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231-avoiding permutations and the Schensted correspondence

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Abstract. The Schensted correspondence gives a bijection between a permutation $\sigma \in S_n$ and a pair of Young tableaux of the same shape λ, which is a partition of n. Given a permutation σ, the corresponding pair of tableaux may be computed by the well-known Robinson-Schensted-Knuth algorithm. We present generating functions for 231-avoiding permutations that contain information about the entire shape of the Schensted correspondence. In particular, this provides information about the longest unions of any number of disjoint increasing or disjoint decreasing subsequences. We also extend this generating function to count the number of consecutive occurrences of certain patterns of length 3.

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1 Introduction

1.1 Permutation Patterns and Pattern Avoidance

Let $\sigma \in S_n$ and $\tau \in S_k$ be permutations. We say that a subsequence $\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_k}$ with $i_1 < i_2 < \cdots < i_k$ is an *occurrence* of the pattern τ in σ when $\sigma_{i_a} < \sigma_{i_b}$ if and only if $\tau_a < \tau_b$ for all $a, b \leq k$. If $i_1, i_2, \ldots, i_k$ are consecutive integers, we call the subsequence a *consecutive occurrence*. We say that σ *avoids* τ if there are no occurrences of τ in σ. For example, in $\sigma = 42153$, 153 and 253 are occurrences of 132, with 153 being a consecutive occurrence, while σ avoids 123. Define $S_n(\tau) \subseteq S_n$ to be the set of permutations that avoid τ.

For $\sigma \in S_n$ and $\tau \in S_m$, define $\sigma \oplus \tau$ to be the permutation $\mu \in S_{n+m}$ such that $\mu_i = \sigma_i$ for $i \leq n$ and $\mu_i = n + \tau_{i-n}$ for $i > n$. Define $\sigma \ominus \tau$ to be the permutation $\nu \in S_{n+m}$ such that $\nu_i = m + \sigma_i$ for $i \leq n$ and $\nu_i = \tau_{i-n}$ for $i > n$. For example, if $\sigma = 42153$ and $\tau = 132$, then $\sigma \oplus \tau = 42153687$ and $\sigma \ominus \tau = 75486132$.

For $\sigma \in S_n(132)$ with $n > 0$, it is easy to see there exist $\mu \in S_j(132)$ and $\nu \in S_k(132)$ for some $j, k \geq 0$ such that $\sigma = \mu \ominus (\nu \oplus 1)$ and that this decomposition is unique. Similarly, for 213-avoiding σ, μ, and ν, $\sigma = \mu \ominus (1 \oplus \nu)$ is unique. For 231-avoiding σ, μ, and ν, $\sigma = \mu \oplus (1 \ominus \nu)$ is unique. Finally, for 312-avoiding σ, μ, and ν, $\sigma = \mu \oplus (\nu \ominus 1)$ is unique. For ease of notation, in this paper we derive generating functions based on the decomposition of 231-avoiding permutations with the understanding that these other three types permit analogous derivations.

1.2 The Schensted Correspondence

The Schensted correspondence gives a bijection between a permutation $\sigma \in S_n$ and a pair of Young tableaux of the same shape λ, which is a partition of n. Given a permutation σ, the corresponding pair of tableaux may be computed by the well-known Robinson-Schensted-Knuth [4] algorithm. Given a partition λ of n, let λ' be the partition such that λ'_i is the number of parts of λ with size at least i. Schensted [5] showed that λ_1 is the length of the longest increasing subsequence in σ and that λ'_1 is the length of the longest decreasing subsequence in σ. Greene [1] later showed that $\lambda_1 + \lambda_2 + \cdots + \lambda_i$ is the size of the largest union of i disjoint increasing subsequences in σ and that $\lambda'_1 + \lambda'_2 + \cdots + \lambda'_i$ is the size of the largest union of i disjoint decreasing subsequences in σ.

