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Smooth compositions and smooth words

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Abstract. A composition of a positive integer n, $\pi = \pi_1\pi_2\cdots\pi_N$, where $\pi_1+\pi_2+\cdots+\pi_N = n$, is said to be smooth if it contains no pair of adjacent letters with difference greater than 1. A smooth composition π is called cyclic if in addition it satisfies $|\pi_1 - \pi_N| \le 1$. In this paper we study the problem of enumerating the smooth compositions of n with parts in a set. We obtain generating functions for the numbers of smooth compositions and smooth cyclic compositions of n with parts in the set $\{1,\ldots,k\}$. We also derive asymptotic estimates for the numbers of the compositions via singularity analysis. Finally, by viewing compositions as a restricted class of words, we deduce several results on smooth words, including previously known ones.

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1 Introduction

A word over an alphabet A, a set of positive integers, is defined as any ordered sequence of possibly repeated elements of A. A word $\sigma = \sigma_1\cdots\sigma_n$ is said to be *smooth* if $\sigma_{i+1} - \sigma_i \in \{0, 1, -1\}$ for all $i = 1, 2, \ldots, n-1$. For example, there are 7 smooth words in $[3]^2$, namely 11, 12, 21, 22, 23, 32 and 33. A smooth word $\sigma \in A^n$ is said to be *smooth cyclic* if it also satisfies $|\sigma_n - \sigma_1| \le 1$.

Recently, Knopfmacher *et al.* [10] found explicit computational formulas and generating functions for the numbers of smooth words, smooth cyclic words and smooth necklaces of length n over the alphabet $A = [k]$, where $[k] = \{1, \ldots, k\}$. (A smooth necklace is an equivalence class on smooth cyclic words under rotation). In particular, the authors obtained the following nice formula for the number of smooth cyclic words in $[k]^n$:

$$\sum_{j=1}^{k} \left[1 + 2\cos\left(\frac{j\pi}{k+1}\right)\right]^n. \tag{1}$$

Later, Mansour [12] presented explicit computational formulas for the number of smooth set partitions of $[n]$ (in context a partition of $[n]$ into k blocks is identified with a unique word $\pi_1\pi_2\cdots\pi_n$ such that $i \in B_{\pi_i}$ for all $i \in [n]$).

In this paper we consider the harder problem of enumerating smooth compositions with parts in a set. A *composition* $\pi = \pi_1\pi_2\cdots\pi_m$ of n is an ordered collection of positive integers whose sum is n. Thus a composition π of n with parts in A is a restricted word over the alphabet A. Thus, for example, there are 10 smooth compositions of 5 with parts in [3]: 23, 32, 122, 212, 221, 1112, 1121, 1211, 2111, 11111. It is well known that the number of compositions of n with parts in $\{1, 2, 3, \ldots\}$ is given by 2^{n-1}, for all $n \geq 1$.

We will derive generating functions for the number of smooth compositions and the number of smooth cyclic compositions of n with m parts in $[k]$. In addition we give corresponding asymptotic estimates for the numbers of the compositions as n grows without bound. We also deduce the theorems in [10], among other results on smooth words. Table 1 and Table 2 show the numbers of smooth compositions and smooth cyclic compositions for some small n.

$k \backslash n$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	3	5	8	13	21	34	55	89	144	233	377	610	987
3	1	2	4	5	10	16	26	45	73	123	205	340	569	946	1577
4	1	2	4	6	10	16	28	46	75	128	212	353	589	983	1641
5	1	2	4	6	11	16	28	46	77	129	212	355	592	988	1646
6	1	2	4	6	11	17	28	46	77	129	214	356	592	988	1648

Table 1: Number of smooth compositions of n with parts in $[k]$

In a seminal work Carlitz [2] studied the compositions of n in which any pair of adjacent parts are different. Such compositions, subsequently called *Carlitz compositions*, have been further explored by several authors (e.g. [11, 5]) with a weighted generalization appearing in [3]. In one sense our paper provides a natural counterpart to the study of Carlitz-type compositions in requiring adjacent parts to differ by no more than a unit.

In another sense this work may be viewed as the latest generalization of that of Alladi and Hoggat [1] who considered various enumerative questions (such as frequency of parts, rises, levels, descents) on the compositions of n with parts in $A = \{1, 2\}$. Observe that every word in $\{1, 2\}^n$ is necessarily smooth (or smooth cyclic). Previous generalizations include Heubach, Chinn and Grimaldi [7] with the enumeration of general compositions and palindromic compositions with respect to rises, levels and descents. However, a more comprehensive analysis of compositions under the three statistics is

$k\backslash n$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	3	5	8	13	21	34	55	89	144	233	377	610	987
3	1	2	4	5	10	14	24	41	65	111	183	304	509	844	1409
4	1	2	4	6	10	14	26	42	65	114	186	309	513	855	1429
5	1	2	4	6	11	14	26	42	67	115	186	309	516	858	1430
6	1	2	4	6	11	15	26	42	67	115	188	310	516	858	1430

Table 2: Number of smooth cyclic compositions of n with parts in $[k]$

provided by Heubach and Mansour [8] with a systematic enumeration of compositions with parts in a given set of positive integers (for details and other references see [9]).

In Section 2, we compute the generating function for the number of smooth compositions of n with m parts in $[k]$ and show that it is a rational function of two variables (Theorem 2.3). Then we apply singularity analysis to the generating function, leading to an asymptotic estimate of $c_k(n)$, the number of smooth compositions of n with parts in $[k]$ (Theorem 2.6). In Section 3 we apply previous results to obtain enumeration formulas for certain classes of smooth words, including the theorems of Knopfmacher *et al.* [10] as special cases. Section 4 and Section 5 are devoted to the enumeration and applications, respectively, of smooth cyclic compositions, according to the agenda of Sections 2 and 4 on smooth compositions.

2 Smooth compositions

Let $c_k(n,m)$ be the number of smooth compositions of n with m parts in $[k]$, and let $c_k(n)$ be the number of all smooth compositions of n with parts in $[k]$. Let $C_k(x,y)$ denote the generating function of the number $c_k(n,m)$,

$$C_k(x,y) = \sum_{n\geq 0}\sum_{m\geq 0} c_k(n,m)x^n y^m$$

In this section we obtain a formula for $C_k(x,y)$. Let π be a composition and denote by $parts_i(\pi)$ the number of occurrences of the integer i as a part of π.

