n. If α and β are cycle avoidance Wilf equivalent, we shall write $\alpha\sim_{ca}\beta$. If α and β are cycle matching Wilf equivalent, we shall write $\alpha\sim_{cm}\beta$. Similarly, for sets of permutations Γ and Δ in S_j, we say that Γ and Δ are cycle avoidance Wilf equivalent (ca-Wilf equivalent) if $CAV_n(\Gamma)=CAV_n(\Delta)$ for all n and we say that Γ and Δ are cycle matching Wilf equivalent (cm-Wilf equivalent) if $NCM_n(\Gamma)=NCM_n(\Delta)$ for all n.

Callan [7] and Vella [29] observed that for n-cycles, ca-Wilf equivalence classes are closed under reverse and complement. This is also true for both ca-Wilf equivalence and cm-Wilf equivalence for general cycle structures. That is, let σ be a permutation in S_n with k cycles $C_1\ldots C_k$. Then we let the *cycle reverse of* σ, denoted by σ^{cr}, be the permutation which arises from σ by replacing each cycle $C_i=(c_{0,i},c_{1,i},\ldots,c_{p_i-1,i})$ by its reverse cycle $C_i^{cr}=(c_{0,i},c_{p_i-1,i},\ldots c_{1,i})$. For example, if $\sigma=(1,10,9)(2,3)(4,7,5,8,6)$, then $\sigma^{cr}=(1,9,10)(2,3)(4,6,8,5,7)$. We let the cycle complement of σ, denoted by σ^{cc}, be the permutation that results from σ by replacing each element i in the cycle structure of σ by $n+1-i$. For example, if $\sigma=(1,10,9)(2,3)(4,7,5,8,6)$, then $\sigma^{cr}=(10,1,2)(9,8)(7,4,6,3,5)=(1,2,10)(3,5,7,4,6)(8,9)$. Note that in general σ^r, σ^c, σ^{cr} and σ^{cc} are all distinct. For example, if

$\sigma = 2\ 3\ 1\ 5\ 4$ so that it cycle structure is $(1,2,3)(4,5)$, then

$$\begin{aligned} \sigma^r &= 4\ 5\ 1\ 3\ 2, \\ \sigma^c &= 4\ 3\ 5\ 1\ 2, \\ \sigma^{cr} &= (1,3,2)(4,5) = 3\ 1\ 2\ 5\ 4, \text{ and} \\ \sigma^{cc} &= (5,4,3)(2,1) = 2\ 1\ 5\ 3\ 4. \end{aligned}$$

It is easy to see that for any permutation $\sigma \in S_n$,

1. σ has a cycle τ-match if and only if σ^{cr} has a cycle τ^r-match,
2. σ has a cycle τ-match if and only if σ^{cc} has a cycle τ^c-match,
3. σ has a cycle τ occurrence if and only if σ^{cr} has a cycle τ^r occurrence, and
4. σ has a cycle τ occurrence if and only if σ^{cc} has a cycle τ^c occurrence.

It then easily follows that for all permutations τ, $NCM_n(\tau) = NCM_n(\tau^r) = NCM_n(\tau^c)$ so that τ, τ^r, and τ^c are all cycle matching Wilf equivalent. Similarly, $CAV_n(\tau) = CAV_n(\tau^r) = CAV_n(\tau^c)$ so that τ, τ^r, and τ^c are all cycle avoidance Wilf equivalent. Finally we observe that our definitions also ensure that for any $\tau = \tau_1 \ldots \tau_j \in S_j$, any cyclic rearrangement of τ, $\tau^{(i)} = \tau_i \ldots \tau_j \tau_1 \ldots \tau_{i-1}$ also has the property that for any $\sigma \in S_n$, τ cycle occurs in σ if and only if $\tau^{(i)}$ cycle occurs in σ. Thus for all $j \geq 1$, $CAV_n(\tau) = CAV_n(\tau^{(i)})$ so that τ and $\tau^{(i)}$ are cycle avoidance Wilf equivalent.

Given a permutation $\sigma = \sigma_1 \ldots \sigma_n \in S_n$, we let $\text{des}(\sigma) = |\{i : \sigma_i > \sigma_{i+1}\}|$. We say that σ_j is a *left-to-right minimum* of σ if $\sigma_j < \sigma_i$ for all $i < j$. We let $\text{LRMin}(\sigma)$ denote the number of left-to-right minima of σ.

The main goal of this paper is to study the generating functions

$$CAV_\Upsilon(t,x,y) = 1 + \sum_{n \geq 1} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{CAV}_n(\Upsilon)} x^{\text{cyc}(\sigma)} y^{\text{cdes}(\sigma)}, \tag{1}$$

and

$$NCM_\Upsilon(t,x,y) = 1 + \sum_{n \geq 1} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{NCM}_n(\Upsilon)} x^{\text{cyc}(\sigma)} y^{\text{cdes}(\sigma)} \tag{2}$$

for $\Upsilon \subseteq S_j$ as well their specializations such as

$$\begin{aligned} CAV_\Upsilon(t,x) &= 1 + \sum_{n \geq 1} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{CAV}_n(\Upsilon)} x^{\text{cyc}(\sigma)}, \\ CAV_\Upsilon(t) &= 1 + \sum_{n \geq 1} CAV_n(\Upsilon) \frac{t^n}{n!}, \\ NCM_\Upsilon(t,x) &= 1 + \sum_{n \geq 1} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{NCM}_n(\Upsilon)} x^{\text{cyc}(\sigma)}, \text{ and} \\ NCM_\Upsilon(t) &= 1 + \sum_{n \geq 1} NCM_n(\Upsilon) \frac{t^n}{n!}. \end{aligned}$$

We know of several ways to approach the this problem. The most direct way is to use the theory of exponential structures to reduce the problem down to studying pattern matching in n-cycles. That is, let $\mathcal{L}_m$ denote the set of m-cycles in S_m and let $L_m = |\mathcal{L}_m|$. Suppose that R is a ring and for each $m \geq 1$, we have a weight function $W_m : \mathcal{L}_m \to R$. We let $W(L_m) = \sum_{C \in \mathcal{L}_m} W_m(C)$. Now suppose that $\sigma \in S_n$ and the cycles of σ are $C_1, \ldots, C_k$. If C_i is of size m, then we let $W(C_i) = W_m(\mathrm{red}(C_i))$. Then we define the weight of σ, $W(\sigma)$, by

$$W(\sigma) = \prod_{i=1}^{k} W(C_i).$$

We let $\mathcal{C}_{n,k}$ denote the set of all permutations of S_n with k cycles. This given, the following theorem easily follows from the theory of exponential structures, see [27].

THEOREM 1.1

$$1 + \sum_{n=1}^{\infty} \frac{t^n}{n!} \sum_{k=1}^{n} x^k \sum_{\sigma \in \mathcal{C}_{n,k}} W(\sigma) = e^{x \sum_{m \geq 1} \frac{W(L_m) t^m}{m!}}. \tag{3}$$

Let $\Upsilon \subseteq S_j$. Then we will be most interested in the special case of weight functions W_m where $W_m(C) = 1$ if C cycle avoids a set of permutations and $W_m(C) = 0$ otherwise, or where $W_m(C) = 1$ if C has no cycle Υ-matches and $W_m(C) = 0$ otherwise. We let $\mathcal{CAV}_{n,k}(\Upsilon)$ denote the set of permutations σ of S_n with k cycles such that σ cycle avoids Υ and we let $CAV_{n,k}(\Upsilon) = |\mathcal{CAV}_{n,k}(\Upsilon)|$. We let $\mathcal{NCM}_{n,k}(\Upsilon)$ denote the set of permutations σ of S_n with k cycles such that σ has no cycle Υ-matches and $NCM_{n,k}(\Upsilon) = |\mathcal{NCM}_{n,k}(\Upsilon)|$. Similarly, we let $\mathcal{L}_m^{cav}(\Upsilon)$ be the set of m-cycles γ in S_m such that γ cycle avoids Υ, $L_m^{cav}(\Upsilon) = |\mathcal{L}_m^{cav}(\Upsilon)|$, $\mathcal{L}_m^{ncm}(\Upsilon)$ denote the set of m-cycles γ in S_m such that γ has no cycle Υ-matches, and $L_m^{ncm}(\Upsilon) = |\mathcal{L}_m^{ncm}(\Upsilon)|$. Then a special case of Theorem 1.1 is the following theorem.

THEOREM 1.2

$$CAV_{\Upsilon}(t,x) = 1 + \sum_{n \geq 1} \frac{t^n}{n!} \sum_{k=1}^{n} CAV_{n,k}(\Upsilon) x^k = e^{x \sum_{m \geq 1} L_m^{cav}(\Upsilon) \frac{t^m}{m!}}, \tag{4}$$

$$NCM_{\Upsilon}(t,x) = 1 + \sum_{n \geq 1} \frac{t^n}{n!} \sum_{k=1}^{n} NCM_{n,k}(\Upsilon) x^k = e^{x \sum_{m \geq 1} L_m^{ncm}(\Upsilon) \frac{t^m}{m!}}, \tag{5}$$

$$CAV_{\Upsilon}(t,x,y) = 1 + \sum_{n \geq 1} \frac{t^n}{n!} \sum_{k=1}^{n} x^k \sum_{\sigma \in \mathcal{CAV}_{n,k}(\Upsilon)} y^{\mathrm{cdes}(\sigma)} = e^{x \sum_{m \geq 1} \frac{t^m}{m!} \sum_{C \in \mathcal{L}_m^{cav}(\Upsilon)} y^{\mathrm{cdes}(C)}}, \tag{6}$$

and

$$NCM_{\Upsilon}(t,x,y) = 1 + \sum_{n \geq 1} \frac{t^n}{n!} \sum_{k=1}^{n} x^k \sum_{\sigma \in \mathcal{NCM}_{n,k}(\Upsilon)} y^{\mathrm{cdes}(\sigma)} = e^{x \sum_{m \geq 1} \frac{t^m}{m!} \sum_{C \in \mathcal{L}_m^{ncm}(\Upsilon)} y^{\mathrm{cdes}(C)}}. \tag{7}$$

For example, suppose that $\tau = 1\ 2$. It is clear that any cycle of length k where $k \geq 2$ has both a cycle occurrence of τ and a cycle τ-match so that $L_m^{cav}(12) = L_m^{ncm}(12) = 0$ if $m \geq 2$. Since 1-cycles can not have any cycle occurrences of τ or any cycle τ-matches by definition, it follows that

$$y = \sum_{C \in \mathcal{L}_1^{cav}(12)} y^{\mathrm{cdes}(C)} = \sum_{C \in \mathcal{L}_1^{ncm}(12)} y^{\mathrm{cdes}(C)}.$$

Thus

$$CAV_{12}(t,x,y) = NCM_{12}(t,x,y) = e^{xyt}.$$

Next consider $\tau = 1\ 2\ 3$. It was observed by both Callan [7] and Vella [29] that for $k \geq 3$, the only k-cycle which cycle avoids τ is the cycle $(1, k, k-1, \ldots, 2)$. Let

$$A_m(y) = \sum_{C \in \mathcal{L}_m^{cav}(123)} y^{\text{cdes}(C)},$$

then clearly $A_1(y) = y$ since $\text{cdes}((1)) = 1$, and for $k \geq 2$, $A_k(y) = y^{k-1}$ since $\text{cdes}((1,k,\ldots,2)) = k-1$. Thus

$$CAV_{123}(t,x,y) = e^{x\ \ yt + \sum_{m\geq 2} \frac{y^{m-1}t^m}{m!}} = e^{x\ \ yt + \frac{1}{y}(e^{yt}-1-yt)}.$$

It turns out that if $\tau \in S_j$ is a permutation that starts with 1, then we can reduce the problem of finding $NCM_\tau(t,x)$ and $NCM_\tau(t,x,y)$ to the usual problem of finding the generating function of permutations that have no τ-matches. That is, we consider the following well-known bijection described in [8]. Suppose we are given a permutation $\sigma \in S_n$ with k cycles $C_1 \cdots C_k$. Assume we have arranged the cycles so that the smallest element in each cycle is on the left and we arrange the cycles by decreasing smallest elements. Then we let $\bar{\sigma}$ be the permutation that arises from $C_1 \cdots C_k$ by erasing all the parentheses and commas. For example, if $\sigma = (7,10,9,11)\ (4,8,6)\ (1,5,3,2)$, then $\bar{\sigma} = 7\ 10\ 9\ 11\ 4\ 8\ 6\ 1\ 5\ 3\ 2$. It is easy to see that the minimal elements of the cycles correspond to left-to-right minima in $\bar{\sigma}$. It is also easy to see that under our bijection $\sigma \to \bar{\sigma}$, $\text{cdes}(\sigma) = \text{des}(\bar{\sigma})+1$ since every left-to-right minima is part of a descent pair in $\bar{\sigma}$. For example, if $\sigma = (7,10,9,11)\ (4,8,6)\ (1,5,3,2)$ so that $\bar{\sigma} = 7\ 10\ 9\ 11\ 4\ 8\ 6\ 1\ 5\ 3\ 2$, $\text{cdes}((7,10,9,11)) = 2$, $\text{cdes}((4,8,6)) = 2$, and $\text{cdes}((1,5,3,2)) = 3$ so that $\text{cdes}(\sigma) = 2+2+3 = 7$ while $\text{des}(\bar{\sigma}) = 6$. This given, we have the following lemma.

LEMMA 1.3 *If $\tau \in S_j$ and τ starts with 1, then for any $\sigma \in S_n$,*

1. *σ has k cycles if and only if $\bar{\sigma}$ has k left-to-right minima,*
2. *$\text{cdes}(\sigma) = 1 + \text{des}(\bar{\sigma})$, and*
3. *σ has no cycle-τ-matches if and only if $\bar{\sigma}$ has no τ-matches.*

Proof. For (3), suppose that $\bar{\sigma} = \bar{\sigma}_1 \ldots \bar{\sigma}_n$ and $\bar{\sigma}_i = 1$. Since τ starts with 1, it is easy to see that any τ-match in $\bar{\sigma}$ must either occur weakly to the right of $\bar{\sigma}_i$ or strictly to the left of $\bar{\sigma}_i$. That is, 1 can be part of τ-match in $\bar{\sigma}$ only if the τ-match starts at position i. If a τ-match occurred weakly to the right of $\bar{\sigma}_i$, then that τ-match would correspond to a cycle-τ-match in C_k in σ.

Next suppose that the τ-match occurred strictly to the left of $\bar{\sigma}_i = 1$. Then we claim that we can make a similar argument with respect to the cycles $C_1 \cdots C_{k-1}$. That is, suppose that C_{k-1} starts with m. Then m must be the smallest element among $\bar{\sigma}_1 \ldots \bar{\sigma}_{i-1}$. Suppose that $\bar{\sigma}_s = m$ where $1 \leq s < i$. Then again we can argue that any τ-match in $\bar{\sigma}_1 \ldots \bar{\sigma}_{i-1}$ must occur either weakly to the right of $\bar{\sigma}_s$ or strictly to left of $\bar{\sigma}_s$. If the τ-match in $\bar{\sigma}_1 \ldots \bar{\sigma}_{i-1}$ occurs weakly to the right of $\bar{\sigma}_s$, then it would correspond to a cycle-τ-match in C_{k-1}. Continuing on in this way, we see that any τ-match in $\bar{\sigma}$ must correspond to a cycle τ-match in C_j for some j.