1.3 Background

A great deal of work has been published on the enumeration of pattern-avoiding permutations. Knuth [2] showed in 1968 that the 231-avoiding permutations of length n are counted by the Catalan numbers. Since then, generating functions have been found to enumerate these permutations along with many other statistics. The most notable example, as it relates to this paper, is Reifegerste's [3] formula for the number of $\sigma \in S_n(132)$ with longest decreasing subsequence of length k. The generating functions in this paper extend that result.

2 Generating Functions

For $\sigma \in S_n$, define $\lambda(\sigma)$ to be the partition of n corresponding to σ in the Schensted correspondence. Then, it is easy to see that $\lambda(\sigma \oplus \tau)_i = \lambda(\sigma)_i + \lambda(\tau)_i$ and that $\lambda(\sigma \ominus \tau)'_i = \lambda(\sigma)'_i + \lambda(\tau)'_i$ for all i and any permutations σ and τ.

Define the *weight* of a partition λ, denoted $w(\lambda)$, by $w(\lambda) = \prod_i x_i^{\lambda_i}$. Note that if you substitute $x_i = x_i(y_i/y_{i-1})$ with $y_0 = 1$, then the exponent on y_i counts the number of parts of size i in λ'. Define the *inversion number* of a permutation σ, denoted $\text{inv}(\sigma)$, to be the number of occurrences of the pattern 21 in σ. Define $S_{n,k}(231)$ to be the set of $\sigma \in S_n(231)$ such that $\lambda(\sigma)_i = 0$ for all $i > k$. The following theorem gives the generating function for permutations σ such that $\lambda(\sigma)$ has at most k parts by the shape of $\lambda(\sigma)$, the inversion number, and the length.

THEOREM 2.1 *Let $f_k(z)$, short for $f_k(\mathbf{x}, q, z)$, be defined by*

$$f_k(z) = \sum_{n \geq 0} \sum_{\sigma \in S_{n,k}(231)} w(\lambda(\sigma)) q^{\text{inv}(\sigma)} z^n.$$

Then, $f_k(z)$ satisfies the following recurrence.

$$f_k(z) = \frac{1}{1 - x_1 z - \sum_{j=1}^{k-1} x_{j+1} z (f_j(zq) - f_{j-1}(zq))}, k > 0$$

$$f_0(z) = 1$$

Proof. For a given $\sigma \in S_{n,k}(231)$ with $n > 0$, let μ, ν be the unique permutations such that $\sigma = \mu \oplus (1 \ominus \nu)$. Clearly, both μ and ν avoid 231. Note that $\lambda(1 \ominus \nu)$ is the same as $\lambda(\nu)$ with an additional part of size 1. Therefore, if $\nu \in S_{i,j}(231) \setminus S_{i,j-1}(231)$ for $0 \le j < k$, then $w(\lambda(1 \ominus \nu)) = x_{j+1} w(\lambda(\nu))$ and $\operatorname{inv}(1 \ominus \nu) = i + \operatorname{inv}(\nu)$. Clearly, $w(\lambda(\mu \oplus (1 \ominus \nu))) = w(\lambda(\mu)) w(\lambda(1 \ominus \nu))$ and $\operatorname{inv}(\mu \oplus (1 \ominus \nu)) = \operatorname{inv}(\mu) + \operatorname{inv}(1 \ominus \nu)$. We then see that $f_k(z)$ satisfies the following functional equation:

$$f_k(z) = 1 + f_k(z) \left(x_1 z + \sum_{j=1}^{k-1} x_{j+1} z (f_j(zq) - f_{j-1}(zq)) \right)$$

Solve for $f_k(z)$ to complete the proof. □

Note that, despite counting inversions, $f_k(z)$ is a rational function for all k. The following corollary gives the generating function without the restriction to $\lambda(\sigma)$ having at most k parts.