Then each composition π of n with m parts in $[k]$ and $parts_k(\pi) = d$ can be represented as

$$\pi^{(0)}k\pi^{(1)}k\cdots\pi^{(d-1)}k\pi^{(d)} \text{ with } d \geq 0,$$

where $\pi^{(j)}$ is a word over $[k-1]$. We refer to this representation as the *d-maximal part decomposition.* The contribution to the generating function $C_k(x,y)$ of the 0-maximal part decomposition is $C_{k-1}(x,y)$, and the contribution of a d-maximal part decomposition, $d \geq 1$, gives

$$(x^k y)^d E_{k-1}(x,y)B_{k-1}(x,y)(EB_{k-1}(x,y))^{d-1},$$

where $E_k(x,y)$ (respectively $B_k(x,y)$, $EB_k(x,y)$) is the generating function for the number of compositions π of n with m parts in $[k]$ such that $\pi(k+1)$ (respectively, $(k+1)\pi$, $(k+1)\pi(k+1)$) is a smooth composition. Clearly, by the reversal operation $\pi_1\cdots\pi_m \mapsto \pi_m\cdots\pi_1$, we have that

$$B_k(x,y) = E_k(x,y).$$

Thus $C_k(x,y)$ satisfies the relation

$$C_k(x,y) = C_{k-1}(x,y) + \frac{x^k y(E_{k-1}(x,y))^2}{1 - x^k y EB_{k-1}(x,y)}. \tag{2}$$

Next we find a relation between the generating functions $E_k(x,y)$ and $EB_k(x,y)$. By rewriting the d-maximal part decomposition of a smooth composition π with parts in $[k]$ such that $\pi(k+1)$ is also a smooth composition, π can be represented as

$$\pi^{(0)} k \pi^{(1)} k \cdots \pi^{(d-1)} k,$$

with $d \geq 0$, so $\pi = \emptyset$ if $d = 0$. Now we consider the function $E_k(x,y)$. The 0-maximal part decomposition contributes just 1 (the empty composition), while the d-maximal part decomposition, $d \geq 1$, gives $(x^k y)^d E_{k-1}(x,y)(EB_{k-1}(x,y))^{d-1}$. Hence

$$E_k(x,y) = 1 + \frac{x^k y E_{k-1}(x,y)}{1 - x^k y EB_{k-1}(x,y)}. \tag{3}$$

Also for $EB_k(x,y)$, we rewrite the d-maximal part decomposition for a smooth composition π with parts in $[k]$ such that $(k+1)\pi(k+1)$ is also a smooth composition, and obtain a representation of π in the form

$$k \pi^{(1)} k \cdots \pi^{(d-1)} k,$$

with $d \geq 0$, so $\pi = \emptyset$ if $d = 0, 1$. Thus the contribution of the 0-maximal part decomposition is 1, and that of a d-maximal part decomposition, $d \geq 1$, is $(x^k y)^d (EB_{k-1}(x,y))^{d-1}$. Thus

$$EB_k(x,y) = 1 + \frac{x^k y}{1 - x^k y EB_{k-1}(x,y)}. \tag{4}$$

On applying the relation (4) a finite number of times, with the initial condition $EB_0(x,y) = 1$, we obtain the following result.

LEMMA 2.1 *The generating function $EB_k(x,y)$ is given by*

$$EB_k(x,y) = 1 + \cfrac{x^k y}{1 - x^k y - \cfrac{x^{2k-1} y^2}{1 - x^{k-1} y - \cfrac{x^{2k-3} y^2}{1 - x^{k-2} y - \cfrac{\ddots}{1 - x^2 y - \cfrac{x^3 y^2}{1 - x^1 y}}}}}.$$

Using Lemma 2.1 together with (3) we find an explicit form for $E_k(x,y)$.

LEMMA 2.2 *The generating function $E_k(x,y)$ is given by*

$$E_k(x,y) = 1 + \sum_{j=1}^{k} \frac{x^{\binom{k+1}{2}-\binom{j}{2}} y^{k-j+1}}{\prod_{i=j}^{k} \left(1 - x^i y - \frac{x^{2i-1}y^2}{1-x^{i-1}y-\frac{x^{2i-3}y^2}{1-x^{i-2}y-\frac{\ddots}{1-x^2y-\frac{x^3y^2}{1-x^1y}}}} \right)}.$$

Then Lemma 2.2 together with (2) gives the following result for the function $C_k(x,y)$. The second, simpler expression, follows from (3) and (2).

THEOREM 2.3 *The generating function $C_k(x,y)$ is rational in x and y and it is given by*

$$C_k(x,y) = 1 + \sum_{j=1}^{k} \frac{x^j y (E_{j-1}(x,y))^2}{1 - x^j y EB_{j-1}(x,y)} = 1 + \sum_{j=1}^{k} E_{j-1}(x,y)(E_j(x,y) - 1),$$

where $EB_j(x,y)$ and $E_j(x,y)$ are given in Lemma 2.1 and Lemma 2.2, respectively.

As an illustration, $C_k(x,y)$ is computed below for $k = 1,2,3,4$.

- $C_1(x,1) = \dfrac{1}{1-x}$,
- $C_2(x,1) = \dfrac{1}{1-x-x^2}$,
- $C_3(x,1) = \dfrac{(1+x)(1+x^2)}{1-x^2-2x^3-x^4-x^5}$,
- $C_4(x,1) = \dfrac{1-x^4-x^5-x^6+x^{10}}{1-x-x^2-x^3+x^5+2x^6+x^9-x^{10}}$.

2.1 Asymptotics for the number of smooth compositions of n

We introduce the following notation. Let $q_k(x,y)$ be a polynomial defined by

$$q_k(x,y) = (1-x^k y) q_{k-1}(x,y) - x^{2k-1} y^2 q_{k-2}(x,y), \; q_0(x,y) = 1, \; q_1(x,y) = 1 - xy. \tag{5}$$

For simplicity we denote the polynomial $q_k(x,1)$ by $q_k(x)$.

LEMMA 2.4 *For all $k \geq 1$,*

(a) $EB_k(x,y) = 1 + x^k y \dfrac{q_{k-1}(x,y)}{q_k(x,y)}$,

(b) $E_k(x,y) = 1 + \frac{1}{q_k(x,y)} \sum_{i=1}^{k} x^{i+(i+1)+\cdots+k} y^{k+1-i} q_{i-1}(x,y)$,

Proof. (a) We proceed to give a proof by induction on k. Clearly, (a) holds for $k=1$. From (4) and the induction hypothesis we obtain that

$$\begin{aligned} EB_k(x,y) &= 1+\frac{x^ky}{1-x^ky-x^{2k-1}y^2\frac{q_{k-2}(x,y)}{q_{k-1}(x,y)}} = 1+\frac{x^kyq_{k-1}(x,y)}{(1-x^ky)q_{k-1}(x,y)-x^{2k-1}y^2q_{k-2}(x,y)} \\ &= 1+\frac{x^kyq_{k-1}(x,y)}{q_k(x,y)}, \end{aligned}$$

as required.