Vice versa, it is easy to see that since τ starts with 1, the only way that a cycle-τ-match in C_i can involve the smallest element $c_{0,i}$ in the cycle C_i is if $c_{0,i}$ corresponds to the 1 in τ in cycle match. But this easily implies that any τ-cycle match in C_i must also correspond to a τ-match in the elements of $\bar{\sigma}$ corresponding to C_i.

Thus we have proved that for any σ, σ has cycle-τ-match if only if $\bar{\sigma}$ has a τ-match. □

We should note that if a permutation τ does not start with 1, then it may be that case that $NCM_n(\tau) \neq NM_n(\tau)$. The pattern $\tau = 3\ 1\ 4\ 2$ is an example such that neither τ, τ^r, τ^c, nor $(\tau^r)^c$ starts with one. Even though we do not know how to compute closed forms for $NCM_\tau(t)$ and $NM_\tau(t)$, we have computed the following table.

n	$L_n^{ncm}(3142)$	$NCM_n(3142)$	$NM_n(3142)$
1	1	1	1
2	1	2	2
3	2	6	6
4	5	23	23
5	20	111	110
6	92	638	632
7	532	4278	4237
8	3565	32784	32465

One consequence of Lemma 1.3 is that we can automatically obtain refinements of generating functions for the number of permutations with no τ-matches when τ starts with 1. That is, let

$$\begin{aligned} NM_\tau(t,x) &= \sum_{n\geq 0} \frac{t^n}{n!} \sum_{\sigma\in\mathcal{NM}_n(\tau)} x^{\mathrm{LRMin}(\sigma)} \text{ and} \\ NM_\tau(t,x,y) &= \sum_{n\geq 0} \frac{t^n}{n!} \sum_{\sigma\in\mathcal{NM}_n(\tau)} x^{\mathrm{LRMin}(\sigma)} y^{1+\mathrm{des}(\sigma)}. \end{aligned}$$

Then we have the following corollary of Lemma 1.3.

Corollary 1.4 *If $\tau \in S_j$ and τ starts with 1, then*

$$NCM_\tau(t,x) = NM_\tau(t,x) \text{ and} \tag{8}$$
$$NCM_\tau(t,x,y) = NM_\tau(t,x,y). \tag{9}$$

Then by Theorem 1.2 and Lemma 1.3, if $\tau \in S_j$ and τ starts with 1, we have that

$$\begin{aligned} NM_\tau(t,1) &= NCM_\tau(t,1) \\ &= e^{\sum_{m\geq 1} L_m^{ncm}(\tau)\frac{t^m}{m!}} \end{aligned}$$

so that

$$\ln(NM_\tau(t,1)) = \sum_{m\geq 1} L_m^{ncm}(\tau)\frac{t^m}{m!}. \tag{10}$$

But then

$$NM_\tau(t,x) = NCM_\tau(t,x) \tag{11}$$
$$= e^{x\sum_{m\geq 1} L_m^{ncm}(\tau)\frac{t^m}{m!}}$$
$$= e^{x\ln(NM_\tau(t,1))} = (NM_\tau(t,1))^x. \tag{12}$$

Thus if we can compute $NM_\tau(t,1)$ for a permutation $\tau \in S_j$ that starts with 1, we automatically can compute $NM_\tau(t,x)$. For example, Goulden and Jackson [11] proved that when $\tau = 1\ 2 \dots k$, then

$$NM_\tau(t) = \frac{1}{\sum_{i\geq 0} \frac{t^{ki}}{(ki)!} - \frac{t^{ki+1}}{(ki+1)!}}. \tag{13}$$

Hence, we automatically have the following refinement of Goulden and Jackson's result.

THEOREM 1.5 *If* $\tau = 1\ 2 \dots k$ *where* $k \geq 2$, *then*

$$NM_\tau(t,x) = \left(\frac{1}{\sum_{i\geq 0} \frac{t^{ki}}{(ki)!} - \frac{t^{ki+1}}{(ki+1)!}}\right)^x. \tag{14}$$

An example, where one can use the full power of Theorem 1.1 is the following. In Section 2, we shall show that

$$\sum_{n\geq 1} \frac{t^n}{n!} \sum_{C \in \mathcal{L}_n^{ncm}(132)} y^{\mathrm{cdes}(C)} = \ln\left(\frac{1}{1 - y\int_0^t e^{(1-y)s - y\frac{s^2}{2}}\, ds}\right). \tag{15}$$

Then it follows that

$$\begin{aligned} NCM_{132}(t,x,y) &= \sum_{n\geq 0} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{\mathrm{cyc}(\sigma)} y^{\mathrm{cdes}(\sigma)} \\ &= \sum_{n\geq 0} \frac{t^n}{n!} \sum_{k=1}^{n} x^k \sum_{\sigma \in \mathcal{NCM}_{n,k}(\tau)} y^{\mathrm{cdes}(\sigma)} \\ &= e^{x \ln\left(\frac{1}{1 - y\int_0^t e^{(1-y)s - y\frac{s^2}{2}}\, ds}\right)} \\ &= \left(\frac{1}{1 - y\int_0^t e^{(1-y)s - y\frac{s^2}{2}}\, ds}\right)^x. \end{aligned} \tag{16}$$

This result is a refinement of theorem of Elizalde and Noy [9].

The outline of this paper is as follows. In Section 2, we determine the generating function $CAV_\tau(t,x,y)$ and $NCM_\tau(t,x,y)$ for all $\tau \in S_3$ as well as compute $CAV_\Upsilon(t,x,y)$ and $NCM_\Upsilon(t,x,y)$ for certain subsets $\Upsilon \subseteq S_3$. In Section 3, we shall compute $NCM_\tau(t,x,y)$ for all $\tau = \tau_1 \dots \tau_j \in S_j$ where $\tau_1 =$ and $\tau_j = 2$ and for all $\tau = \tau_1 \dots \tau_{j+p} \in S_{j+p}$ of the form $\tau = 1\ 2 \dots j-1\ \gamma\ j$ where $j \geq 3$ and γ is a permutation of $j+1, \dots, j+p$. Finally, in Section 4, we shall briefly describe two other approaches to computing the generating function $NCM_\tau(t,x,y)$.

2 Patterns of length 3

In this section, we study $CAV_\tau(t,x,y)$ and $NCM_\tau(t,x,y)$ for $\tau \in S_3$.

First we consider $CAV_\tau(t,x)$ for $\tau \in S_3$. It follows from our remarks in the introduction that both cycle avoidance Wilf equivalence and cycle matching Wilf equivalence are closed under the operation of reverse and complement. Thus

1. $1\ 2\ 3 \sim_{ca} 3\ 2\ 1$ and $1\ 2\ 3 \sim_{cm} 3\ 2\ 1$ and
2. $1\ 3\ 2 \sim_{ca} 2\ 3\ 1 \sim_{ca} 2\ 1\ 3 \sim_{ca} 3\ 1\ 2$ and $1\ 3\ 2 \sim_{cm} 2\ 3\ 1 \sim_{cm} 2\ 1\ 3 \sim_{cm} 3\ 1\ 2$.

Now since cycle avoidance Wilf equivalence is closed under cyclic rearrangements, it follows that $1\ 2\ 3 \sim_{ca} 2\ 3\ 1$ which means that all permutations of length three are cycle avoidance Wilf equivalent. Thus for all permutations τ of length three, we have

$$CAV_\tau(t) = CAV_{123}(t) = e^{e^t-1}$$

which is also the exponential generating function for Bell numbers B_n that count the number of partitions of a set with n elements. But since

$$CAV_\tau(t) = e^{\sum_{m\geq 1} L_m^{cav}(\tau)\frac{t^m}{m!}}$$

for all $\tau \in S_3$, it must be the case that

$$\sum_{m\geq 1} L_m^{cav}(\tau)\frac{t^m}{m!} = e^t - 1$$

for all $\tau \in S_3$ and, hence,

$$CAV_\tau(t,x) = e^{x\sum_{m\geq 1} L_m^{cav}(\tau)\frac{t^m}{m!}} = e^{x(e^t-1)}$$

for all $\tau \in S_3$. However it is not the case that the generating functions $CAV_\tau(t,x,y)$ are equal for all $\tau \in S_3$. That is, suppose that α is a cyclic rearrangement of β. Then it is easy to see that $\mathcal{L}_m^{cav}(\alpha) = \mathcal{L}_m^{cav}(\beta)$ for all $m \geq 1$ so that

$$\sum_{C\in\mathcal{L}_m^{cav}(\alpha)} y^{\mathrm{cdes}(C)} = \sum_{C\in\mathcal{L}_m^{cav}(\beta)} y^{\mathrm{cdes}(C)}. \tag{17}$$

But then it follows from Theorem 1.2 that we must have $CAV_\alpha(t,x,y) = CAV_\beta(t,x,y)$. It thus follows from our results in the introduction that

$$CAV_{123}(t,x,y) = CAV_{312}(t,x,y) = CAV_{231}(t,x,y) = e^{x\left(yt+\frac{1}{y}(e^{yt}-1-yt)\right)}.$$

Next consider $\tau = 1\ 3\ 2$. It is easy to see that for $k \geq 3$, the only k-cycle which cycle avoids τ is the cycle $(1,2,\ldots,k)$. Thus

$$\sum_{C\in\mathcal{L}_m^{cav}(132)} y^{\mathrm{cdes}(C)} = y,$$

for all $k \geq 1$. Hence

$$CAV_{132}(t,x,y) = CAV_{213}(t,x,y) = CAV_{321}(t,x,y) = e^{x\left(\sum_{m\geq 1}\frac{yt^m}{m!}\right)} = e^{xy(e^t-1)}.$$

Next we shall consider the generating functions $NCM_\tau(t,x,y)$ for $\tau \in S_3$. We claim that is enough to compute $NCM_{123}(t,x,y)$ and $NCM_{132}(t,x,y)$. That is, for any $j \geq 2$ and $\tau \in S_j$, we can compute $NCM_{\tau^r}(t,x,y)$ and $NCM_{\tau^c}(t,x,y)$ from $NCM_\tau(t,x,y)$. Note that it follows from Theorem 1.2 that

$$\sum_{n\geq 1}\frac{t^n}{n!}\sum_{C\in\mathcal{L}_n^{ncm}(\tau)} y^{\mathrm{cdes}(C)} = \ln\left(NCM_\tau(t,1,y)\right). \tag{18}$$

Since $\sum_{C \in \mathcal{L}_1^{ncm}(\tau)} y^{\text{cdes}(C)} = y$, it follows that

$$\sum_{n \geq 2} \frac{t^n}{n!} \sum_{C \in \mathcal{L}_n^{ncm}(\tau)} y^{\text{cdes}(C)} = \ln\left(NCM_\tau(t,1,y)\right) - yt. \tag{19}$$

Given any n-cycle C in S_n, recall C^{cr} denotes its cycle-reverse and C^{cc} denotes its cycle-complement. Then $C \in \mathcal{L}_n^{ncm}(\tau)$ if and only if $C^{cr} \in \mathcal{L}_n^{ncm}(\tau^r)$ and $C \in \mathcal{L}_n^{ncm}(\tau)$ if and only if $C^{cc} \in \mathcal{L}_n^{ncm}(\tau^c)$. Now if $n \geq 2$, then it is easy to see that $n - \text{cdes}(C) = \text{cdes}(C^{cr}) = \text{cdes}(C^{cc})$. That is, each descent as we read once around the cycle C becomes a rise as we read around the cycles of C^{cr} and C^{cc} and each rise as we read once around the cycle C becomes a descent as we read around the cycles of C^{cr} and C^{cc}. Note, however, that if C is a one-cycle, then $C^{cr} = C^{cc} = C$ and $\text{cdes}(C) = \text{cdes}(C^{cr}) = \text{cdes}(C^{cc}) = 1$ so that it is not the case that $\text{cdes}(C^{cr}) = \text{cdes}(C^{cr}) = 1 - \text{cdes}(C)$. Thus we have to treat the one-cycles separately. Thus we have that

$$\begin{aligned} \sum_{n \geq 2} \frac{t^n}{n!} \sum_{C \in \mathcal{L}_n^{ncm}(\tau)} y^{n-\text{cdes}(C)} &= \sum_{n \geq 2} \frac{t^n}{n!} \sum_{C \in \mathcal{L}_n^{ncm}(\tau^r)} y^{\text{cdes}(C)} \\ &= \sum_{n \geq 2} \frac{t^n}{n!} \sum_{C \in \mathcal{L}_n^{ncm}(\tau^c)} y^{\text{cdes}(C)}. \end{aligned}$$

It follows that if $\tau \in S_j$ where $j \geq 2$ and

$$G(t,y) = \sum_{n \geq 2} \frac{t^n}{n!} \sum_{C \in \mathcal{L}_n^{ncm}(\tau)} y^{\text{cdes}(C)}, \tag{20}$$

then

$$G(ty, y^{-1}) = \sum_{n \geq 2} \frac{t^n}{n!} \sum_{C \in \mathcal{L}_n^{ncm}(\tau^r)} y^{\text{cdes}(C)} = \sum_{n \geq 2} \frac{t^n}{n!} \sum_{C \in \mathcal{L}_n^{ncm}(\tau^c)} y^{\text{cdes}(C)}. \tag{21}$$

Thus by (19), we have that

$$\begin{aligned} \ln\left(NCM_\tau(ty,1,y^{-1})\right) - t &= \sum_{n \geq 2} \frac{t^n}{n!} \sum_{C \in \mathcal{L}_n^{ncm}(\tau^r)} y^{\text{cdes}(C)} \\ &= \sum_{n \geq 2} \frac{t^n}{n!} \sum_{C \in \mathcal{L}_n^{ncm}(\tau^c)} y^{\text{cdes}(C)} \end{aligned}$$

so that

$$\begin{aligned} ty - t + \ln\left(NCM_\tau(ty,1,y^{-1})\right) &= \sum_{n \geq 1} \frac{t^n}{n!} \sum_{C \in \mathcal{L}_n^{ncm}(\tau^r)} y^{\text{cdes}(C)} \\ &= \sum_{n \geq 1} \frac{t^n}{n!} \sum_{C \in \mathcal{L}_n^{ncm}(\tau^c)} y^{\text{cdes}(C)}. \end{aligned}$$

Then we can apply Theorem 1.2 to obtain the following result.