Corollary 2.2 *Let $f(z)$, short for $f(\mathbf{x}, q, z)$, be defined by*

$$f(z) = \sum_{n \ge 0} \sum_{\sigma \in S_n(231)} w(\lambda(\sigma)) q^{\operatorname{inv}(\sigma)} z^n.$$

Then,

$$f(z) = \frac{1}{1 - x_1 z - \sum_{j \ge 1} x_{j+1} z (f_j(zq) - f_{j-1}(zq))}.$$

Proof. Take the limit as $k \to \infty$ in Theorem 2.1. □

Define $w_r(\lambda) = \prod_{i=1}^{r} x_i^{\lambda_i}$. The following corollary gives the generating function that tracks the first r parts of $\lambda(\sigma)$.

Corollary 2.3 *Let $f^{(r)}(z)$, short for $f^{(r)}(\mathbf{x}, q, z)$, be defined by*

$$f^{(r)}(z) = \sum_{n \ge 0} \sum_{\sigma \in S_n(231)} w_r(\lambda(\sigma)) q^{\operatorname{inv}(\sigma)} z^n.$$

Then,

$$f^{(r)}(z) = \frac{1}{1 - x_1 z - z(f^{(r)}(zq) - f_{r-1}(zq)) - \sum_{j=1}^{r-1} x_{j+1} z (f_j(zq) - f_{j-1}(zq))}.$$

Proof. In Theorem 2.1, let $x_i = 1$ for all $i > r$, then take the limit as $k \to \infty$. □

In particular, note that $f^{(r)}(z)$ can be expressed as a continued fraction and that, when $q = 1$, $f^{(r)}(z)$ is the solution to a quadratic equation.

The decomposition of 231-avoiding permutations allows us to generalize the above results to count the number of consecutive occurrences of some patterns. For $\tau \in S_m$ and $\sigma \in S_n$, let $\tau(\sigma)$ denote the number of times τ occurs as a consecutive pattern in σ. The following theorem generalizes Theorem 2.1 by counting the number of consecutive occurrences of 132 and 312 in σ.

Theorem 2.4 *Let $f_k(z)$, short for $f_k(\mathbf{x}, y_1, y_2, q, z)$, be defined by*

$$f_k(z) = \sum_{n \ge 0} \sum_{\sigma \in S_{n,k}(231)} w(\lambda(\sigma)) y_1^{132(\sigma)} y_2^{312(\sigma)} q^{\operatorname{inv}(\sigma)} z^n.$$

Then, $f_k(z)$ satisfies the following recurrence.

$$f_k(z) = \frac{1+\sum_{j=1}^{k-1} x_{j+1}(1-y_1)z(1+(y_2-1)z)(f_j(zq)-f_{j-1}(zq))}{1-x_1z+\sum_{j=1}^{k-1} x_{j+1}y_1z(1+(y_2-1)z)(f_j(zq)-f_{j-1}(zq))}, k>0$$
$$f_0(z) = 1.$$

Proof. Proceed as in the proof of Theorem 2.1. Note that $132(\mu\oplus(1\ominus\nu)) = 132(\mu)+132(\nu)$, except there is one additional consecutive occurrence if both μ and ν are non-empty. Similarly, $312(\mu\oplus(1\ominus\nu)) = 312(\mu)+312(\nu)$, except there is one additional consecutive occurrence if ν has size at least 2 and $\nu_1 < \nu_2$. However, 231-avoiding permutations ν with $\nu_1<\nu_2$ must decompose as $\nu = 1\oplus\rho$, so such permutations are easy to count. Modifying the functional equation obtained in Theorem 2.1, we obtain

$$f_k(z) = 1+\left(x_1z+\sum_{j=1}^{k-1} x_{j+1}z(1+y_2z-z)(f_j(zq)-f_{j-1}(zq))\right)$$
$$+(f_k(z)-1)\left(x_1z+\sum_{j=1}^{k-1} x_{j+1}y_1z(1+y_2z-z)(f_j(zq)-f_{j-1}(zq))\right).$$

Solve for $f_k(z)$ to complete the proof. □

It is clear that we may obtain similar corollaries to this result. In particular, the numerator in the recurrence for $f_k(z)$ may be cleared of all f_j terms, so $f^{(r)}(z)$ may still be written as a continued fraction. With $q=1$, we can still solve for $f^{(r)}(z)$ as the solution to a quadratic equation.

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