(b) We also give a proof by induction on k. Clearly, (b) holds for $k=1$. From (3), (4), (a), and the induction hypothesis we get that

$$\begin{aligned} E_k(x,y) &= 1+x^ky\frac{q_{k-1}(x,y)}{q_k(x,y)}E_{k-1}(x,y) \\ &= 1+x^ky\frac{q_{k-1}(x,y)}{q_k(x,y)}\left(1+\frac{1}{q_{k-1}(x,y)}\sum_{i=1}^{k-1}x^{i+(i+1)+\cdots+k-1}y^{k-i}q_{i-1}(x,y)\right) \\ &= 1+x^ky\frac{q_{k-1}(x,y)}{q_k(x,y)}+\frac{1}{q_k(x,y)}\sum_{i=1}^{k-1}x^{i+(i+1)+\cdots+k}y^{k+1-i}q_{i-1}(x,y) \\ &= 1+\frac{1}{q_k(x,y)}\sum_{i=1}^{k}x^{i+(i+1)+\cdots+k}y^{k+1-i}q_{i-1}(x,y), \end{aligned}$$

as required. □

Lemma 2.5 *For $k\geq 2$, the polynomial $q_k(x)$ has only one positive simple zero in the domain $|x| < r_0 = \frac{7}{10}$.*

Proof. It is not hard to check that the lemma holds for $k=2,3,4$, by using, for example, the principle of the argument. Thus from now we assume that $k\geq 5$.

First let us prove that $|q_{k-1}(x)|\leq 2|q_k(x)|$ for all $k\geq 5$ and $|x|=0.7$. We proceed by induction on k. From (5) we have that

$$\begin{aligned} q_4(x) &= -x^{10}+x^9+2x^6+x^5-x^3-x^2-x+1, \\ q_5(x) &= -x^{14}-x^{13}+x^{12}-x^{11}-x^{10}+x^8+x^7+3x^6-x^3-x^2-x+1. \end{aligned}$$

It is not hard to see that $h(x)=2|q_5(x)|-|q_4(x)|\geq 0$ for all $|x|=0.7$. This may be clarified by plotting the graph of $h(0.7e^{it})$ with $t\in[0,2\pi]$, see Figure 2.1.

From the definitions and the inductive hypothesis we have that

$$\begin{aligned} |q_k(x)| &\geq |q_{k-1}(x)|-|x^k||q_{k-1}(x)|-|x^{2k-1}||q_{k-2}(x)| \\ &\geq |q_{k-1}(x)|-0.7^k|q_{k-1}(x)|-2\cdot 0.7^{2k-1}|q_{k-1}(x)|, \end{aligned}$$

which implies that

$$2|q_k(x)|\geq 2(1-0.7^k-2\cdot 0.7^{2k-1})|q_{k-1}(x)|\geq |q_{k-1}(x)|.$$

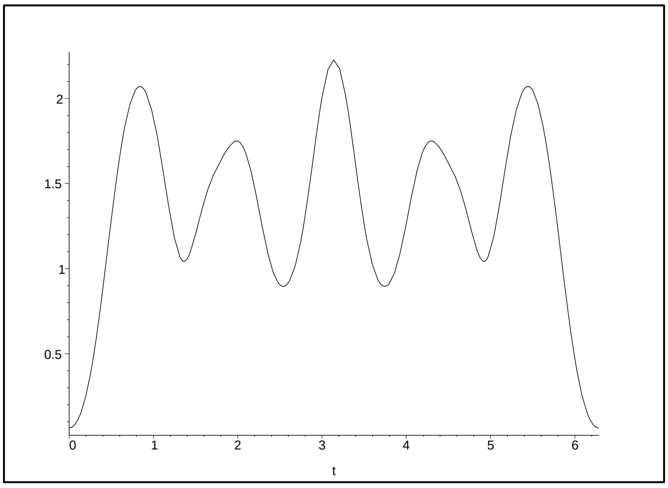

Figure 1: The graph of the function $2|q_5(x)| - |q_4(x)|$ where $|x| = 0.7$.

For all $k \geq 5$ and $x,\ |x| = 0.7$, we have

$$\begin{aligned}|q_k(x) - q_{k-1}(x)| &= |x^k q_{k-1}(x) + x^{2k-1} q_{k-2}(x)| = 0.7^k |q_{k-1}(x) + x^{k-1} q_{k-2}(x)| \\ &\leq 0.7^k |q_{k-1}(x)| + 0.7^{2k-1} |q_{k-2}(x)| \leq 0.7^k |q_{k-1}(x)| + 0.7^{2k-1} 2 |q_{k-1}(x)| \\ &\leq 0.7^k (1 + 2 \cdot 0.7^{k-1}) |q_{k-1}(x)| \leq |q_{k-1}(x)|.\end{aligned}$$

Thus by Rouche's theorem (see for example [6]), the number of zeros of the polynomial $q_k(x)$ on the disc $|x| < 0.7$ is the same as the number of zeros of the polynomial $q_{k-1}(x)$ on the disc $|x| < 0.7$. By induction and the fact that $q_k(x)$, $k = 2, 3, 4$, has a unique zero on the disc $|x| < 0.7$, we deduce that $q_k(x)$ has exactly one zero on the disc $|x| < 0.7$. Note that the unique zero of $q_k(x)$ on the disc $|x| < 0.7$ is also the dominant pole of the rational function $EB_k(x, 1)$. Finally, since the coefficients of $EB_k(x, 1)$ are non-negative, Pringsheim's Theorem ([4], Theorem $IV.6$) implies that this dominant pole is real and positive. □