THEOREM 2.1 *Let $\tau \in S_j$ where $j \geq 2$. Then*

$$NCM_{\tau^r}(t,x,y) = NCM_{\tau^c}(t,x,y) = e^{x(yt-t+\ln(NCM_\tau(ty,1,y^{-1})))}. \tag{22}$$

Next we shall show that we can find an explicit expression $NCM_\tau(t,x,y)$ where $\tau = 1\,2\ldots j$ for any $j \geq 3$ using some results of Mendes and Remmel [19]. Suppose that we want to compute the generating function

$$\begin{aligned} NCM_\tau(t,x,y) &= \sum_{n\geq 0} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{NCM}_n(\tau)} x^{\mathrm{cyc}(\sigma)} y^{\mathrm{cdes}(\sigma)} \\ &= e^{x\sum_{n\geq 1} \frac{t^n}{n!} \sum_{C \in \mathcal{L}_n^{ncm}(\tau)} y^{\mathrm{cdes}(\sigma)}} \end{aligned} \tag{23}$$

in the case where τ starts with 1. Then by Corollary 1.4, we know that

$$NCM_\tau(t,x,y) = NM_\tau(t,x,y) = \sum_{n\geq 0} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{NM}_n(\tau)} x^{\mathrm{LRMin}(\sigma)} y^{1+\mathrm{des}(\sigma)}. \tag{24}$$

Now suppose that we can compute

$$NM_\tau(t,1,y) = \sum_{n\geq 0} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{NM}_n(\tau)} y^{1+\mathrm{des}(\sigma)}. \tag{25}$$

Then we know that

$$NM_\tau(t,1,y) = e^{\sum_{n\geq 1} \frac{t^n}{n!} \sum_{C \in \mathcal{L}_n^{ncm}(\tau)} y^{\mathrm{cdes}(\sigma)}}$$

so that

$$\sum_{n\geq 1} \frac{t^n}{n!} \sum_{C \in \mathcal{L}_n^{ncm}(\tau)} y^{\mathrm{cdes}(\sigma)} = \ln\left(NM_\tau(t,1,y)\right).$$

But then it follows that

$$NCM_\tau(t,x,y) = NM_\tau(t,x,y) = e^{x\ln(NM_\tau(t,1,y))}. \tag{26}$$

Thus we need only compute (25). However, Mendes and Remmel [19] proved the following theorem.

THEOREM 2.2 ([19]) *If $\tau = j\ldots 2\,1$ where $j \geq 2$, then*

$$\sum_{n\geq 0} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{NM}_n(\tau)} y^{\mathrm{des}(\sigma)} = \left(\sum_{n\geq 0} \frac{t^n}{n!} \sum_{i\geq 0} (-1)^i \mathcal{R}_{n-1,i,j-1} y^i \right)^{-1} \tag{27}$$

where $\mathcal{R}_{n,i,j}$ is the number of rearrangements of i zeroes and $n-i$ ones such that j zeroes never appear consecutively.

Replacing y by $1/y$ and then replacing t by yt in (27) yields

$$\sum_{n\geq 0}\frac{t^n}{n!}\sum_{\sigma\in\mathcal{NM}_n(\tau)} y^{n-\text{des}(\sigma)} = \left(\sum_{n\geq 0}\frac{t^n}{n!}\sum_{i\geq 0}(-1)^i\mathcal{R}_{n-1,i,j-1}y^{n-i}\right)^{-1}. \tag{28}$$

It is easy to see that if $\sigma\in S_n$ has no $j\ldots 2\,1$-matches, then the reverse of σ, σ^r has no $1\,2\ldots j$-matches and that $n-\text{des}(\sigma)$ equals $1+\text{des}(\sigma^r)$. Thus it follows that if $\alpha = 1\,2\ldots j$, then

$$\sum_{n\geq 0}\frac{t^n}{n!}\sum_{\sigma\in\mathcal{NM}_n(\alpha)} y^{1+\text{des}(\sigma)} = \left(\sum_{n\geq 0}\frac{t^n}{n!}\sum_{i\geq 0}(-1)^i\mathcal{R}_{n-1,i,j-1}y^{n-i}\right)^{-1}. \tag{29}$$

Thus we have the following theorem.

THEOREM 2.3 *For $j\geq 2$ and $\tau = 1\,2\ldots j$,*

$$\begin{aligned} NCM_\tau(t,x,y) &= \sum_{n\geq 0}\frac{t^n}{n!}\sum_{\sigma\in\mathcal{NCM}_n(\tau)} x^{\text{cyc}(\sigma)}y^{\text{cdes}(\sigma)} \\ &= e^{x\ln\frac{1}{\sum_{n\geq 0}\frac{t^n}{n!}\sum_{i\geq 0}(-1)^i\mathcal{R}_{n-1,i,j-1}y^{n-i}}} \\ &= \left(\frac{1}{\sum_{n\geq 0}\frac{t^n}{n!}\sum_{i\geq 0}(-1)^i\mathcal{R}_{n-1,i,j-1}y^{n-i}}\right)^x. \end{aligned} \tag{30}$$

where $\mathcal{R}_{n,i,j}$ is the number of rearrangements of i zeroes and $n-i$ ones such that j zeroes never appear consecutively.

In the case where $\tau = 1\,2\,3$, we can obtain an explicit formula $NCM_{123}(t,x,y)$ by another method. We start with a general observation. Suppose $\tau=\tau_1\ldots\tau_j\in S_j$ where $\tau_1 = 1$. We can write any n-cycle C in the form $C=(\alpha_1,\ldots,\alpha_n)$ where $\alpha_1 = 1$. It is easy to see that the only cycle τ-match in C that can involve $\alpha_1 = 1$ is $\alpha_1\,\alpha_2\ldots\alpha_j$. This means that the only possible cycle τ-matches in C must be of the form $\alpha_i\,\alpha_{i+1}\ldots\alpha_{i+j-1}$ where $i\leq n-j+1$. Thus the problem of finding n-cycles with no cycle τ-matches is equivalent to the problem of finding permutations $\sigma=\sigma_1\ldots\sigma_n$ where $\sigma_1 = 1$ and σ has no τ-matches. Let S_n^1 denote the set of all permutations $\sigma=\sigma_1\ldots\sigma_n\in S_n$ such that $\sigma_1 = 1$ and let $\mathcal{NM}_n^1(\tau) = S_n^1\cap\mathcal{NM}_n(\tau)$ be the set of permutations of S_n^1 with no τ-matches. Then

$$A_{n,\tau}(y) = \sum_{\sigma\in\mathcal{NM}_n^1(\tau)} y^{1+\text{des}(\sigma)} = \sum_{C\in\mathcal{L}_n^{ncm}(\tau)} y^{\text{cdes}(C)}. \tag{31}$$

It turns out that in many cases we can find recurrences for $A_{n,\tau}(y)$ by classifying the permutations $\sigma=\sigma_1\ldots\sigma_n\in S_n$ such that $\sigma_1 = 1$ according to the position of 2 in σ. Let $\mathcal{E}_{n,k,\tau}$ denote the set of permutations $\sigma=\sigma_1\ldots\sigma_n\in\mathcal{NM}_n^1(\tau)$ such that $\sigma_k = 2$.

Now fix $\tau = 1\,2\,3$ and let $A_m(y) = A_{m,\tau}(y)$ and $\mathcal{E}_{n,k} = \mathcal{E}_{n,k,\tau}$. Our goal is to compute $A(t,y) = \sum_{m\geq 1}\frac{A_m(y)t^m}{m!}$. Now $A_1(y) = A_2(y) = y$ since the permutation 1 has no τ-matches, $1+\text{des}(1) = 1$, the permutation 1 2 has no τ-matches, and $1+\text{des}(12) = 1$. There are two permutations in S_3 that start with 1, namely, 1 2 3 and 1 3 2 and only 1 3 2 has no τ-matches so that $A_3(y) = y^2$ since $1 + \mathrm{des}(132) = 2$. Now suppose that $n \geq 3$. Every permutation in $\mathcal{E}_{n,2}$ is of the form $1\ 2\ \sigma_3 \ldots \sigma_n$. Clearly, $1\ 2\ \sigma_3$ is a 1 2 3-match so that the elements in $\mathcal{E}_{n,2}$ do not contribute to $A_n(y)$. For $3 \leq k \leq n$, the elements of the $\mathcal{E}_{n,k}$ are of the form

$$1\ \sigma_2 \ldots \sigma_{k-1}\ 2\ \sigma_{k+1} \ldots \sigma_n.$$

In such a case, the only way that 2 can be part of a 1 2 3-match is if the τ-match is $2\ \sigma_{k+1}\ \sigma_{k+2}$. It follows that an element of $\mathcal{E}_{n,k}$ contributes to $A_n(y)$ only if there is no τ-match in $\sigma_1 \ldots \sigma_{k-1}$ and there is no τ-match in $2\ \sigma_{k+1} \ldots \sigma_n$. Note that since $(\sigma_{k-1}, 2)$ is a descent pair,

$$1 + \mathrm{des}(1\ \sigma_2 \ldots \sigma_{k-1}\ 2\ \sigma_{k+1} \ldots \sigma_n) = 1 + \mathrm{des}(1\ \sigma_2 \ldots \sigma_{k-1}) + 1 + \mathrm{des}(2\ \sigma_{k+1} \ldots \sigma_n).$$

Hence the contribution of $\mathcal{E}_{n,k}$ to $A_n(y)$ is just $\binom{n-2}{k-2} A_{k-1}(y) A_{n-k+1}(y)$ since there are $\binom{n-2}{k-2}$ ways to choose the elements which make up $\sigma_2, \ldots, \sigma_{k-1}$.

It then follows that for $n \geq 3$,

$$A_n(y) = \sum_{k=3}^{n} \binom{n-2}{k-2} A_{k-1}(y) A_{n-k+1}(y). \tag{32}$$

Multiplying both sides of (32) by $\frac{t^{n-2}}{(n-2)!}$ and summing for $n \geq 3$, we see that

$$\begin{aligned} \frac{\partial^2 A(t,y)}{\partial t^2} - y &= \sum_{n \geq 3} t^{n-2} \sum_{k=3}^{n} \frac{A_{k-1}(y)}{(k-2)!} \frac{A_{n-k+1}(y)}{(n-k)!} \\ &= \frac{\partial A(t,y)}{\partial t} \left(\frac{\partial A(t,y)}{\partial t} - y \right). \end{aligned}$$

Thus thinking of $A(t,y)$ as a function of t, we see that $A(t,y)$ satisfies the differential equation

$$A''(t,y) - (A'(t,y))^2 + yA'(t,y) - y = 0 \tag{33}$$

where $A(0,y) = 0$ and $A'(0,y) = y$. If we let $A(y,t) = -\ln(U(t,y))$, then thinking of $U(t,y)$ as a function of t, it follows that

$$\begin{aligned} A'(t,y) &= -\frac{U'(t,y)}{U(t,y)} \text{ and} \\ A''(t,y) &= -\frac{U''(t,y)U(t,y) - (U'(t,y))^2}{(U(t,y))^2} = -\frac{U''(t,y)}{U(t,y)} + \left(\frac{U'(t,y)}{U(t,y)}\right)^2. \end{aligned}$$

Substituting these expressions into (33), one can easily show that $U(t,y)$ satisfies the differential equation

$$U''(t,y) + yU'(t,y) + yU(t,y) = 0 \tag{34}$$

where $U(0,y) = 1$ and $U'(0,y) = -y$. One can use Mathematica to solve this differential equation to conclude that

$$U(t,y) = e^{-\frac{yt}{2}} \left(\cosh\left(\frac{t\sqrt{y^2-4y}}{2} \right) - \frac{\sqrt{y}}{\sqrt{y-4}} \sinh\left(\frac{t\sqrt{y^2-4y}}{2} \right) \right)$$

so that

$$A(t,y) = -\ln\left(e^{-\frac{yt}{2}}\cosh\left(\frac{t\sqrt{y^2-4y}}{2}\right) - \frac{\sqrt{y}}{\sqrt{y-4}}\sinh\left(\frac{t\sqrt{y^2-4y}}{2}\right)\right). \quad (35)$$

We can then apply Theorem 1.2 to obtain the following theorem.

THEOREM 2.4

$$NCM_{123}(t,x,y) = e^{\frac{xyt}{2}}\left(\frac{1}{\left(\cosh\left(\frac{t\sqrt{y^2-4y}}{2}\right) - \frac{\sqrt{y}}{\sqrt{y-4}}\sinh\left(\frac{t\sqrt{y^2-4y}}{2}\right)\right)}\right)^x. \quad (36)$$

One can use Mathematica to compute that

$$\begin{aligned} A(y,t) &= yt + y\frac{t^2}{2!} + y^2\frac{t^3}{3!} + (2y^2+y^3)\frac{t^4}{4!} + \\ & (8y^3+y^4)\frac{t^5}{5!} + (16y^3+22y^4+y^5)\frac{t^6}{6!} \\ & (136y^4+52y^5+y^6)\frac{t^7}{7!} + (272y^4+720y^5+114y^6+y^7)\frac{t^8}{8!} + \\ & (3968y^5+3072y^6+240y^7+y^8)\frac{t^9}{9!} + \cdots \end{aligned}$$

and

$$\begin{aligned} & NCM(t,x,y) = 1 + xyt + xy(1+xy)\frac{t^2}{2!} + xy^2(1+3x+x^2y)\frac{t^3}{3!} + \\ & xy^2(2+3x+y+4xy+6x^2y+x^3y^2)\frac{t^4}{4!} + \\ & xy^3(8+20x+15x^2+y+5xy+10x^2y+10x^3y+x^4y^2)\frac{t^5}{5!} + \\ & xy^3\left(16+30x+15x^2+22y+73xy+90x^2y+45x^3y+y^2+\right. \\ & \left. 6xy^2+15x^2y^2+20x^3y^2+15x^4y^2+x^5y^3\right)\frac{t^6}{6!} + \\ & xy^4\left(136+350x+315x^2+105x^3+52y+210xy+343x^2y+280x^3y+\right. \\ & \left. 105x^4y+y^2+7xy^2+21x^2y^2+35x^3y^2+35x^4y^2+21x^5y^2+x^6y^3\right)\frac{t^7}{7!} + \cdots \end{aligned}$$

We note that the sequence $\{A_n(t)\}_{n\geq 1}$ starts out

$$1, 1, 1, 3, 9, 39, 189, 1107, 7281, \ldots$$

which is sequence A080635 in the OIES and counts the number of permutations in S_n without double falls and without an initial fall. Bergeron, Flajolet, and Salvy [4] showed that the exponential generating function of this sequence which starts $1 + t + \frac{t^2}{2} + \cdots$ is

$$\frac{1}{2}\left(1+\sqrt{3}\tan\left(\frac{1}{2}\left(\frac{\pi}{3}+\sqrt{3}t\right)\right)\right).$$

Thus our generating function $A(y,t)$ which starts at t can be viewed as a refinement of the result of Begeron, Flajolet, and Salvy.