Denote the positive zero of $q_k(x)$ in the domain $|x| < 0.7$ by x_k. From the recurrence relations (3) and (4) we see that $EB_k(x, 1)$ and $E_k(x, 1)$ have the same denominator polynomial $q_k(x)$ when expressed as rational functions, and the denominator of $C_k(x, 1)$ is, by Theorem 2.3, a product of the polynomials $q_i(x)$ for $1 \leq i \leq k$. By Lemma 2.5 the denominators $q_k(x)$ of $E_k(x, 1)$ (or, equivalently of $EB_k(x, 1)$), have a dominant positive real zero x_k which is simple and with nonzero numerators at this point, for any fixed $k \geq 2$. Now $[x^n]EB_{k+1}(x, 1) \geq [x^n]EB_k(x, 1)$ shows that $x_k \leq x_{k-1}$; but the case $x_k = x_{k-1}$ would imply by Lemma 2.5 that $q_k(x)$ and $q_{k-1}(x)$ have a common zero, which would also be a zero of $q_{k-2}(x)$ (from (5)). Thus by induction, this zero would also be a zero of $q_2(x)$ which gives a contradiction. Hence, since $x_i < x_{i-1}$ for $1 \leq i \leq k$, x_k is also the dominant zero of $C_k(x, 1)$. It follows that $C_k(x, 1)$ has a simple pole at $x = x_k$, from which we find that $[x^n]C_k(x, 1) = c_k(n) \sim A_k \rho_k^n$, where $\rho_k = 1/x_k$ and

$$A_k = -\frac{1}{\rho_k} \lim_{x \to x_k} (x - x_k) C_k(x, 1).$$

Since for $k \geq 2$, $c_k(n)$ is majorized by the number of all compositions 2^{n-1}, and $c_k(n)$ is minorized by $c_2(n)$, the number of compositions of n into parts in $\{1,2\}$, we see that $\frac{\sqrt{5}+1}{2} \leq \rho_k \leq 2$. Table 3 gives the values of A_k and ρ_k for $k = 1, 2, \cdots, 10$.

k	A_k	ρ_k
1	1.0000000000000	1.0000000000000000000
2	0.7236067977499	1.6180339887498948482
3	0.7440162935258	1.6663019373129334689
4	0.7602926572950	1.6681776473625616534
5	0.7623308902084	1.6682019659165787194
6	0.7624337682585	1.6682020668770706106
7	0.7624366320296	1.6682020670183924214
8	0.7624366795734	1.6682020670184611046
9	0.7624366800475	1.6682020670184611164
10	0.7624366800503	1.6682020670184611164

Table 3: Values of A_k and ρ_k for $k = 1, \cdots, 10$.

We compare the asymptotic estimate $A_k \rho_k^n$ with the exact value of $c_k(n)$ for $n = 50$ below, see Table 4.

k	$A_k \rho_k^{50}$	$c_k(50)$
1	1.000000000	1
2	$2.036501107 \times 10^{10}$	20365011074
3	$9.104737076 \times 10^{10}$	91047370756
4	$9.842279096 \times 10^{10}$	98422791219
5	$9.875860598 \times 10^{10}$	98758609330
6	$9.877223253 \times 10^{10}$	98772238776
7	$9.877260395 \times 10^{10}$	98772611178
8	$9.877261011 \times 10^{10}$	98772617594
9	$9.877261017 \times 10^{10}$	98772617723
10	$9.877261017 \times 10^{10}$	98772617738

Table 4: Values of $A_k \rho_k^{50}$ and $c_k(50)$ for $k = 1, \cdots, 10$.

In addition as $2 \geq \rho_{k+1} > \rho_k$ for each $k \geq 2$ it follows that $\rho = \lim_{k \to \infty} \rho_k$ exists. Also as $k \to \infty$, $A_k \to A$, where via numerical computations we find $A = 0.76243668005040263843980 67$ and $\rho = 1.6682020670184611163610704$. For $n = 50$ this gives an estimate of $9.877261017 \times 10^{10}$ for the number of smooth compositions, as compared with the exact value of 98772617767.

In summary we have proved the following assertion.

THEOREM 2.6 *The number $c_n(n)$ of smooth compositions of n is asymptotically given by $A\rho^n$ as $n \to \infty$, where numerically, $A = 0.7624366800504026384398067$ and $\rho = 1.6682020670184611163610704$.*

3 Application to Smooth words

As an immediate application of the results of the previous section, we obtain explicit formulas for several families of smooth words, including the results of [10].

3.1 Smooth words of the form $(k+1)\sigma(k+1)$

As a corollary of Lemma 2.1 we deduce the generating function for the number of words σ of length n over $[k]$ such that $(k+1)\sigma(k+1)$ is a smooth word. Lemma 2.1 gives that

$$EB_k(1,t) = 1 + \frac{t}{1 - tEB_{k-1}(1,t)},$$

with $EB_0(1,t) = 1$. Using the fact that Chebyshev polynomials $U_s(x)$ of the second kind satisfy the recurrence relation

$$U_s(x) = 2xU_{s-1}(x) - U_{s-2}(x), \tag{6}$$

with the initial conditions $U_0(x) = 1$ and $U_1(x) = 2x$, we obtain

$$EB_k(1,t) = \frac{U_k\left(\frac{1-t}{2t}\right) - tU_{k+1}\left(\frac{1-t}{2t}\right)}{tU_k\left(\frac{1-t}{2t}\right)} = \frac{1}{t} - \frac{U_{k+1}\left(\frac{1-t}{2t}\right)}{U_k\left(\frac{1-t}{2t}\right)},$$

which is equivalent to the identity

$$EB_k(1,t) = 1 + \frac{U_{k-1}\left(\frac{1-t}{2t}\right)}{U_k\left(\frac{1-t}{2t}\right)}. \tag{7}$$

COROLLARY 3.1 *The number of words σ of length n, $n \geq 2$, over the alphabet $[k]$ such that $(k+1)\sigma(k+1)$ is a smooth word is given by*

$$\frac{2}{k+1}\sum_{j=1}^{k} \sin^2\left(\frac{j\pi}{k+1}\right)\left(1 + 2\cos\left(\frac{j\pi}{k+1}\right)\right)^{n-1}.$$

Proof. From [10, Lemma 2.3] we get that

$$\frac{U_{k-1}(x)}{U_k(x)} = \frac{1}{k+1}\sum_{j=1}^{k} \frac{\sin^2\left(\frac{j\pi}{k+1}\right)}{x - \cos\left(\frac{j\pi}{k+1}\right)}.$$

Therefore, by (7) the coefficient of t^n, $n \geq 2$, in the generating function $EB_k(1,t)$ is given by

$$\begin{aligned}[t^n]EB_k(1,t) &= [t^n]\frac{2t}{k+1}\sum_{j=1}^{k} \frac{\sin^2\left(\frac{j\pi}{k+1}\right)}{1 - t\left(1 + 2\cos\left(\frac{j\pi}{k+1}\right)\right)} \\ &= \frac{2}{k+1}\sum_{j=1}^{k} \sin^2\left(\frac{j\pi}{k+1}\right)\left(1 + 2\cos\left(\frac{j\pi}{k+1}\right)\right)^{n-1},\end{aligned}$$

as claimed. □

This corollary implies that the number, asymptotically, of words σ of length n over $[k]$ such that $(k+1)\sigma(k+1)$ is a smooth word is given by $\frac{2}{k+1}\sin^2\left(\frac{\pi}{k+1}\right)\left(1 + 2\cos\left(\frac{\pi}{k+1}\right)\right)^{n-1}$.

3.2 Smooth words of the form $\sigma(k+1)$

We apply Lemma 2.2 together with (7) to get

$$\begin{aligned}E_k(1,t) &= 1+\sum_{j=1}^{k}\frac{t^{k-j+1}}{\prod_{i=j}^{k}(1-tEB_{i-1}(1,t))} = 1+\sum_{j=1}^{k}\frac{1}{\prod_{i=j}^{k}\frac{U_i\left(\frac{1-t}{2t}\right)}{U_{i-1}\left(\frac{1-t}{2t}\right)}}\\ &= 1+\sum_{j=1}^{k}\frac{U_{j-1}\left(\frac{1-t}{2t}\right)}{U_k\left(\frac{1-t}{2t}\right)} = \frac{\sum_{j=0}^{k}U_j\left(\frac{1-t}{2t}\right)}{U_k\left(\frac{1-t}{2t}\right)}\\ &= \frac{t}{1-3t}\frac{U_{k+1}\left(\frac{1-t}{2t}\right)-U_k\left(\frac{1-t}{2t}\right)-1}{U_k\left(\frac{1-t}{2t}\right)},\end{aligned}$$

where the last equality[1] can be proved easily from the fact that

$$\sum_{j=0}^{p}\sin(jt) = \frac{\sin((p+1)t)(\cos(t)-1)+\sin(t)\cos((p+1)t)-\sin(t)}{2(\cos(t)-1)}.$$

Thus by (6) we have

$$E_k(1,t) = \frac{t}{1-3t}\frac{\frac{1-2t}{t}U_k\left(\frac{1-t}{2t}\right)-U_{k-1}\left(\frac{1-t}{2t}\right)-1}{U_k\left(\frac{1-t}{2t}\right)}. \tag{8}$$

Corollary 3.2 *The number of words σ of length n, $n \geq 1$, over the alphabet $[k]$ such that $\sigma(k+1)$ is a smooth word is given by*

$$\frac{2}{k+1}\sum_{j=1}^{k}(1+(-1)^{j+1})\cos^2\left(\frac{j\pi}{2(k+1)}\right)\left(1+2\cos\left(\frac{j\pi}{k+1}\right)\right)^{n-1}$$

Proof. Using [10, Lemma 2.3] and (8) we obtain

$$\begin{aligned}[t^n]E_k(1,t) &= [t^n]\frac{1-2t}{1-3t}-[t^n]\frac{t}{1-3t}\frac{U_{k-1}\left(\frac{1-t}{2t}\right)+1}{U_k\left(\frac{1-t}{2t}\right)}\\ &= 3^{n-1}-[t^{n-2}]\frac{1}{1-3t}\frac{2}{k+1}\sum_{j=1}^{k}\frac{(1+(-1)^{j+1})\sin^2\left(\frac{j\pi}{k+1}\right)}{1-t\left(1+2\cos\left(\frac{j\pi}{k+1}\right)\right)}.\end{aligned}$$

[1] The verification of such identities is *a priori* trivial and can be done by a computer, since, upon rewriting the trigonometric functions via Euler's formulæ, one only has to sum some finite geometric series.

Note that $\frac{1}{(1-3t)(1-\omega t)} = -\frac{3}{(1-3t)(\omega-3)} + \frac{\omega}{(1-\omega t)(\omega-3)}$ and $1+2\cos\frac{j\pi}{k+1} - 3 = -4\sin^2\frac{j\pi}{2(k+1)}$, thus

$$\begin{aligned}
&[t^n]E_k(1,t)\\
&= 3^{n-1} - [t^{n-2}]\frac{2}{k+1}\sum_{j=1}^{k}(1+(-1)^{j+1})\cos^2\left(\frac{j\pi}{2(k+1)}\right)\left[\frac{3}{1-3t} - \frac{1+2\cos\left(\frac{j\pi}{k+1}\right)}{1-t\left(1+2\cos\left(\frac{j\pi}{k+1}\right)\right)}\right]\\
&= 3^{n-1} - \frac{2}{k+1}\sum_{j=1}^{k}(1+(-1)^{j+1})\cos^2\left(\frac{j\pi}{2(k+1)}\right)\left[3^{n-1} - \left(1+2\cos\left(\frac{j\pi}{k+1}\right)\right)^{n-1}\right]
\end{aligned}$$

Using the identities $\frac{2}{k+1}\sum_{j=1}^{k}(1+(-1)^{j+1})\cos^2\left(\frac{j\pi}{2(k+1)}\right) = 1$, we get that

$$[t^n]E_k(1,t) = \frac{2}{k+1}\sum_{j=1}^{k}(1+(-1)^{j+1})\cos^2\left(\frac{j\pi}{2(k+1)}\right)\left(1+2\cos\left(\frac{j\pi}{k+1}\right)\right)^{n-1},$$

as claimed. □

This corollary implies that the number, asymptotically, of words σ of length n over $[k]$ such that $\sigma(k+1)$ is a smooth word is given by $\frac{4}{k+1}\cos^2\left(\frac{\pi}{2(k+1)}\right)\left(1+2\cos\left(\frac{\pi}{k+1}\right)\right)^{n-1}$.

3.3 All smooth words

As an application of Theorem 2.3 we get the following result.