We can use the same method to compute $NCM_{132}(t,x,y)$. In this case, we will directly compute

$$B(t,y) = \sum_{m\geq 1} \frac{t^m}{m!} \sum_{C \in \mathcal{L}_m^{ncm}(132)} y^{\text{cdes}(C)}. \tag{37}$$

Let $B_m(y) = B_{m,132}(y)$ and $\mathcal{E}_{n,k} = \mathcal{E}_{n,k,132}$. Our goal is to compute $B(t,y) = \sum_{m\geq 1} \frac{B_m(y)t^m}{m!}$. Now $B_1(y) = B_2(y) = y$ since the permutation 1 has no τ-matches, $1+\text{des}(1) = 1$, the permutation 1 2 has no τ-matches, and $1+\text{des}(12) = 1$. There are two permutations in S_3 that start with 1, namely, 1 2 3 and 1 3 2, and only 1 2 3 has no τ-matches so that $B_3(y) = y$ since $1+\text{des}(123) = 1$. Now suppose that $n \geq 4$. Every permutation in $\mathcal{E}_{n,2}$ is of the form $1\ 2\ \sigma_3 \ldots \sigma_n$. Clearly, the only τ-matches must be of the form $\sigma_i\ \sigma_{i+1}\ \sigma_{i+2}$ where $i \geq 2$ so that $\mathcal{E}_{n,2}$ contributes $B_{n-1}(y)$ to $B_n(y)$. Every permutation in $\mathcal{E}_{n,3}$ is of the form $1\ \sigma_2\ 2 \ldots \sigma_n$ where $\sigma_2 \geq 3$. Thus all such permutations have a τ-match so that $\mathcal{E}_{n,3}$ contributes nothing to $B_n(y)$. For $4 \leq k \leq n$, the elements of the $\mathcal{E}_{n,k}$ are of the form

$$1\ \sigma_2 \ldots \sigma_{k-1}\ 2\ \sigma_{k+1} \ldots \sigma_n.$$

In such a case, the only way that 2 can be part of a τ-match is if the τ-match is $2\ \sigma_{k+1}\ \sigma_{k+2}$. It follows that an element of $\mathcal{E}_{n,k}$ contributes to $B_n(y)$ only if there is no τ-match in $\sigma_1 \ldots \sigma_{k-1}$ and there is no τ-match in $2\ \sigma_{k+1} \ldots \sigma_n$. Note that since $(\sigma_{k-1}, 2)$ is a descent pair,

$$1+\text{des}(1\ \sigma_2 \ldots \sigma_{k-1}\ 2\ \sigma_{k+1} \ldots \sigma_n) = 1+\text{des}(1\ \sigma_2 \ldots \sigma_{k-1}) + 1 + \text{des}(2\ \sigma_{k+1} \ldots \sigma_n).$$

Hence the contribution of $\mathcal{E}_{n,k}$ to $B_n(y)$ is just $\binom{n-2}{k-2} B_{k-1}(y) B_{n-k+1}(y)$ since there are $\binom{n-2}{k-2}$ ways to choose the elements which make up $\sigma_2, \ldots, \sigma_{k-1}$. Thus for $n \geq 4$,

$$B_n(y) = B_{n-1}(y) + \sum_{k=4}^{n} \binom{n-2}{k-2} B_{k-1}(y) B_{n-k+1}(y). \tag{38}$$

Dividing both sides of (38) by $(n-2)!$, we obtain that for all $n \geq 4$,

$$\frac{B_n(y)}{(n-2)!} = \frac{B_{n-1}(y)}{(n-2)!} + \sum_{k=2}^{n-2} \frac{B_{k+1}(y)}{k!} \frac{B_{n-k-1}(y)}{(n-2-k)!}. \tag{39}$$

If we multiply both sides of (39) by t^{n-2} and sum, we obtain the differential equation

$$\frac{\partial^2 B(t,y)}{\partial t^2} - y - yt = \frac{\partial B(t,y)}{\partial t} - y - yt + \left(\frac{\partial B(t,y)}{\partial t} - y - yt \right) \frac{\partial B(t,y)}{\partial t}$$

Let $B'(t,y) = \frac{\partial B(t,y)}{\partial t}$, then $B(t,y)$ satisfies the differential equation

$$B''(t,y) = B'(t,y)(1-y-yt+B'(t,y)) \tag{40}$$

with initial conditions $B_0(y) = 0$ and $B_1(y) = y$. One can check that the solution to (40) is

$$B(t,y) = \ln \left(\frac{1}{1 - y \int_0^t e^{(1-y)s - ys^2/2} ds} \right). \tag{41}$$

Hence

$$\begin{aligned} L_{132}(t,y) &= \sum_{m\geq 1} \frac{t^m}{m!} \sum_{C\in \mathcal{L}^{ncm}_m(132)} y^{\mathrm{cdes}(C)}. \\ &= \ln\left(\frac{1}{1-y\int_0^t e^{(1-y)s-ys^2/2}ds}\right). \end{aligned} \tag{42}$$

We can then apply Theorem 1.2 to obtain the following theorem.

THEOREM 2.5

$$\begin{aligned} NCM_{132}(t,x,y) &= e^{x\ln\left(\frac{1}{1-y\int_0^t e^{(1-y)s-ys^2/2}ds}\right)} \\ &= \frac{1}{\left(1-y\int_0^t e^{(1-y)s-ys^2/2}ds\right)^x}. \end{aligned} \tag{43}$$

We note that specialization

$$NCM_{132}(t,1,1) = \frac{1}{1-\int_0^t e^{-s^2/2}ds}$$

has been proved by Elizalde and Noy [9].

One can use our generating functions for $NCM_{132}(t,x,y)$ to compute the initial values of $L_n^{ncm}(132)$ and $NCM_n(132)$.

n	$L_n^{ncm}(132)$	$NCM_n(132)$
1	1	1
2	1	2
3	1	5
4	2	16
5	7	63
6	28	296
7	131	1623
8	720	10176
9	4513	71793
10	31824	562848

If one looks in the OEIS [25], then both the sequences for $L_n^{ncm}(132)$ and $NCM_n(132)$ occur. The sequence for $L_n^{ncm}(132)$ is sequence A052319 which counts the number of increasing rooted trimmed trees with n nodes. Here an increasing tree is a tree labeled with $1,\ldots,n$ where the numbers increase as you move away from the root. A tree with a forbidden limb of length k is a tree where the path from any leaf inward hits a branching node or another leaf within k steps. A trimmed tree is a tree with a forbidden limb of length 2. The sequence for $NCM_n(132)$ is the number of permutations that have no 1 3 2-matches as expected.

We end this section with some results on $CAV_\Upsilon(t,x,y)$ and $NCM_\Upsilon(t,x,y)$ where $\Upsilon \subseteq S_3$. For certain Υ's, this problem is uninteresting. For example, if Υ contains both 1 2 3 and 1 3 2, then any k-cycle $C = (\sigma_1, \sigma_2, \ldots, \sigma_k)$ where $\sigma_1 = 1$ and $k \geq 3$ will have a cycle Υ-match since $\sigma_1\ \sigma_2\ \sigma_3$ must be either a cycle 1 2 3-match or a cycle 1 3 2-match. Thus in this case $\mathcal{L}_1^{ca}(\Upsilon) = \mathcal{L}_1^{ncm}(\Upsilon) = \{(1)\}$, $\mathcal{L}_2^{ca}(\Upsilon) = \mathcal{L}_2^{ncm}(\Upsilon) = \{(1,2)\}$, and $\mathcal{L}_k^{ca}(\Upsilon) = \mathcal{L}_k^{ncm}(\Upsilon) = \emptyset$ for $k \geq 3$. It then follows from Theorem 1.2 that

$$CAV_\Upsilon(t,x,y) = NCM_\Upsilon(t,x,y) = e^{x\ \ yt+\frac{yt^2}{2}}\ .$$

Similarly, suppose that $\Upsilon = \{123, 213\}$. Then we claim that $\mathcal{L}_k^{ncm}(\Upsilon) = \emptyset$ for $k \geq 3$. That is, for $k \geq 3$, a k-cycle $C = (1, c_2, \ldots, c_k)$ has no cycle $\{123, 213\}$-matches. Then consider the possible positions for k in $c_2, \ldots, c_k$. Clearly, we can not have $k = c_2$ since then $c_k\ 1\ k$ would be a cycle match of 2 1 3. We can not have $k = c_3$ since then $1\ c_2\ k$ would be a cycle match of 1 2 3. Now suppose that $k = c_i$ where $i > 3$. Then we either have (i) $c_{i-2} < c_{i-1} < k$ or (ii) $c_{i-2} > c_{i-1} < k$. But in case (i), C would contain a cycle 1 2 3-match and in case (ii), C would contain a cycle 2 1 3-match. Thus such a C can not exist and we can conclude that

$$NCM_{\{123,213\}}(t,x,y) = e^{x\ \ yt+\frac{yt^2}{2}}\ .$$

A more interesting case is when $\Upsilon = \{123, 321\}$. First observe that since any cycle contains a cycle occurrence of 1 3 2 if and only if it contains a cycle occurrence of 3 2 1, then it is the case that any k-cycle C where $k \geq 3$ must have a cycle occurrence of either 1 2 3 or 3 2 1. Thus

$$CAV_\Upsilon(t,x,y) = e^{x\ \ yt+\frac{yt^2}{2}}\ .$$

Next consider the case of computing $NCM_\Upsilon(t,x,y)$. Let $C = (\sigma_1, \ldots, \sigma_n)$ be an n-cycle such that $\sigma_1 = 1$. If $n \geq 3$, then we must have $\sigma_2 > \sigma_3$ since otherwise there will be a cycle 1 2 3-match. But then we must have $\sigma_3 < \sigma_4$ since otherwise there would be cycle 3 2 1-match. Continuing on in this way, we see that $\sigma_2 \ldots \sigma_n$ must be an alternating permutation. That is, we must have

$$\sigma_2 > \sigma_3 < \sigma_4 > \sigma_5 < \sigma_6 > \sigma_7 \cdots .$$

However, this means that if $n = 2k+1 \geq 3$, then there are no n cycles which have no cycle Υ-matches since we are forced to have $\sigma_{2k} > \sigma_{2k+1} > \sigma_1$ which is a cycle 3 2 1-match. If $n = 2k$ and $\sigma_2 \ldots \sigma_n$ is alternating, then C will have no cycle Υ-matches. For such σ it is easy to see that $1+\text{des}(\sigma) = k$. Thus in this case, $L_{2k+1}^{ncm}(\Upsilon) = 0$ for $k \geq 1$ and $L_{2k}^{ncm}(\Upsilon)$ is just the number of odd alternating permutations of length $2k-1$ for $k \geq 1$. Deutsch and Elizalde [8] called n-cycles $(1, \sigma_2, \ldots, \sigma_n)$ such that

$$\sigma_2 > \sigma_3 < \sigma_4 > \sigma_5 < \sigma_6 > \sigma_7 \cdots ,$$

cycle up-down permutations. One can follow the methods of [8] to find an explicit formula for $NCM_\Upsilon(t,x,y)$. That is, we let Alt_n denote the number of down-up permutations of length n, then André [1, 2] proved that

$$\sum_{n\geq 0} Alt_{2n+1} \frac{t^{2n+1}}{(2n+1)!} = \frac{\sin(t)}{\cos(t)}. \tag{44}$$

Thus

$$\begin{aligned}\sum_{n\geq 1} L_{2n}^{ncm}(\Upsilon)\frac{t^{2n}}{(2n)!} &= \sum_{n\geq 1} Alt_{2n-1}\frac{t^{2n}}{(2n)!}\\ &= \int_0^t \frac{\sin(z)}{\cos(z)}dz = -\ln|\cos(t)|.\end{aligned}$$

Hence,

$$\begin{aligned}\sum_{n\geq 1}\frac{t^{2n}}{(2n)!}\sum_{C\in\mathcal{L}_{2n}^{ncm}(\Upsilon)} y^{\mathrm{cdes}(C)} &= \sum_{n\geq 1} y^n L_{2n}^{ncm}(\Upsilon)\frac{t^{2n}}{(2n)!}\\ &= -\ln|\cos(t\sqrt{y})|.\end{aligned}$$

and

$$\sum_{n\geq 1}\frac{t^n}{n!}\sum_{C\in\mathcal{L}_{n}^{ncm}(\Upsilon)} y^{\mathrm{cdes}(C)} = ty - \ln|\cos(t\sqrt{y})|. \tag{45}$$

Thus it follows from Theorem 1.2

$$NCM_{\Upsilon}(t,x,y) = e^{x(ty-\ln|\cos(t\sqrt{y})|)} = \frac{e^{xyt}}{\cos(t\sqrt{y})^x} = e^{xyt}\sec(t\sqrt{y})^x.$$

We end section with another non-trivial example which is the case where $\Gamma = \{123, 231\}$. Let

$$G_n(y) = \sum_{C\in\mathcal{L}_n^{ncm}(\Gamma)} y^{\mathrm{cdes}(C)} \tag{46}$$

and

$$G(t,y) = \sum_{n\geq 1} G_n(y)\frac{t^n}{n!}. \tag{47}$$

Note that $\mathcal{L}_1^{ncm}(\Gamma) = \{(1)\}$ and $\mathcal{L}_2^{ncm}(\Gamma) = \{(1,2)\}$ so that $G_1(y) = G_2(y) = y$.