COROLLARY 3.3 *(see [10, Theorem 2.2]) The generating function $C_k(1,t)$ for the number of smooth words of length n over the alphabet $[k]$ is given by*

$$1 + \frac{t(k-(3k+2)t)}{(1-3t)^2} + \frac{2t^2}{(1-3t)^2}\frac{1+U_{k-1}\left(\frac{1-t}{2t}\right)}{U_k\left(\frac{1-t}{2t}\right)}.$$

Moreover, the number of smooth words of length n over the alphabet $[k]$ is given by

$$\frac{1}{k+1}\sum_{j=1}^{k}(1+(-1)^{j+1})\cot^2\frac{j\pi}{2(k+1)}\left(1+2\cos\frac{j\pi}{k+1}\right)^{n-1}.$$

Proof. Setting $x=1$ and $y=t$ in Theorem 2.3, we have

$$C_k(1,t) = 1 + \sum_{j=1}^{k}\frac{t(E_{j-1}(x,y))^2}{1-tEB_{j-1}(1,t)}.$$

From (7) we obtain that

$$C_k(1,t) = 1 + \sum_{j=1}^{k}\frac{U_{j-1}\left(\frac{1-t}{2t}\right)(E_{j-1}(x,y))^2}{U_j\left(\frac{1-t}{2t}\right)},$$

and from (3) we have

$$\begin{aligned}
&C_k(1,t)\\
&= 1 + \frac{t^2}{(1-3t)^2}\sum_{j=1}^{k}\frac{\left[U_j\left(\frac{1-t}{2t}\right) - U_{j-1}\left(\frac{1-t}{2t}\right) - 1\right]^2}{U_j\left(\frac{1-t}{2t}\right)U_{j-1}\left(\frac{1-t}{2t}\right)}\\
&= 1 + \frac{t^2}{(1-3t)^2}\sum_{j=1}^{k}\frac{U_j^2\left(\frac{1-t}{2t}\right) + U_{j-1}^2\left(\frac{1-t}{2t}\right) + 1 - 2U_j\left(\frac{1-t}{2t}\right) + 2U_{j-1}\left(\frac{1-t}{2t}\right) - 2U_j\left(\frac{1-t}{2t}\right)U_{j-1}\left(\frac{1-t}{2t}\right)}{U_j\left(\frac{1-t}{2t}\right)U_{j-1}\left(\frac{1-t}{2t}\right)}\\
&= 1 + \frac{t^2}{(1-3t)^2}\left[\frac{k-(3k+2)t}{t} + \frac{U_{k-1}\left(\frac{1-t}{2t}\right)+2}{U_k\left(\frac{1-t}{2t}\right)} + \sum_{j=1}^{k}\frac{1}{U_j\left(\frac{1-t}{2t}\right)U_{j-1}\left(\frac{1-t}{2t}\right)}\right].
\end{aligned}$$

Using the fact that $\sum_{j=1}^{k}\frac{1}{U_j\left(\frac{1-t}{2t}\right)U_{j-1}\left(\frac{1-t}{2t}\right)} = \frac{U_{k-1}\left(\frac{1-t}{2t}\right)}{U_k\left(\frac{1-t}{2t}\right)}$, we obtain

$$C_k(1,t) = 1 + \frac{t(k-(3k+2)t)}{(1-3t)^2} + \frac{2t^2}{(1-3t)^2}\frac{1+U_{k-1}\left(\frac{1-t}{2t}\right)}{U_k\left(\frac{1-t}{2t}\right)},$$

as claimed. The second part now follows from this generating function. The details may be found in [10, Theorem 2.4]. □

Observe that the generating function for the number of smooth words of length n over $[k+1]$ such that the letter $k+1$ occurs exactly $\ell > 0$ times can be derived from (7) and (8) to be precisely

$$(E_k(1,t))^2(EB_k(1,t))^{\ell-1}.$$

Hence we obtain the following refinement of the foregoing corollary.

COROLLARY 3.4 *The generating function for the number of smooth words of length n over the alphabet $[k+1]$ such that the letter $k+1$ occurs exactly ℓ times is given by*

$$\left(\frac{1-2t}{1-3t} - \frac{t}{1-3t}\frac{U_{k-1}\left(\frac{1-t}{2t}\right)+1}{U_k\left(\frac{1-t}{2t}\right)}\right)^2\left(\frac{1-2t^2}{t} + \frac{U_{k-1}\left(\frac{1-t}{2t}\right)}{U_k\left(\frac{1-t}{2t}\right)}\right)^{\ell-1}.$$

Our previous results and this corollary imply that the number, asymptotically, of smooth words σ of length n over $[k+1]$ such that the letter $k+1$ occurs exactly ℓ times is given by

$$\frac{2^{\ell+4}}{(k+1)^{\ell+2}}\binom{n}{\ell}\cos^4\left(\frac{\pi}{2(k+1)}\right)\sin^{2\ell}\left(\frac{\pi}{k+1}\right)\left(1+2\cos\left(\frac{\pi}{k+1}\right)\right)^{n-2\ell-2}.$$

4 Smooth cyclic compositions

Let $D_k(n,m)$ be the number of smooth cyclic compositions of n with m parts in $[k]$, and let $d_k(n)$ be the number of smooth cyclic compositions of n with parts in $[k]$. Denote the generating function of $d_k(n,m)$ by $D_k(x,y)$.

$$D_k(x,y) = \sum_{n\geq 0}\sum_{m\geq 0} d_k(n,m)x^n y^m$$

In order to compute the generating function $D_k(x,y)$ we introduce the following notation: Let $M_k(x,y)$ be the generating function for the number of smooth cyclic compositions $\pi'(k+1)\pi''$ of n with m parts such that $\pi'\pi''$ is a word over $[k]$.

Let $\pi = \pi'(k+1)\pi''$ be any smooth cyclic composition such that $\pi'\pi''$ is a word over $[k]$. Then the maximal part decomposition of π has the form

$$\pi = \pi'^{(0)}k\pi'^{(1)}k\cdots\pi'^{(s')}k(k+1)k\pi''^{(s'')}\cdots k\pi''^{(1)}k\pi''^{(0)}$$

with $s', s'' \geq 0$. Thus, π is a smooth cyclic composition if and only if $\pi'^{(0)}k\pi''^{(0)}$ is a smooth cyclic composition, $k\pi'^{(j)}k$ ($j = 1, 2, \ldots, s'$) and $k\pi''^{(j)}k$ ($j = 1, 2, \ldots, s''$) are smooth compositions, and $\pi'^{(j)}$ and $\pi''^{(j)}$ are words over $[k-1]$ for all j. Now, if we consider the cases where $\pi' = \pi'' = \emptyset$, $\pi' = \emptyset$ and $\pi'' \neq \emptyset$, $\pi'' = \emptyset$ and $\pi' \neq \emptyset$, $\pi', \pi'' \neq \emptyset$, it follows from Lemma 2.1 that the function $M_k(x,y)$ satisfies

$$M_k(x,y) = x^{k+1}y + 2x^{k+1}y\frac{x^k y}{1 - x^k yEB_{k-1}(x,y)} + x^{k+1}y\frac{x^k y}{(1 - x^k yEB_{k-1}(x,y))^2}M_{k-1}(x,y),$$

which is equivalent to

$$M_k(x,y) = x^{k+1}y + \frac{2x^{2k+1}y^2}{1 - x^k yEB_{k-1}(x,y)} + \frac{x^{2k+1}y^2}{(1 - x^k yEB_{k-1}(x,y))^2}M_{k-1}(x,y), \tag{9}$$

with the initial condition $M_0(x,y) = xy$.