Now suppose that $n \geq 3$ and $C = (1,\sigma_2,\ldots,\sigma_n)$ is an n-cycle in S_n which has no cycle Γ-matches. Then it cannot be the case that $\sigma_{n-1} < \sigma_n$ since otherwise $\sigma_{n-1}\ \sigma_n\ 1$ would be a cycle 2 3 1-match in C. Thus it must be the case that $\sigma_{n-1} > \sigma_n$. It cannot be that $\sigma_i = 2$ since then $1\ \sigma_2\ \sigma_3$ would be a cycle 1 2 3-match in C and it cannot be that $\sigma_{n-1} = 2$ since then $\sigma_{n-1}\ \sigma_n\ 1$ would be a cycle 2 3 1-match in C. If $\sigma_n = 2$, it easy to see that $C = (1,\sigma_2,\ldots,\sigma_{n-1},2)$ has no cycle Γ-matches if and only if $C' = (1,\sigma_2-1,\ldots,\sigma_{n-1}-1)$ has no cycle Γ-matches. Note $\mathrm{cdes}(C) = \mathrm{cdes}(C')+1$ so that the n-cycles of the form $C = (1,\sigma_2,\ldots,\sigma_{n-1},2)$ contribute $yG_{n-1}(y)$ to $G_n(y)$. Thus consider the cases where $\sigma_k = 2$ where $3 \leq k \leq n-2$. We claim that it must be the case that neither $(1,\sigma_2,\ldots,\sigma_{k-1})$ nor $(2,\sigma_{k+1},\ldots,\sigma_n)$ have any cycle Γ-matches. That is, it is easy to see that the only possible cycle Γ-match in $(1,\sigma_2,\ldots,\sigma_{k-1})$ that does not occur in C is if $k-1 \geq 3$ and $\sigma_{k-2}\ \sigma_{k-1}\ 1$ is a cycle 2 3 1-match. But in that case, $\sigma_{k-2},\sigma_{k-1} > 2$ so that $\sigma_{k-2}\ \sigma_{k-1}\ 2$ would have been a cycle 2 3 1-match in C. Similarly, the only possible cycle Γ-match in $(2,\sigma_{k+1},\ldots,\sigma_n)$ that does not occur in C is if $\sigma_{n-1}\ \sigma_n\ 2$ is cycle 2 3 1-match. But in that case, $\sigma_{n-1}\sigma_n 1$ would have been a cycle 231-match in C. Vice versa, it is easy to see that if $\sigma_k = 2$ where $3 \leq k \leq n-2$ and neither $(1,\sigma_2,\ldots,\sigma_{k-1})$ nor

$(2, \sigma_{k+1}, \ldots, \sigma_n)$ have any cycle Γ-matches, then C does not have any cycle Γ-matches. That is, the only possible cycle Γ-match in C that does not occur in either $(1, \sigma_2, \ldots, \sigma_{k-1})$ nor $(2, \sigma_{k+1}, \ldots, \sigma_n)$ is if $\sigma_{k-2}\ \sigma_{k-1}\ 2$ is a cycle 2 3 1-match. This is not possible if $k = 3$ since in that case $\sigma_{k-2} = 1$. Similarly if $3 < k \leq n-2$ and $\sigma_{k-2}\ \sigma_{k-1}\ 2$ is a cycle 2 3 1-match in C, then $\sigma_{k-2}\ \sigma_{k-1}\ 1$ would be a cycle 2 3 1-match in $(1, \sigma_2, \ldots, \sigma_{k-1})$. Note that it is also the case that

$$\text{cdes}((1, \sigma_2, \ldots, \sigma_{k-1})) + \text{cdes}((2, \sigma_{k+1}, \ldots, \sigma_n)) = \text{cdes}(C).$$

Thus for $3 \leq k \leq n-2$, the cycles of the form $C = (1, \sigma_2, \ldots, \sigma_n)$ where $\sigma_k = 2$ contribute $\binom{n-2}{k-2} G_{k-1}(y) G_{n-k+1}(y)$ to $G_n(y)$ since we have $\binom{n-2}{k-2}$ ways to choose the elements $\sigma_2, \ldots, \sigma_{k-1}$.

It then follows that for $n \geq 3$,

$$G_n(y) = yG_{n-1}(y) + \sum_{k=3}^{n} \binom{n-2}{k-2} G_{k-1}(y) G_{n-k+1}(y). \tag{48}$$

Multiplying both sides of (48) by $\frac{t^{n-2}}{(n-2)!}$ and summing for $n \geq 3$, we see that

$$\begin{aligned}
\frac{\partial^2 G(t,y)}{\partial t^2} - y &= \sum_{n \geq 3} G_n(t,y) \frac{t^{n-2}}{(n-2)!} \\
&= y \sum_{n \geq 3} G_{n-1}(y) \frac{t^{n-2}}{(n-2)!} + \sum_{n \geq 3} t^{n-2} \sum_{i=3}^{n} \frac{G_{i-1}(y)}{(i-2)!} \frac{G_{n-i+1}(y)}{(n-2)!} \\
& y \left(\frac{\partial G(t,y)}{\partial t} - y \right) + \left(\frac{\partial G(t,y)}{\partial t} - y \right) \left(\frac{\partial G(t,y)}{\partial t} - y - yt \right).
\end{aligned}$$

Thus thinking of $G(t,y)$ as a function of t, we see that $G(t,y)$ satisfies the differential equation

$$G''(t,y) - (G'(t,y))^2 + (y + yt) y G'(t,y) - (y + y^2 t) = 0 \tag{49}$$

where $G(0,y) = 0$ and $G'(0,y) = y$. If we let $G(y,t) = -\ln(U(t,y))$, then thinking of $U(t,y)$ as a function of t, one can easily show that $U(t,y)$ satisfies the differential equation

$$U''(t,y) + (y + yt) U'(t,y) + (y + y^2 t) U(t,y) = 0 \tag{50}$$

where $U(0,y) = 1$ and $U'(0,y) = -y$. We used Mathematica to solve this differential equation which gave the formula

$$U(t,y) = e^{-\frac{yt^2}{2}} \left(1 - \frac{e^{-\frac{y}{2}} \sqrt{2\pi y}}{2} \text{erfi} \left(\sqrt{\frac{y}{2}} \right) - \frac{e^{-\frac{y}{2}} \sqrt{2\pi y}}{2} \text{erfi} \left((t-1) \sqrt{\frac{y}{2}} \right) \right)$$

so that

$$G(t,y) = -ln \left(e^{-\frac{yt^2}{2}} \left(1 - \frac{e^{-\frac{y}{2}} \sqrt{2\pi y}}{2} \text{erfi} \left(\sqrt{\frac{y}{2}} \right) - \frac{e^{-\frac{y}{2}} \sqrt{2\pi y}}{2} \text{erfi} \left((t-1) \sqrt{\frac{y}{2}} \right) \right) \right) \tag{51}$$

where $\text{erfi}(z)$ is imaginary error function defined by the series

$$\text{erfi}(z) = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{z^{2k+1}}{k!(2k+1)}.$$

We can then apply Theorem 1.2 to obtain the following theorem.

THEOREM 2.6

$$NCM_{\{123,231\}}(t,x,y) = e^{xG(t,y)} \tag{52}$$

$$= e^{\frac{xyt^2}{2}} \left(\frac{1}{\left(1 - \frac{e^{-\frac{y}{2}}\sqrt{2\pi y}}{2}\text{erfi}\left(\sqrt{\frac{y}{2}}\right) - \frac{e^{-\frac{y}{2}}\sqrt{2\pi y}}{2}\text{erfi}\left((t-1)\sqrt{\frac{y}{2}}\right)\right)} \right)^x.$$

One can use Mathematica to compute that

$$\begin{aligned} G(y,t) &= yt + y\frac{t^2}{2!} + y^2\frac{t^3}{3!} + (y^3)\frac{t^4}{4!} + \\ &(3y^3+y^4)\frac{t^5}{5!} + (13y^4+y^5)\frac{t^6}{6!} + (15y^4+38y^5+y^6)\frac{t^7}{7!} + \\ &(183y^5+94y^6+y^7)\frac{t^8}{8!} + (105y^5+1205y^6+213y^7+y^8)\frac{t^9}{9!} + \cdots \end{aligned}$$

and

$$\begin{aligned} &NCM_{\{123,231\}}(t,x,y) = 1 + xyt + xy(1+xy)\frac{t^2}{2!} + xy^2(1+3x+x^2y)\frac{t^3}{3!} + \\ &xy^2(3x+y+4xy+6x^2y+x^3y^2)\frac{t^4}{4!} + \\ &xy^3(3+10x+15x^2++y+5xy+10x^2y+10x^3y+x^4y^2)\frac{t^5}{5!} + \\ &xy^3(15x^2+13y+43xy+60x^2y+45x^3y+y^2+6xy^2+15x^2y^2+20x^3y^2+15x^4y^2+x^5y^3)\frac{t^6}{6!} + \\ &xy^4\left(15+63x+105x^2+105x^3+38y+147xy+238x^2y+210x^3y+\right. \\ &\left.105x^4y+y^2+7xy^2+21x^2y^2+35x^3y^2+35x^4y^2+21x^5y^2+x^6y^3\right)\frac{t^7}{7!} + \cdots \end{aligned}$$

Neither the sequences $\{G_n(1)\}_{n\geq 1}$ nor the sequences $\{NCM_n(\{123,231\})\}_{n\geq 1}$ appear in the OEIS.

3 General results

In this section, we shall describe how we can compute $NCM_\tau(t,x,y)$ for certain general classes of permutations τ. We start by considering permutations $\tau = \tau_1 \dots \tau_j$ where $\tau_1 = 1$ and $\tau_j = 2$. In that case, we have the following theorem.

THEOREM 3.1 *Let $\tau = \tau_1 \dots \tau_j \in S_j$ where $j \geq 3$ and $\tau_1 = 1$ and $\tau_j = 2$. Then*

$$NCM_\tau(t,x,y) = \frac{1}{\left(1 - \int_0^t e^{(y-1)s - \frac{y^{\text{des}(\tau)}s^{j-1}}{(j-1)!}}ds\right)^x} \tag{53}$$

Proof. Note that in the special case where $j = 3$, the only permutation satisfying the hypothesis of the theorem is $\tau = 1\ 3\ 2$. In that special case, the result follows from Theorem 2.5. Thus assume that we fix a $\tau = \tau_1 \dots \tau_j \in S_j$ where $\tau_1 = 1$ and $\tau_j = 2$ and $j \geq 4$.

Our first goal is to compute

$$A(t,y) = \sum_{n\geq 1} A_n(y)\frac{t^n}{n!} \tag{54}$$

where $A_n(y) = \sum_{\sigma\in\mathcal{NM}_n^1(\tau)} y^{\mathrm{des}(\sigma)+1}$. Now it is easy to see that $A_n(y) = \sum_{\sigma\in S_n^1} y^{\mathrm{des}(\sigma)+1}$ for $1 \leq n \leq j-1$. Thus

$$\begin{aligned} A(t,y) &= yt + y\frac{t^2}{2} + (y+y^2)\frac{t^3}{3!} + \cdots \\ \frac{\partial A(t,y)}{\partial t} &= y + yt + (y+y^2)\frac{t^2}{2!} + \cdots \text{ and} \\ \frac{\partial^2 A(t,y)}{\partial t^2} &= y + (y+y^2)t + \cdots . \end{aligned}$$

For $n \geq j$, we shall prove a recursive formula for $A_n(y)$. We consider three cases for $\sigma = \sigma_1 \dots \sigma_n \in \mathcal{NM}_n^1(\tau)$ depending on the position of 2 in σ.

Case 1. $\sigma_2 = 2$.
In this case because $j \geq 4$, the only possible τ-matches in σ must occur in $\sigma_2 \dots \sigma_n$. Since $\mathrm{des}(\sigma)+1 = \mathrm{des}(\sigma_2 \dots \sigma_n) + 1$, it follows that the contribution of the permutations in this case to $A_n(y)$ is just $A_{n-1}(y)$.

Case 2. $\sigma_k = 2$ where $k \notin \{2, j\}$.
In this case, we have $\binom{n-2}{k-2}$ ways to choose the elements D_k that will constitute $\sigma_2 \dots \sigma_{k-1}$. Once we have chosen D_k, we have to consider the ways in which we can arrange the elements of D_k to form $\sigma_2 \dots \sigma_{k-1}$ and the ways that we can arrange $[n] - (D_k \cup \{1,2\})$ to form $\sigma_{k+1} \dots \sigma_n$ so that

$$\sigma = 1\ \sigma_2 \dots \sigma_{k-1}\ 2\ \sigma_{k+1} \dots \sigma_n \tag{55}$$

has no τ-matches. However it is easy to see that since $k \notin \{2, j\}$ the only τ-matches for σ of the form (55) can occur either entirely in $1\ \sigma_2 \dots \sigma_{k-1}$ or entirely in $2\ \sigma_{k+1} \dots \sigma_n$. Moreover it is the case that

$$\mathrm{des}(\sigma) + 1 = \mathrm{des}(1\ \sigma_2 \dots \sigma_{k-1}) + 1 + \mathrm{des}(2\ \sigma_{k+1} \dots \sigma_n) + 1$$

since $\sigma_{k-1} > 2$. Thus the contribution to $A_n(y)$ of the permutations in this case is

$$\binom{n-2}{k-2} A_{k-1}(y) A_{n-k+1}(y).$$

Case 3. $\sigma_j = 2$.
In this case, we have $\binom{n-2}{j-2}$ ways to choose the elements D_j that will constitute $\sigma_2 \dots \sigma_{j-1}$. Once we

have chosen D_j, we have to consider the ways in which we can arrange the elements of D_j to form $\sigma_2 \dots \sigma_{j-1}$ and we can arrange $[n] - (D_j \cup \{1,2\})$ to form $\sigma_{j+1} \dots \sigma_n$ so that

$$\sigma = 1\ \sigma_2 \dots \sigma_{j-1}\ 2\ \sigma_{j+1} \dots \sigma_n \tag{56}$$

has no τ-matches. Unlike Case 2, it is not enough just to ensure that $1\ \sigma_2 \dots \sigma_{j-1}$ and $2\ \sigma_{j+1} \dots \sigma_n$ have no τ-matches. That is, we must also ensure that $\mathrm{red}(\sigma_2 \dots \sigma_{j-1}) \neq \mathrm{red}(\tau_2 \dots \tau_{j-1})$ since otherwise $1\ \sigma_2 \dots \sigma_{j-1}\ 2$ would be a τ-match. Note that in such a situation $\mathrm{des}(1\ \sigma_2 \dots \sigma_{j-1}) + 1 = \mathrm{des}(\tau)$. Thus the contributions to $A_n(y)$ of the permutations in this case is

$$\binom{n-2}{j-2}(A_{j-1}(y) - y^{\mathrm{des}(\tau)})A_{n-j+1}(y).$$

It follows that for $n \geq j$,

$$A_n(y) = A_{n-1}(y) + \sum_{k=3}^{n} \binom{n-2}{k-2} A_{k-1}(y) A_{n-k+1}(y) - \binom{n-2}{j-2} y^{\mathrm{des}(\tau)} A_{n-j+1}(y) \tag{57}$$

or, equivalently,

$$\frac{A_n(y)}{(n-2)!} = \frac{A_{n-1}(y)}{(n-2)!} + \left(\sum_{k=3}^{n} \frac{A_{k-1}(y)}{(k-2)!} \frac{A_{n-k+1}(y)}{(n-k)!}\right) - \frac{y^{\mathrm{des}(\tau)}}{(j-2)!} \frac{A_{n-j+1}(y)}{(n-j)!}. \tag{58}$$

For any formal power series $f(t) = \sum_{n\geq 1} f_n t^n$, let $f(t)|_{t\leq j}$ denote $f_0 + f_1 t + \cdots + f_j t^j$. Multiplying both sides of (58) by t^{n-2} and summing, we obtain the differential equation where $A'(t,y) = \frac{\partial A(t,y)}{\partial t}$

$$\begin{aligned}
&A''(t,y) - A''(t,y)|_{t\leq j-3} \\
&= A'(t,y) - A'(t,y)|_{t\leq j-3} + \\
&\quad \big(A'(t,y) - y\big)\, A'(t,y) - \big(\big(A'(t,y) - y\big)\, A'(t,y)\big)\,|_{t\leq j-3} - \frac{y^{\mathrm{des}(\tau)}}{(j-2)!} A'(t,y).
\end{aligned}$$

Thus

$$\begin{aligned}
A''(t,y) \;=\;& (1 - y - y^{\mathrm{des}(\tau)})A'(t,y) + \big(A'(t,y)\big)^2 + \\
& \big(A''(t,y)|_{t\leq j-3}\big) - \big(A'(t,y)|_{t\leq j-3}\big) - \big(\big(A'(t,y) - y\big)\, A'(t,y)\big)\,|_{t\leq j-3}.
\end{aligned}$$

We claim that

$$0 = \big(A''(t,y)|_{t\leq j-3}\big) - \left(\frac{\partial A(t,y)}{\partial t}|_{t\leq j-3}\right) - \left(\left(\frac{\partial A(t,y)}{\partial t} - y\right) \frac{\partial A(t,y)}{\partial t}\right)|_{t\leq j-3}$$

or, equivalently, that

$$A''(t,y)|_{t\leq j-3} = \big(A'(t,y) + \big(A'(t,y) - y\big)\, A'(t,y)\big)\,|_{t\leq j-3}. \tag{59}$$

If we take the coefficient of t^s where $0 \leq s \leq t^{j-3}$ on both sides of (59), then we must show that

$$\begin{aligned} \frac{A_{s+2}(y)}{s!} &= \frac{A_{s+1}(y)}{s!} + \sum_{k=1}^{s} \frac{A_{k+1}(y)}{k!} \frac{A_{s-k+1}(y)}{(s-k)!} \\ &= \frac{A_{s+1}(y)}{s!} + \sum_{k=3}^{s+2} \frac{A_{k-1}(y)}{(k-2)!} \frac{A_{s+2-(k-1)}(y)}{(s+2-k)!}. \end{aligned}$$

Thus if we multiply both sides by $s!$, we see that we must show that for $0 \leq s \leq j-3$,

$$A_{s+2}(y) = A_{s+1}(y) + \sum_{k=3}^{s+2} \binom{s+2}{k-2} A_{k-1}(y) A_{s+2-(k-1)}(y). \tag{60}$$

However this follows from our analysis of Cases 1, 2, and 3 above for the recursion of $A_{s+2}(y)$. That is, since $s+2 \leq j-1$, Case 2 does not apply so that we only get the contributions from Cases 1 and 3 which is exactly (60).

Thus we have shown that $A(y,t)$ satisfies the partial differential equation

$$A''(t,y) = (1 - y - y^{\mathrm{des}(\tau)}) A'(t,y) + \big(A'(t,y)\big)^2 \tag{61}$$

with initial conditions that $A(y,0) = 0$, $A(y,t)|_t = y$, and $A(y,t)|_{\frac{t^2}{2!}} = y$. It is then easy to check that the solution to this differential equation is

$$A(y,t) = \ln \left(\frac{1}{1 - \int_0^t e^{(1-y)s + y^{\mathrm{des}(\tau)} \frac{s^{j-1}}{(j-1)!}} ds} \right). \tag{62}$$

Thus

$$\begin{aligned} A(y,t) &= \sum_{n \geq 1} \frac{t^n}{n!} \sum_{C \in \mathcal{L}_n^{ncm}(\tau)} y^{\mathrm{cdes}(C)} \\ &= \ln \left(\frac{1}{1 - \int_0^t e^{(1-y)s + y^{\mathrm{des}(\tau)} \frac{s^{j-1}}{(j-1)!}} ds} \right). \end{aligned} \tag{63}$$

But then we know by Theorem 1.2, that

$$\begin{aligned} NCM_\tau(t,x,y) &= e^{x \sum_{n \geq 1} \frac{t^n}{n!} \sum_{C \in \mathcal{L}_n^{ncm}(\tau)} y^{\mathrm{cdes}(C)}} \\ &= e^{x \ln \left(\frac{1}{1 - \int_0^t e^{(1-y)s + y^{\mathrm{des}(\tau)} \frac{s^{j-1}}{(j-1)!}} ds} \right)} \\ &= \left(\frac{1}{1 - \int_0^t e^{(1-y)s + y^{\mathrm{des}(\tau)} \frac{s^{j-1}}{(j-1)!}} ds} \right)^x \end{aligned}$$

which is what we wanted to prove. □

We end this section by showing how one can compute $NCM_\tau(t,x,y)$ where $\tau \in S_m$ is of the form $\tau = 1\ 2\ \dots\ (j-1)\ \gamma\ j$ where γ is a permutation of the elements $j+1,\dots,m$ where $m \geq j+1$. We let $p = m-j$ so that $\mathrm{red}(\gamma) \in S_p$. We shall assume that $j \geq 3$ since we have already dealt with permutations that start with 1 and end with 2.

Using our previous theorems as a guide, we shall assume that $NCM_\tau(t,x,y)$ is of the form

$$NCM_\tau(t,x,y) = e^{x\sum_{n\geq 1}\frac{t^n}{n!}\sum_{C\in\mathcal{L}_n^{ncm}(\tau)} y^{\mathrm{cdes}(C)}} = \frac{1}{(U_\tau(t,y))^x}$$

where

$$U_\tau(t,y) = \sum_{n\geq 0} U_{n,\tau}(y)\frac{t^n}{n!}. \tag{64}$$

We have been unable to find a closed form for $U_\tau(t,y)$. However, we can show that the coefficients of $U_{n,\tau}(y)$ satisfy a simple recursion. That is, we shall prove the following.

THEOREM 3.2 *Suppose that $\tau = 1\ 2\ \dots j-1\ \gamma\ j$ where γ is a permutation of $j+1,\dots,j+p$ and $j \geq 3$. Then*

$$NCM_\tau(t,x,y) = \frac{1}{(U_\tau(t,y))^x}$$

where

$$U_\tau(t,y) = \sum_{n\geq 0} U_{n,\tau}(y)\frac{t^n}{n!} \tag{65}$$

and

$$U_{n+j,\tau}(y) = (1-y)U_{n+j-1,\tau}(y) - y^{\mathrm{des}(\tau)}\binom{n}{p}U_{n-p+1,\tau}(y). \tag{66}$$

Proof. Taking the natural logarithm of both sides (64) and using (31), we see

$$-\ln(U_\tau(t,y)) = \sum_{n\geq 1}\frac{t^n}{n!}\sum_{C\in\mathcal{L}_n^{ncm}(\tau)} y^{\mathrm{cdes}(C)} = \sum_{n\geq 1}\frac{t^n}{n!}\sum_{\sigma\in\mathcal{NM}_n^1(\tau)} y^{\mathrm{des}(\sigma)+1}. \tag{67}$$

Before proceeding, we need to establish some notation. Fix τ of the form $1\ 2\ \dots j-1\gamma j$ where $j \geq 3$. For any $\sigma \in S_n^1$, we let τ-imch(σ) be the indicator function that the initial segment of size m in σ is a τ-match. Thus τ-imch$(\sigma) = 1$ if $\mathrm{red}(\sigma_1\dots\sigma_m) = \tau$ and τ-imch$(\sigma) = 0$ otherwise. For $i = 1,\dots,j-1$, we let $\tau^{(i)} = \mathrm{red}(i\ i+1\ \dots j-1\ \gamma\ j)$. Our first goal is to compute

$$A(t,y) = \sum_{n\geq 1} A_n(y)\frac{t^n}{n!} \tag{68}$$

where

$$A_n(y) = \sum_{\sigma\in\mathcal{NM}_n^1(\tau)} y^{1+\mathrm{des}(\sigma)}.$$

For $i = 2,\dots,k-1$, we shall also need the following functions

$$B_i(t,y) = 1 + \sum_{n\geq 1} B_{i,n}(y)\frac{t^n}{n!} \tag{69}$$

where

$$B_{i,n}(y) = \sum_{\substack{\sigma \in S_n^1 \\ \tau\text{-mch}(\sigma)=0 \\ \tau^{(2)}\text{imch}(\sigma)=0 \\ \tau^{(3)}\text{imch}(\sigma)=0 \\ \vdots \\ \tau^{(i)}\text{imch}(\sigma)=0}} y^{1+\text{des}(\sigma)}.$$

Thus $B_{i,n}(y)$ is the sum of $y^{1+\text{des}(\sigma)}$ over all permutation σ in S_n^1 such that σ has no τ-matches and σ does not start with a $\tau^{(j)}$-match for $j = 2, \dots, i$.

First we develop recursions for $A_n(y)$ for $n \geq 2$. Let $\mathcal{E}_{n,k,\tau}$ denote the set of all $\sigma = \sigma_1 \dots \sigma_n \in \mathcal{NM}_n^1(\tau)$ such that $\sigma_k = 2$. We then consider two cases for $\sigma \in \mathcal{NM}_n^1(\tau)$ depending on which $\mathcal{E}_{n,k,\tau}$ contains σ.

Case 1. $\sigma \in \mathcal{E}_{n,2,\tau}$.
Thus $\sigma = 1\ 2\ \sigma_3 \dots \sigma_n$. To ensure that σ has no τ-matches, we must ensure that there are no τ matches in $2\ \sigma_3 \dots \sigma_n$ and that σ does not start with a τ-match which is equivalent to ensuring that $2\ \sigma_3 \dots \sigma_n$ does not start with a $\tau^{(2)}$-match. Thus in this case, the permutations of $\mathcal{E}_{n,2,\tau}$ contribute $B_{2,n-1}(y)$ to $A_n(y)$.

Case 2 $\sigma \in \mathcal{E}_{n,k,\tau}$ where $3 \leq k \leq n$.
In this case, it is easy to see that the only possible τ-matches must occur in $\sigma_k \dots \sigma_n$ or in $\sigma_1 \dots \sigma_{k-1}$. Thus we have $\binom{n-2}{k-2}$ ways to choose the elements that will constitute $\sigma_2 \dots \sigma_{k-1}$ and $A_{k-1}(1)$ ways to order them so that there are no τ-matches in $\sigma_1 \dots \sigma_{k-1}$. Once we have picked $\sigma_2 \dots \sigma_{k-1}$, there are $A_{n-k+1}(1)$ ways to order the remaining elements so that there are no τ-matches in $\sigma_k \dots \sigma_n$. Having picked σ, we have that

$$\text{des}(\sigma) + 1 = \text{des}(\sigma_1 \dots \sigma_{k-1}) + 1 + \text{des}(\sigma_k \dots \sigma_n) + 1$$

since $\sigma_{k-1} > 2$. Hence in this case, the permutations in $\mathcal{E}_{n,k,\tau}$ will contribute

$$\binom{n-2}{k-2} A_{k-1}(y) A_{n-k+1}(y)$$

to $A_n(y)$.

It follows that for $n \geq 2$,

$$A_n(y) = B_{2,n-1}(y) + \sum_{k=3}^{n} \binom{n-2}{k-2} A_{k-1}(y) A_{n-k+1}(y). \tag{70}$$

We can develop similar recursions for $B_{2,n}(y)$ for $n \geq 2$. However we have to consider the cases $j = 3$ and $j > 3$ separately.

First consider, the case where $j = 3$. Note in this case $\tau^{(2)} = \text{red}(2\ \gamma\ 3) = 1\ \alpha\ 2$ where α is a permutation of $3, \dots, p+2$ such that $\text{red}(\alpha) = \text{red}(\gamma)$. We then consider three cases for $\sigma \in \mathcal{NM}_n^1(\tau)$

depending on which $\mathcal{E}_{n,k,\tau}$ contains σ.

Case 1. $\sigma \in \mathcal{E}_{n,2,\tau}$.
Thus $\sigma = 1\ 2\ \sigma_3 \ldots \sigma_n$. To guarantee that σ has no τ-matches, we must ensure there are no τ matches in $2\ \sigma_3 \ldots \sigma_n$ and that σ does not start with a τ-match which is equivalent to ensuring that $2\ \sigma_3 \ldots \sigma_n$ does not start with a $\tau^{(2)}$-match. It might seem that to ensure σ does not start with a $\tau^{(2)}$-match then we must ensure that $2\ \sigma_3 \ldots \sigma_n$ does start a $\tau^{(3)}$-match. However, in this case $\tau^{(3)} = \mathrm{red}(\gamma\ 3)$ does not start with 1 so then it is automatically true that $2\ \sigma_3 \ldots \sigma_n$ does start with a $\tau^{(3)}$-match. Thus the permutations in $\mathcal{E}_{n,2,\tau}$ contribute $B_{2,n-1}(y)$ to $B_{2,n}(y)$.

Case 2. $\sigma \in \mathcal{E}_{n,p+2,\tau}$.
In this case, it is easy to see that the only possible τ-matches must occur in $\sigma_{p+1} \ldots \sigma_n$ or in $\sigma_1 \ldots \sigma_p$. Now we have $\binom{n-2}{p}$ ways to choose the elements that will constitute $\sigma_2 \ldots \sigma_{p+1}$. We can order these elements in any way that we want except that we cannot have $\mathrm{red}(\sigma_2 \ldots \sigma_{p+1}) = \mathrm{red}(\gamma)$ since otherwise σ would start with a $\tau^{(2)}$ match. Note that $B_{2,p+1}(y) = \sum_{\beta \in S^1_{p+1}} y^{\mathrm{des}(\beta)+1}$ since no permutation of length $p+1$ can contain a τ-match or start with a $\tau^{(2)}$-match. Since

$$\mathrm{des}(1\ \sigma_2 \ldots \sigma_{p+1}) + 1 + \mathrm{des}(2\ \sigma_{p+2} \ldots \sigma_n) + 1 = \mathrm{des}(\sigma)$$

and $\mathrm{des}(1\ \gamma) + 1 = \mathrm{des}(\tau)$, the permutations in $\mathcal{E}_{n,p+2,\tau}$ will contribute

$$\binom{n-2}{p}(B_{2,p+1}(y) - y^{\mathrm{des}(\tau)})A_{n-p-1}(y)$$

to $B_{2,n}(y)$.

Case 3. $\sigma \in \mathcal{E}_{n,k,\tau}$ where $3 \leq k \leq n$ and $k \notin \{2, p+2\}$.
In this case, it is easy to see that the only possible τ-matches must occur in $\sigma_k \ldots \sigma_n$ or in $\sigma_1 \ldots \sigma_{k-1}$. Thus we have $\binom{n-2}{k-2}$ ways to choose that elements that will constitute $\sigma_2 \ldots \sigma_{k-1}$ and $B_{2,k-1}(1)$ ways to order them so that there are no τ-matches in $\sigma_1 \ldots \sigma_{k-1}$ and $\sigma_1 \ldots \sigma_{k-1}$ does not start with a $\tau^{(2)}$ match and $A_{n-k+1}(1)$ ways to order $\sigma_k \ldots \sigma_n$ so it contains no τ-match. It follows that the permutations in $\mathcal{E}_{n,k,\tau}$ will contribute $\binom{n-2}{k-2}B_{2,k-1}(y)A_{n-k+1}(y)$ to $B_{2,n}(y)$.