LEMMA 4.1 *The generating function $M_k(x,y)$ is given by*

$$\frac{x^{k(k+2)}y^k}{\prod_{i=1}^k (1 - x^i yEB_{i-1}(x,y))^2} + \sum_{j=1}^k \left(1 + \frac{2x^j y}{1 - x^j yEB_{j-1}(x,y)}\right)\frac{x^{k(k+2)-j(j+1)}y^{k-j+1}}{\prod_{i=j+1}^k (1 - x^i yEB_{i-1}(x,y))^2},$$

where $EB_k(x,y)$ is given by Lemma 2.1. Alternatively,

$$M_k(x,y) = \frac{x^{k(k+2)}y^k}{q_k^2(x,y)}\left[1 + \sum_{j=1}^k x^{-j(j+1)}y^{-j+1}q_j^2(x,y) + 2\sum_{j=1}^k x^{-j^2}y^{-j+2}q_j(x,y)q_{j-1}(x,y)\right].$$

Proof. Iterating (9), we get the first expression for $M_k(x,y)$. Then by Lemma 2.4 and (4) we obtain the second expression. □

Now we find an expression for the generating function $D_k(x,y)$. Using the *d-maximal part decomposition* of a smooth cyclic composition π, we get that π can also be written as

$$\pi = \pi^{(0)}k\pi^{(1)}k\cdots\pi^{(s)}k\pi^{(s+1)},$$

where $k\pi^{(j)}k$ is a smooth composition, $\pi^{(0)}k\pi^{(s+1)}$ is a smooth cyclic composition, and $\pi^{(j)}$ has parts in $[k-1]$, for all j. Thus, the generating function $D_k(x,y)$ satisfies the relation

$$D_k(x,y) = D_{k-1}(x,y) + \frac{M_{k-1}(x,y)}{1 - x^k yEB_{k-1}(x,y)}. \tag{10}$$

Thus iterating (10) we obtain the first expression in the following theorem. Using Lemma 4.1 together with Lemma 2.4 we obtain the second expression.

THEOREM 4.2 *The generating function $D_k(x,y)$ for the number of smooth cyclic compositions of n with m parts in $[k]$ is given by*

$$D_k(x,y) = 1 + \sum_{j=1}^{k} \frac{M_{j-1}(x,y)}{1 - x^j y EB_{j-1}(x,y)},$$

where $EB_k(x,y)$ and $M_k(x,y)$ are given by Lemma 2.1 and Lemma 4.1, respectively. Alternatively, $D_k(x,y)$ is given by

$$1 + \sum_{j=1}^{k} \frac{x^{j^2-1}y^{j-1}}{q_j(x,y)q_{j-1}(x,y)} \left[1 + \sum_{i=1}^{j-1} x^{-i(i+1)}y^{-i+1}q_i^2(x,y) + 2\sum_{i=1}^{j-1} x^{-i^2}y^{-i+2}q_i(x,y)q_{i-1}(x,y)\right].$$

For instance, the generating function $D_k(x,y)$, $k = 1,2,3,4$, is shown below

- $D_1(x,1) = \dfrac{1}{1-x}$,
- $D_2(x,1) = \dfrac{1}{1-x-x^2}$,
- $D_3(x,1) = \dfrac{1-x^4-2x^6}{(1-x)(1-x^2-2x^3-x^4-x^5)}$,
- $D_4(x,1) = \dfrac{1-x^4-x^5-3x^6-2x^9+3x^{10}}{1-x-x^2-x^3+x^5+2x^6+x^9-x^{10}}$.

4.1 Asymptotics for the number of smooth cyclic compositions of n

As before we denote the positive zero of $q_k(x)$ in the domain $|x| < 0.7$ by x_k, and observe from Theorem 4.2 that the denominator of $D_k(x,1)$ is a product of the polynomials $q_i(x)$ for $1 \le i \le k$.

k	B_k	ρ_k
1	1.00000000000000	1.0000000000000000000
2	0.72360679774997	1.6180339887498948482
3	0.66413814247853	1.6663019373129334689
4	0.65850941719622	1.6681776473625616534
5	0.6583712284018	1.6682019659165787194
6	0.65837031233556	1.6682020668770706106
7	0.65837031047395	1.6682020670183924214
8	0.65837031047271	1.6682020670184611046
9	0.65837031047271	1.6682020670184611164
10	0.65837031047271	1.6682020670184611164

Table 5: Values of B_k and ρ_k for $k = 1, \cdots, 10$.

The same line of reasoning as in the case of general smooth compositions shows that x_k is the dominant zero of $D_k(x,1)$. Hence $D_k(x,1)$ has a simple pole at $x = x_k$, from which we find that $[x^n]D_k(x,1) = d_k(n) \sim B_k \rho_k^n$ where $\rho_k = 1/x_k$ and

$$B_k = -\frac{1}{\rho_k} \lim_{x \to x_k} (x - x_k) D_k(x,1).$$

Table 5 shows the values of B_k and ρ_k for $k = 1, 2, \cdots, 10$.

The asymptotic estimate $B_k \rho_k^n$ and the exact value of $d_k(n)$ are compared for $n = 50$, see Table 6.

k	$B_k \rho_k^{50}$	$[x^{50}]D_k(x,1)$
1	1.000000000	1
2	$2.036501107 \times 10^{10}$	20365011074
3	$8.127245629 \times 10^{10}$	81272456282
4	$8.524656142 \times 10^{10}$	85246564557
5	$8.529081737 \times 10^{10}$	85290831724
6	$8.529095679 \times 10^{10}$	85290974920
7	$8.529095691 \times 10^{10}$	85290975524
8	$8.529095691 \times 10^{10}$	85290975604
9	$8.529095691 \times 10^{10}$	85290975625
10	$8.529095691 \times 10^{10}$	85290975626

Table 6: Values of $B_k \rho_k^{50}$ and $d_k(50)$ for $k = 1, \cdots, 10$.

We know from previous analysis in section 2.1 that $\rho = \lim_{k \to \infty} \rho_k$ exists and has the numerical value $\rho = 1.6682020670184611163610704$. Also as $k \to \infty$, $B_k \to B$, where we obtain by numerical computations $B = 0.65837031047271348106443473 52$. For $n = 50$ this gives an estimate of $8.529095691 \times 10^{10}$ for the number of smooth cyclic compositions, as compared with the exact value of 85290975647.