Thus if $n \geq p+2$, we have the recursion

$$B_{2,n}(y) = B_{2,n-1}(y) + \left(\sum_{k=3}^{n} \binom{n-2}{p-2} B_{2,k-1}(y)A_{n-k+1}(y)\right) - \binom{n-2}{p} y^{\mathrm{des}(\tau)} A_{n-p-1}(y). \tag{71}$$

For $2 \leq n \leq p+1$, Case 2 does not apply so that we have the recursion

$$B_{2,n}(y) = B_{2,n-1}(y) + \left(\sum_{k=3}^{n} \binom{n-2}{p-2} B_{2,k-1}(y)A_{n-k+1}(y)\right). \tag{72}$$

Before considering the case where $j > 3$, we shall show how we can derive a recursion (66) for the

$U_{n,\tau}(y)$s in this case. We have shown that for all $n \geq 2$,

$$\begin{aligned} A_n(y) &= B_{2,n-1}(y) + \sum_{k=3}^{n} \binom{n-2}{k-2} A_{k-1}(y) A_{n-k+1}(y) \text{ and} \\ B_{2,n}(y) &= B_{2,n-1}(y) + \left(\sum_{k=3}^{n} \binom{n-2}{p-2} B_{2,k-1}(y) A_{n-k+1}(y) \right) - \\ & \qquad \chi(n \geq p+2) y^{\mathrm{des}(\tau)} \binom{n-2}{p} A_{n-p-1}(y) \end{aligned}$$

where for any statement A, we let $\chi(A)$ equal 1 if A is true and equal 0 if A is false. Thus we have that for all $n \geq 2$,

$$\begin{aligned} \frac{A_n(y)}{(n-2)!} &= \frac{B_{2,n-1}(y)}{(n-2)!} + \sum_{k=3}^{n} \frac{A_{k-1}(y)}{(k-2)!} \frac{A_{n-k+1}(y)}{(n-k)!} \text{ and} \\ \frac{B_{2,n}(y)}{(n-2)!} &= \frac{B_{2,n-1}(y)}{(n-2)!} + \left(\sum_{k=3}^{n} \frac{B_{2,k-1}(y)}{(k-2)!} \frac{A_{n-k+1}(y)}{(n-k)!} \right) - \chi(n \geq p+2) \frac{y^{\mathrm{des}(\tau)}}{p!} \frac{A_{n-p-1}(y)}{(n-p)!}. \end{aligned}$$

Multiplying by t^{n-2} and summing, we obtain the following differential equations when we think of $A = A(t,y)$ and $B_2 = B_2(t,y)$ as just functions of t:

$$\begin{aligned} A'' &= B_2' + (A' - y)A' \text{ and} \\ B_2'' &= B_2' + (B_2' - y)A' - \frac{y^{\mathrm{des}(\tau)} t^p}{p!} A'. \end{aligned}$$

Now if $U = U(t,y) = U_\tau(t,y)$, then $A = -ln(U)$. Thus

$$A' = \frac{-U'}{U} \text{ and} \tag{73}$$

$$A'' = \frac{-U''}{U} + \left(\frac{U'}{U} \right)^2. \tag{74}$$

Making these substitutions in our first differential equation and solving for B_2', we see that

$$B_2' = -\frac{U'' + yU'}{U}. \tag{75}$$

Thus

$$B_2'' = -\frac{U''' + yU''}{U} + \frac{(U'' + yU')U'}{U^2}. \tag{76}$$

Substituting these expressions into our second differential equation and simplifying, we obtain the following differential equation for U,

$$U''' = (1-y)U'' - \frac{y^{\mathrm{des}(\tau)} t^p}{p!} U'. \tag{77}$$

Taking the coefficient of $\frac{t^n}{n!}$ on both sides of (77), we see that

$$U_{n+3,\tau}(y) = (1-y)U_{n+2}(y) - \binom{n}{p} y^{\mathrm{des}(\tau)} U_{n-p+1}(y) \tag{78}$$

in the case where $\tau = 1\ 2\ \gamma\ 3$ and γ is a permutation of $4, \ldots, 3+p$.

Now consider the recursion for $B_{2,n}(y)$ where $j > 3$. We then consider two cases for $\sigma \in \mathcal{NM}_n^1(\tau)$ depending on which set $\mathcal{E}_{n,k,\tau}$ contains σ.

Case 1. $\sigma \in \mathcal{E}_{n,2,\tau}$.
Thus $\sigma = 1\ 2\ \sigma_3 \ldots \sigma_n$. To ensure that σ has no τ-matches, we must ensure that there are no τ matches in $2\ \sigma_3 \ldots \sigma_n$ and that σ does not start with a τ-match which is equivalent to ensuring that $2\ \sigma_3 \ldots \sigma_n$ does not start with a $\tau^{(2)}$-match. However in this case, we must also ensure that σ does not start with a $\tau^{(2)}$ which means that $2\ \sigma_3 \ldots \sigma_n$ must not start with a $\tau^{(3)}$-match. Thus in this case, the $\sigma \in \mathcal{E}_{n,2,\tau}$ contribute $B_{3,n-1}(y)$ to $B_{2,n}(y)$.

Case 2 $\sigma \in \mathcal{E}_{n,k,\tau}$ where $3 \le k \le n$.
In this case, it is easy to see that the only possible τ-matches must occur in $\sigma_k \ldots \sigma_n$ or in $\sigma_1 \ldots \sigma_{k-1}$. Thus we have $\binom{n-2}{k-2}$ ways to choose the elements that will constitute $\sigma_2 \ldots \sigma_{k-1}$ and $B_{2,k-1}(1)$ ways to order them so that there are no τ-matches in $\sigma_1 \ldots \sigma_{k-1}$ and $\sigma_1 \ldots \sigma_{k-1}$ does not start with a $\tau^{(2)}$ match and there are $A_{n-k+1}(1)$ ways to order $\sigma_k \ldots \sigma_n$ so that there is no τ-match. It follows that the permutations in $\mathcal{E}_{n,k}$ will contribute $\binom{n-2}{k-2} B_{2,k-1}(y) A_{n-k+1}(y)$ elements to $B_{2,n}(y)$.

It follows that if $j \ge 3$, then for $n \ge 2$,

$$B_{2,n}(y) = B_{3,n-1}(y) + \sum_{k=3}^{n} \binom{n-2}{k-2} B_{2,k-1}(y) A_{n-k+1}(y). \tag{79}$$

One can repeat this type of argument to show that in general, for $2 \le i \le j-2$

$$B_{i,n}(y) = B_{i+1,n-1}(y) + \sum_{k=3}^{n} \binom{n-2}{k-2} B_{i,k-1}(y) A_{n-k+1}(y). \tag{80}$$

The recursion for $B_{j-1,n}(y)$ is similar to the recursion for $B_{2,n}(y)$ when $j = 3$. That is, $\tau^{(j-1)} = \mathrm{red}(j-1\ \gamma\ j) = 1\ \alpha\ 2$, where α is a permutation of $3, \ldots, p+2$ and $\mathrm{red}(\gamma) = \mathrm{red}(\alpha)$. Then we have to consider three cases depending on which set $\mathcal{E}_{n,k,\tau}$ contains σ.

Case 1. $\sigma \in \mathcal{E}_{n,2,\tau}$.
Thus $\sigma = 1\ 2\ \sigma_3 \ldots \sigma_n$. To ensure that σ has no τ-matches and does not start with a $\tau^{(i)}$-match for $i = 2, \ldots, j-1$, we clearly have to ensure that $2\ \sigma_3 \ldots \sigma_n$ has no τ-matches and does not start with a $\tau^{(i)}$-match for $i = 2, \ldots, j-1$. However, we do not have to worry about $2\ \sigma_3 \ldots \sigma_n$ starting with a $\tau^{(j)} = \mathrm{red}(\sigma\ j)$ since $\tau^{(j)}$ does not start with its least element. Thus in this case, the permutations in $\mathcal{E}_{n,2,\tau}$ contribute $B_{j-1,n-1}(y)$ to $B_{j-1,n}(y)$.

Case 2. $\sigma \in \mathcal{E}_{n,p+2,\tau}$ In this case, it is easy to see that the only possible τ-matches must occur in $\sigma_{p+1} \ldots \sigma_n$ or in $\sigma_1 \ldots \sigma_p$. Now we have $\binom{n-2}{p}$ ways to choose that elements that will constitute $\sigma_2 \ldots \sigma_{p+1}$. We can order these elements in any way that we want except that we cannot have $\mathrm{red}(\sigma_2 \ldots \sigma_{p+1}) = \mathrm{red}(\gamma)$ since otherwise σ would start with at $\tau^{(j-1)}$ match. Note that $B_{j-1,p+1}(y) = \sum_{\beta \in S^1_{p+1}} y^{\mathrm{des}(\beta)+1}$ since no permutation of length $p+1$ can contain a τ-match or start with a $\tau^{(i)}$-match for $i = 2, \ldots j-1$. Thus since

$$\mathrm{des}(1\ \sigma_2 \ldots \sigma_{p+1}) + 1 + \mathrm{des}(2\ \sigma_{p+2} \ldots \sigma_n) + 1 = \mathrm{des}(\sigma)$$

and $\mathrm{des}(1\ \gamma) + 1 = \mathrm{des}(\tau)$, the permutations in $\mathcal{E}_{n,p+2,\tau}$ will contribute

$$\binom{n-2}{p}(B_{j-1,p+1}(y) - y^{\mathrm{des}(\tau)})A_{n-p+1}(y)$$

to $B_{j-1,n}(y)$.

Case 3. $\sigma \in \mathcal{E}_{n,k,\tau}$ where $3 \leq k \leq n$ and $k \notin \{2, p+2\}$.
In this case, it is easy to see that the only possible τ-matches must occur in $\sigma_k \ldots \sigma_n$ or in $\sigma_1 \ldots \sigma_{k-1}$. Thus we have $\binom{n-2}{k-2}$ ways to choose that elements that will constitute $\sigma_2 \ldots \sigma_{k-1}$ and $B_{j-1,k-1}(1)$ ways to order them so that there are no τ-matches in $\sigma_1 \ldots \sigma_{k-1}$ and $\sigma_1 \ldots \sigma_{k-1}$ does not start with a $\tau^{(i)}$-match for $i = 2, \ldots, j-1$ and there are $A_{n-k+1}(1)$ ways to order $\sigma_k \ldots \sigma_n$ so that there is no τ-match. Thus the permutations in $\mathcal{E}_{n,k,\tau}$ will contribute $\binom{n-2}{k-2}B_{j-1,k-1}(y)A_{n-k+1}(y)$ to $B_{j-1,n}(y)$.

It follows that for $n \geq 2$,

$$\begin{aligned} B_{j-1,n}(y) &= B_{j-1,n-1}(y) + \sum_{k=3}^{n} \binom{n-2}{k-2} B_{j-1,k-1}(y)A_{n-k+1}(y) - \\ &\quad \chi(n \geq p+2)\binom{n-2}{p} y^{\mathrm{des}(\tau)} A_{n-p-1}(y). \end{aligned} \tag{81}$$

Thus for all $n \geq 2$, we have proved that in general

$$
\begin{aligned}
A_n(y) &= B_{2,n-1}(y) + \sum_{k=3}^{n} \binom{n-2}{k-2} A_{k-1}(y) A_{n-k+1}(y) \\
B_{2,n}(y) &= B_{3,n-1}(y) + \sum_{k=3}^{n} \binom{n-2}{k-2} B_{2,k-1}(y) A_{n-k+1}(y) \\
B_{3,n}(y) &= B_{4,n-1}(y) + \sum_{k=3}^{n} \binom{n-2}{k-2} B_{3,k-1}(y) A_{n-k+1}(y) \\
&\vdots \\
B_{j-2,n}(y) &= B_{j-1,n-1}(y) + \sum_{k=3}^{n} \binom{n-2}{k-2} B_{j-2,k-1}(y) A_{n-k+1}(y) \\
B_{j-1,n}(y) &= B_{j-1,n-1}(y) + \left(\sum_{k=3}^{n} \binom{n-2}{k-2} B_{j-1,k-1}(y) A_{n-k+1}(y) \right) - \\
&\qquad \chi(n \geq p+2) \binom{n-2}{p} y^{\mathrm{des}(\tau)} A_{n-p-1}(y).
\end{aligned}
$$

As in the case for $j = 3$, if we multiply everything by $\frac{t^{n-2}}{(n-2)!}$ and then sum over $n \geq 2$ we get the following system of differential equations where we think of $A(t, y)$ and $B_i(t, y)$ for $i = 2, \ldots, j-1$ as functions of t.

$$
\begin{aligned}
&(D_1) \quad A'' = B_2' + A'^2 - yA' \\
&(D_2) \quad B_2'' = B_3' + B_2'A' - yA' \\
&(D_3) \quad B_3'' = B_4' + B_3'A' - yA' \\
&\vdots \\
&(D_{j-2}) \; B_{j-2}'' = B_{j-1}' + B_{j-2}'A' - yA' \\
&(D_{j-1}) \; B_{j-1}'' = B_{j-1}' + B_{j-1}'A' - yA' - \frac{t^p}{(p)!} y^{\mathrm{des}(\tau)} A'.
\end{aligned}
$$

As in the case $j = 3$, we let $A(t, y) = -\log(U(t, y))$ so that $A' = \frac{-U'}{U}$ and $A'' = \frac{-U''}{U} + \frac{U'^2}{U^2}$. Thus under this substitution, the first differential equation becomes

$$\frac{-U''}{U} + \frac{U'^2}{U^2} = B_2' + \frac{U'^2}{U^2} + y\frac{U'}{U}$$

so that

$$B_2' = \frac{-U'' - yU'}{U}. \tag{82}$$

In fact, we have the following lemma.

Lemma 3.3 *For* $2 \leq i \leq j-1$,

$$B_i' = \frac{-U^{(i)} - y\sum_{k=1}^{i-1} U^{(k)}}{U}. \tag{83}$$

Proof. We proceed by induction on i. We have already shown that (83) in the case where $i = 2$. Now suppose that

$$B_i' = \frac{-U^{(i)} - y\sum_{k=1}^{i-1} U^{(k)}}{U}. \tag{84}$$

Then we must show that

$$B_{i+1}' = \frac{-U^{(i+1)} - y\sum_{k=1}^{i} U^{(k)}}{U}. \tag{85}$$

Taking the derivative of both sides of (84) with respect to t, we see that

$$B_i'' = \frac{-U^{(i+1)} - y\sum_{k=2}^{i} U^{(k)}}{U} + \left(\frac{U^{(i)} + y\sum_{k=1}^{i-1} U^{(k)}}{U} \cdot \frac{U'}{U}\right).$$

Plugging our expression for B_i'' and B_i' into the differential equation

$$(D_i)\ B_i'' = B_{i+1}' + B_i'A' - yA',$$

we see that

$$\frac{-U^{(i+1)} - y\sum_{k=2}^{i} U^{(k)}}{U} + \left(\frac{U^{(i)} + y\sum_{k}^{i-1} U^{(k)}}{U} \cdot \frac{U'}{U}\right)$$
$$= B_{i+1}' + \left(\frac{-U^{(i)} - y\sum_{k=1}^{i-1} U^{(k)}}{U} \cdot \frac{-U'}{U}\right) - \left(y \cdot \frac{-U'}{U}\right).$$

Solving for B_{i+1}' we see that

$$B_{i+1}' = \frac{-U^{(i+1)} - y\sum_{k=1}^{i} U^{(k)}}{U}.$$

□

By the Lemma, we know that

$$B_{j-1}' = \frac{-U^{(j-1)} - y\sum_{k=1}^{j-2} U^{(k)}}{U},$$

and, hence,

$$B_{j-1}'' = \frac{-U^{(j)} - y\sum_{k=2}^{j-1} U^{(k)}}{U} + \left(\frac{U^{(j-1)} + y\sum_{k}^{j-2} U^{(k)}}{U} \cdot \frac{U'}{U}\right).$$

Thus plugging these expressions into the differential equation (D_{j-1}), we obtain that

$$\frac{-U^{(j)} - y\sum_{k=2}^{j-1} U^{(k)}}{U} + \left(\frac{U^{(j-1)} + y\sum_{k}^{j-2} U^{(k)}}{U} \cdot \frac{U'}{U}\right)$$
$$= \frac{-U^{(j-1)} - y\sum_{k=1}^{j-2} U^{(k)}}{U} +$$
$$\left(\frac{-U^{(j-1)} - y\sum_{k=1}^{j-2} U^{(k)}}{U} \cdot \frac{-U'}{U}\right) - \left(y \cdot \frac{-U'}{U}\right) - \left(\frac{t^p}{p!} y^{\mathrm{des}(\tau)} \cdot \frac{-U'}{U}\right).$$

Simplifying this expression yields that

$$U^{(j)} = (1-y)U^{(j-1)} - \frac{t^p}{p!} y^{\mathrm{des}(\tau)} U'. \tag{86}$$

Then taking the coefficient of $\frac{t^n}{n!}$ on both side of (86) gives that

$$U_{n+j} = (1-y)U_{n+j-1} - y^{\mathrm{des}(\tau)} \binom{n}{p} U_{n-p+1}$$

which is what we wanted to prove. □

We end this section with an example of the use of Theorem 3.2. Let $\tau = 1243$ and

$$A_{n,\tau}(t,y) = \sum_{n\geq 1} A_{n,\tau}(y)\frac{t^n}{n!} = \sum_{n\geq 1} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{NM}_n^1(\tau)} y^{\mathrm{des}(\sigma)+1} = \sum_{n\geq 1} \frac{t^n}{n!} \sum_{C \in \mathcal{L}_n^{ncm}(\tau)} y^{\mathrm{cdes}(\sigma)}.$$

It is easy to check that $A_{1,\tau}(y) = y$, $A_{2,\tau}(y) = y$, $A_{3,\tau}(y) = y + y^2$, and $A_{4,\tau}(y) = y + 3y^2 + y^3$. Now

$$U_\tau(t,y) = \sum_{n\geq 0} U_{n,\tau}(y) = e^{-A_\tau(t,y)}$$

so that one can use Mathematica to compute that $U_{0,\tau}(y) = 1$, $U_{1,\tau}(y) = -y$,
$U_{2,\tau}(y) = -y + y^2$, $U_{3,\tau}(y) = -y + 2y^2 - y^3$, and $U_{4,\tau}(y) = -y + 4y^2 - 3y^3 + y^4$.

By Theorem 3.2, we know that we have the recursion that

$$U_{n+3,\tau}(y) = (1-y)U_{n+2,\tau}(y) - yU_{n,\tau}(y).$$

Thus we can use this recursion to compute that
$U_{5,\tau}(y) = -y + 6y^2 - 8y^3 + 4y^4 - y^5$,
$U_{6,\tau}(y) = -y + 8y^2 - 16y^3 + 13y^4 - 5y^5 + y^6$,
$U_{7,\tau}(y) = -y + 10y^2 - 28y^3 + 32y^4 - 19y^5 + 6y^6 - y^7$, and
$U_{8,\tau}(y) = -y + 12y^2 - 44y^3 + 68y^4 - 55y^5 + 26y^6 - 7y^7 + y^8$.

But then we know that $NCM_\tau(t,x,y) = \frac{1}{(U_\tau(t,y))^x}$. Let $NCM_{\tau,n}(x,y)$ be the coefficient of $\frac{t^n}{n!}$ in $NCM_\tau(t,x,y)$. That is, let

$$NCM_\tau(t,x,y) = \sum_{n\geq 0} NCM_{\tau,n}(x,y)\frac{t^n}{n!}.$$

Thus one can use Mathematica to compute the polynomials $NCM_{\tau,n}(x,y)$ where

$NCM_{\tau,0}(x,y) = 1$,

$NCM_{\tau,1}(x,y) = xy$,

$NCM_{\tau,2}(x,y) = xy + x^2y^2,$

$NCM_{\tau,3}(x,y) = xy + xy^2 + 3x^2y^2 + x^3y^3,$

$NCM_{\tau,4}(x,y) = xy + 3xy^2 + 7x^2y^2 + xy^3 + 4x^2y^3 + 6x^3y^3 + x^4y^4,$

$NCM_{\tau,5}(x,y) = xy + 9xy^2 + 15x^2y^2 + 8xy^3 + 25x^2y^3 + 25x^3y^3 + xy^4 + 5x^2y^4 + 10x^3y^4 + 10x^4y^4 + x^5y^5,$

$NCM_{\tau,6}(x,y) = xy + 23xy^2 + 31x^2y^2 + 45xy^3 + 119x^2y^3 + 90x^3y^3 + 20xy^4 + 73x^2y^4 + 105x^3y^4 + 65x^4y^4 + xy^5 + 6x^2y^5 + 15x^3y^5 + 20x^4y^5 + 15x^5y^5 + x^6y^6,$

$NCM_{\tau,7}(x,y) = xy + 53xy^2 + 63x^2y^2 + 217xy^3 + 490x^2y^3 + 301x^3y^3 + 192xy^4 + 623x^2y^4 + 749x^3y^4 + 350x^4y^4 + 47xy^5 + 196x^2y^5 + 343x^3y^5 + 315x^4y^5 + 140x^5y^5 + xy^6 + 7x^2y^6 + 21x^3y^6 + 35x^4y^6 + 35x^5y^6 + 21x^6y^6 + x^7y^7$, and

$NCM_{\tau,8}(x,y) = xy + 115xy^2 + 127x^2y^2 + 916xy^3 + 1838x^2y^3 + 966x^3y^3 + 1500xy^4 + 4333x^2y^4 + 4466x^3y^4 + 1701x^4y^4 + 765xy^5 + 2810x^2y^5 + 4214x^3y^5 + 3164x^4y^5 + 1050x^5y^5 + 105xy^6 + 495x^2y^6 + 1008x^3y^6 + 1148x^4y^6 + 770x^5y^6 + 266x^6y^6 + xy^7 + 8x^2y^7 + 28x^3y^7 + 56x^4y^7 + 70x^5y^7 + 56x^6y^7 + 28x^7y^7 + x^8y^8.$

4 Conclusions

As mentioned in the introduction, we know of two other ways to compute $NCM_\tau(t,x,y)$ and $NCM_\Upsilon(t,y)$ for various τ's and Υ's.

Our second approach again uses the function $U_\tau(t,y)$ as defined in the previous section where

$$NCM_\tau(t,x,y) = \sum_{n\geq 0} NCM_{\tau,n}(x,y)\frac{t^n}{n!} = \frac{1}{(U_\tau(t,y))^x}.$$

It follows that

$$U_\tau(t,y) = \frac{1}{NCM_\tau(t,1,y)} = \frac{1}{\sum_{n\geq 0} NCM_{\tau,n}(1,y)\frac{t^n}{n!}}. \tag{87}$$

Remmel and his coauthors [3, 15, 18, 19, 20, 21, 24, 30] developed a method called the homomorphism method to show that many generating functions involving permutation statistics can be applied to simple symmetric function identities such as

$$H(t) = 1/E(-t) \tag{88}$$

where

$$H(t) = \sum_{n\geq 0} h_n t^n = \prod_{i\geq 1}\frac{1}{1-x_i t}$$

is the generating function of the homogeneous symmetric functions h_n in infinitely many variables $x_1, x_2, \ldots$ and

$$E(t) = \sum_{n\geq 0} e_n t^n = \prod_{i\geq 1} 1 + x_i t$$

is the generating function of the elementary symmetric functions e_n in infinitely many variables $x_1, x_2, \ldots$. Now if we define a homomorphism θ and the ring of symmetric function so that

$$\theta(e_n) = \frac{(-1)^n}{n!} NCM_{\tau,n}(1,y),$$

then

$$\theta(E(-t)) = \frac{1}{\sum_{n\geq 0} NCM_{\tau,n}(1,y)\frac{t^n}{n!}}.$$

Thus $\theta(H(t))$ should equal $U_\tau(t,y)$. One can then use the combinatorial methods associated with the homomorphism method to develop recursions for the coefficient of $U_\tau(t,y)$ much like we did in Theorem 3.2. For example, we can show that

$$U_{n,1324}(y) = (1-y)U_{n-1,1324}(y) + \sum_{k=2}^{\lfloor n/2 \rfloor} (-y)^{k-1} C_{k-1} U_{n-2k+1,1324}(y)$$

where C_k is the k-th Catalan number and

$$U_{n,1423}(y) = (1-y)U_{n-1,1423}(y) + \sum_{k=2}^{\lfloor n/2 \rfloor} (-y)^{k-1} \binom{n-k-1}{k-1} U_{n-2k+1,1423}(y).$$

The first author has developed a third way to approach the problem of computing $NCM_\Upsilon(t)$ which is completely different from the other two approaches. That method involves defining a certain bijection between the set of derangements and certain fillings of brick tabloids. That bijection allows one to compute generating functions for the number derangements that have no cycle Υ-matches by applying an appropriate ring homomorphism defined on the ring of symmetric functions Λ in infinitely many variables $x_1, x_2, \ldots$ to certain simple symmetric function identities as described above. This approach is generally much more complicated than the first two approaches. However, it allows us to compute $NCM_\Upsilon(t)$ for a number of sets of permutations Υ which seem beyond the techniques employed in this paper. For example, one can show that

$$NCM_\Upsilon(t) = \frac{e^{t+t^2/2+t^4/12}}{1-\sum_{n\geq 3}(n-1)\frac{t^n}{n!}}$$

where Υ is the set of patterns that contains 1234 and all patterns $\tau = \tau_1\tau_2\tau_3\tau_4\tau_5$ such that $\tau_1 < \tau_2 > \tau_3 < \tau_4 > \tau_5$. This approach will be described in a forth coming paper.

References

[1] D. André, *Développements de sec x et de tang x*, C. R. Math. Acad. Sci. Paris, 88 (1879) 965–967.

[2] D. André, *Mémoire sur les permutations alternées*, J. Math. Pures Appl. (9), 7 (1881) 167–184.

[3] D. Beck and J. B. Remmel, *Permutation enumeration of the symmetric group and the combinatorics of symmetric functions*, J. Combin. Theory Ser. A, 72 (1995) 1–49.

[4] F. Bergeron, P. Flajolet and B. Salvy, *Varieties of increasing trees*, CAAP '92 (Rennes 1992), Lecture Notes in Comput. Sci., 581 (1992) 24–48.

[5] F. Brenti, *Unimodal polynomials arising from symmetric functions*, Proc. Amer. Math. Soc., 108 (1990) 1133–1141.

[6] F. Brenti, *Permutation enumeration, symmetric functions, and unimodality*, Pacific J. Math., 157 (1993) 1–28.

[7] D. Callan, *Pattern avoidance in circular permutations*, arXiv:math/0210014v1.

[8] E. Deutsch and S. Elizalde, *Cycle-up-down permutations*, Australas. J. Combin., to appear.

[9] S. Elizalde and M. Noy, *Consecutive patterns in permutations*,quad Adv. in Appl. Math., 30 (2003) 110–125.

[10] Ö. Eğecioğlu and J. B. Remmel, *Brick tabloids and connection matrices between bases of symmetric functions*, Discrete Appl. Math., 34 (1991) 107–120.

[11] I. Goulden and D. Jackson, *Combinatorial enumeration*, A Wiley-Interscience Publication, John Wiley & Sons Inc., New York, 1983, With a foreword by Gian-Carlo Rota, Wiley-Interscience Series in Discrete Mathematics.

[12] S. Kitaev, *Introduction to partially ordered patterns*, Discrete Appl. Math., 155 (2007) 929–944.

[13] S. Kitaev, A. Niedermaier, J. B. Remmel and A. Riehl, *New pattern matching conditions for wreath products of the cyclic groups with symmetric groups*, preprint.

[14] D. E. Knuth, *The Art of Computer Programming. Volume 1. Fundamental Algorithms*, Addison-Wesley, Reading, MA 1973.

[15] T. Langley and J. B. Remmel, *Enumeration of m-tuples of permutations and a new class of power bases for the space of symmetric functions*, submitted.

[16] I. G. MacDonald, *Symmetric Functions and Hall Polynomials*, Oxford Univ. Press, London/New York, 1979.

[17] A. Mendes, *Building generating functions brick by brick*, Ph.D. thesis, University of California, San Diego, 2004.

[18] A. Mendes and J. B. Remmel, *Generating functions for statistics on $C_k \wr S_n$*, Sém. Lothar. Combin., 54A (2005/07), Art. B54At, 40 pp. (electronic).

[19] A. Mendes and J. B. Remmel, *Permutations and words counted by consecutive patterns*, Adv. in Appl. Math., 37 (2006) 443–480.

[20] A. Mendes and J. B. Remmel, *Descents, major indices, and inversions in permutation groups*, Discrete Math., 308 (2008) 2509–2524.

[21] A. Mendes, J. B. Remmel and A. Riehl, *Permutations with k-regular descent patterns*, Permutation Patterns (S. Linton, N. Ruškuc, V. Vatter, eds.), London. Math. Soc. Lecture Notes 376, 2010, pp. 259–286.

[22] A. Mendes and J. B. Remmel, *Generating Functions from Symmetric Functions*, preprint.

[23] A. Neidermaier and J. B. Remmel, *Analogues of up-down permutations for colored permutations*, J. Integer Seq., 13 (2010) Article 10.5.6, 32 pp.

[24] J. B Remmel and A. Riehl, *Generating functions for permutations which contain a given descent set*, Electron. J. Combin., 17 (2010) R27, 33 pp.

[25] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, available online from `http://oeis.org`.

[26] R. Simion and F. W. Schmidt, *Restricted permutations*, European J. Combin., 6 (1985) 383–406.

[27] R. P. Stanley, *Enumerative Combinatorics, volume 2*, Cambridge University Press, 1999.

[28] E. Steingrímsson, *Generalized permutation patterns – a short survey*, Permutation Patterns, St Andrews 2007, S.A. Linton, N. Ruskuc, V. Vatter (eds.), LMS Lecture Note Series, Cambridge University Press, to appear.

[29] A. Vella, *Pattern avoidance in permutations: linear and cyclic orders*, Electron. J. Combin., 10 (2003) R18.

[30] J. D. Wagner, *The permutation enumeration of wreath products and cyclic and symmetric groups*, Adv. in Appl. Math., 30 (2003) 343–368.