In summary we have established the theorem

THEOREM 4.3 *The number of smooth cyclic compositions of* n *is asymptotically given by* $B\rho^n$ *as* $n \to \infty$*, where numerically,* $B = 0.6583703104727134810644347352$ *and* $\rho = 1.6682020670184611163610704$.

5 Application to smooth cyclic words

In this section we apply our results on smooth cyclic compositions to derive enumeration formulas for some classes of smooth cyclic words.

5.1 Smooth cyclic words of the form $\sigma'(k+1)\sigma''$

COROLLARY 5.1 *The generating function $M_k(1,t)$ for the number of smooth cyclic words π of length n over the alphabet $[k+1]$ such that π contains the letter $k+1$ exactly once is given by*

$$t+\frac{t^2}{(1+t)(1-3t)}\left[2-\frac{2(k+1)}{U_k^2\left(\frac{1-t}{2t}\right)}-\frac{(1+3t)U_{k-1}\left(\frac{1-t}{2t}\right)}{U_k\left(\frac{1-t}{2t}\right)}\right].$$

Proof. From (9) we have

$$M_k(x,y)=t+\frac{2t^2}{1-tEB_{k-1}(1,t)}+\frac{t^2}{(1-tEB_{k-1}(1,t))^2}M_{k-1}(1,t),$$

then, (7) gives

$$M_k(1,t)=t+\frac{2tU_{k-1}\left(\frac{1-t}{2t}\right)}{U_k\left(\frac{1-t}{2t}\right)}+\frac{U_{k-1}^2\left(\frac{1-t}{2t}\right)}{U_k^2\left(\frac{1-t}{2t}\right)}M_{k-1}(1,t),$$

with $M_k(1,t)=t$. Iterating this recurrence we obtain

$$M_k(1,t)=t+\frac{t}{U_k^2\left(\frac{1-t}{2t}\right)}\left[\sum_{j=0}^{k-1}U_{j-1}^2\left(\frac{1-t}{2t}\right)+2\sum_{j=0}^{k-1}U_j\left(\frac{1-t}{2t}\right)U_{j+1}\left(\frac{1-t}{2t}\right)\right].$$

Applying the identities

$$\sum_{j=0}^{k}U_j^2\left(\frac{1-t}{2t}\right)=\frac{t^2U_{k+1}\left(\frac{1-t}{2t}\right)\left[U_{k+1}\left(\frac{1-t}{2t}\right)-U_{k-1}\left(\frac{1-t}{2t}\right)\right]-2(k+2)t^2}{(1+t)(1-3t)}$$

and

$$2\sum_{j=0}^{k}U_j\left(\frac{1-t}{2t}\right)U_{j+1}\left(\frac{1-t}{2t}\right)=\frac{1-t}{t}\sum_{j=0}^{k}U_j^2\left(\frac{1-t}{2t}\right)+U_k\left(\frac{1-t}{2t}\right)U_{k+1}\left(\frac{1-t}{2t}\right),$$

we get that

$$\begin{aligned}&M_k(1,t)\\&=t+\frac{t}{U_k^2\left(\frac{1-t}{2t}\right)}\left[\frac{tU_k\left(\frac{1-t}{2t}\right)\left(U_k\left(\frac{1-t}{2t}\right)-U_{k-2}\left(\frac{1-t}{2t}\right)\right)-2(k+1)t}{(1+t)(1-3t)}+U_{k-1}\left(\frac{1-t}{2t}\right)U_k\left(\frac{1-t}{2t}\right)\right].\end{aligned}$$

Thus on using (6) we obtain the desired result. □

5.2 The theorem on smooth cyclic words

Using Corollary 5.1, (2) and (7) we get the following result, which is [10, Theorem 3.2]. We omit some details in the proof.

THEOREM 5.2 *The generating function $D_k(1,t)$ for the number of smooth cyclic words of length n over alphabet $[k]$ is given by*

$$1+\frac{kx(1+3x)}{(1+x)(1-3x)}-\frac{2(k+1)x}{(1+x)(1-3x)}\frac{U_{k-1}\left(\frac{1-x}{2x}\right)}{U_k\left(\frac{1-x}{2x}\right)}.$$

Proof. Notice that (2) and (7) together imply

$$D_k(1,t)=1+\sum_{j=1}^{k}\frac{U_{j-1}\left(\frac{1-t}{2t}\right)}{U_j\left(\frac{1-t}{2t}\right)}M_{j-1}(1,t).$$

Using Corollary 5.1 with the identity

$$\sum_{j=1}^{k}\frac{U_{j-2}\left(\frac{1-t}{2t}\right)}{U_j\left(\frac{1-t}{2t}\right)}=-k+\frac{1-t}{t}\sum_{j=1}^{k}\frac{U_{j-1}\left(\frac{1-t}{2t}\right)}{U_j\left(\frac{1-t}{2t}\right)},$$

we obtain that

$$D_k(1,t)=1+\frac{k(1+3t)t}{(1+t)(1-3t)}-\frac{2t}{(1+t)(1-3t)}\left[\sum_{j=1}^{k}\frac{U_{j-1}^2\left(\frac{1-t}{2t}\right)+j}{U_j\left(\frac{1-t}{2t}\right)U_{j-1}\left(\frac{1-t}{2t}\right)}\right].$$

Therefore, by the identity

$$\sum_{j=1}^{k}\frac{U_{j-1}^2\left(\frac{1-t}{2t}\right)+j}{U_{j-1}\left(\frac{1-t}{2t}\right)U_j\left(\frac{1-t}{2t}\right)}=\frac{(k+1)U_{k-1}\left(\frac{1-t}{2t}\right)}{U_k\left(\frac{1-t}{2t}\right)},$$

we obtain the desired result. □

We remark that Theorem 5.2 implies the formula for the number of smooth cyclic words stated earlier (see equation (1)).

Lastly, we state a generalization of Theorem 5.2 below (a proof uses the ℓ-maximal part decomposition of a smooth cyclic word over $[k]$, together with (7) and Corollary (5.1)).

COROLLARY 5.3 *The generating function for the number of smooth cyclic words π of length n over the alphabet $[k]$ such that π contains the letter k exactly ℓ times is given by*

$$t^{\ell-1}EB_{k-1}(1,t)^{\ell-1}M_{k-1}(1,t),$$

where the generating functions $EB_k(1,t)$ and $M_k(1,t)$ are given by (7) and Corollary 5.1.

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