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Six bijections between deco polyominoes and permutations

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To the memory of Alberto Del Lungo (1965–2003)

Abstract. In this paper we establish six bijections between a particular class of polyominoes, called deco polyominoes, enumerated according to their directed height by $n!$, and permutations. Each of these bijections allows us to establish different correspondences between classical statistics on deco polyominoes and on permutations.

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1 Introduction

Deco polyominoes were introduced in [3] with the aim to find a class of polyominoes counted by the factorial.

In this study we establish six different bijections between deco polyominoes and permutations giving correspondences between certain statistics on deco polyominoes and on permutations.

In Section 2 we recall some definitions and properties regarding deco polyominoes and permutations. Sections 3, 4, 5, 6, 7, and 8 describe and exemplify the six bijection we establish between deco polyominoes and permutations, pointing out correspondences between certain statistics defined on them.

2 Notations and Definitions

We introduce some notations and definitions regarding deco polyominoes and permutation, respectively.

2.1 Deco polyominoes

In the $\Re^2$ plane, a *cell* is a unitary square $[i, i+1] \times [j, j+1]$, $i, j \in \aleph$, and a *polyomino* is a connected set of pairs of cells having one side in common. Polyominoes are defined up to a translation. We can obtain a *directed* polyomino by starting out from a cell, called *source*, and by adding other cells in predetermined directions, such as East and North, that is, to the right of or over existing cells. In this way, a polyomino grows in a *privileged direction.* A *column* (resp. *row)* is the intersection of a polyomino with an infinite vertical (resp. horizontal) strip $[i, i+1] \times \Re$ (resp. $\Re \times [j, j+1]$). A *directed column-convex* polyomino is a directed polyomino whose columns are connected (see Figure 1 a)). Finally, the *directed height* of a directed polyomino is the number of lines orthogonal to the privileged direction that go through the cell center (from here on we call it simply the *height*), while its *vertical height* is the number of rows and its *width* is the number of columns. The *area* of a polyomino is defined to be the number of its cells. The *level* of a column of a directed polyomino is the number of rows lying between the top of the column itself and the bottom of the source. The *bottom border* of a directed polyomino is made up of the cells lying on the lowest path going from the source to the highest rightmost cell (see Figure 1 b)). A *parallelogram polyomino* can be defined as an array of unit cells that is bounded by two lattice paths which use the steps $(1,0)$ and $(0,1)$, and which intersect only at their endpoints. They form a subclass of convex (i.e. row- and column-convex) polyominoes. It is well-known that parallelogram polyominoes of semiperimeter $n+1$ are counted by the Catalan number $\frac{1}{n+1}\binom{2n}{n}$, see [7] for example. We will consider a particular class of directed column-convex polyominoes, called *deco polyominoes* after the French *dernière colonne*: last column. They are defined as directed column-convex polyominoes in which the height is attained only in the last column (see Figure 1 b) where the height 16 is attained only in the last column; the polyomino in Figure 1 a) is not deco because it has 8 columns but the height 14 is attained in column 5). Deco polyominoes have been introduced by E. Barcucci, A. Del Lungo, and R. Pinzani [3] (see also [2]). We denote the class of deco polyominoes of height n by D_n. Clearly, the height of a deco polyomino is equal to the number of cells in its bottom border. Also, for any deco polyomino of height n we have *width* + *level of last column* $= n+1$. It is easy to see that the parallelogram polyominoes of semiperimeter $n+1$ form a subset of the deco polyominoes of height n.

As shown in [3], the set D_n of deco polyominoes of height n $(n \geq 2)$ admits the following decomposition. If a deco polyomino of height n has no cell to the right of the source, then it is made up of the source and a deco polyomino of height $n-1$ attached to the North side of the source (see Figure 2 a)). Otherwise, when there is a cell to the right of the source, the polyomino is made up of the first column, containing k cells , where $k \leq n-1$ (because height n is attained only in the last column), and a deco polyomino of height $n-1$, attached to the East side of the source (see Figure 2 b)).

From the above decomposition of D_n, taking into account that in the first case we have $|D_{n-1}|$ possibilities and in the second case we have $(n-1)|D_{n-1}|$ possibilities, we obtain $|D_n| = n|D_{n-1}|$; this, together with $|D_1| = 1$, yields $|D_n| = n!$.

A deco polyomino of height n can be built step by step by a sequence of n steps of two kinds:

elevation: add a cell at the bottom of the leftmost column of the previous deco polyomino; the first step is always of this kind, the "previous" deco polyomino being the empty one;

column pasting: add a new column to the left of the previous deco polyomino in such a way that the bottoms of the first two columns lie at the same level.

For a deco polyomino of height n we define $a_j = 0$ if the j-th step of the above described construction is an elevation and $a_j = k$ if it is a pasting of a column of length k. Thus, a deco polyomino of height n can be coded by the sequence $(a_n, a_{n-1}, \ldots, a_2, a_1)$ (it is convenient to list the a_j's in this order). Since after step j of the step-by-step construction we obtain a deco polyomino of height j, it follows that $0 \leq a_j \leq j-1$ for $j = 1, \ldots, n$. For example, for the deco polyomino of Figure 3 a), the step-by-step construction is shown in Figure 3 b) (from right to left) and, consequently, the corresponding code is $(5, 5, 3, 0, 1, 0, 0)$.

A different coding of a deco polyomino is given in [3]; the reader may derive the simple connection between the two codings.

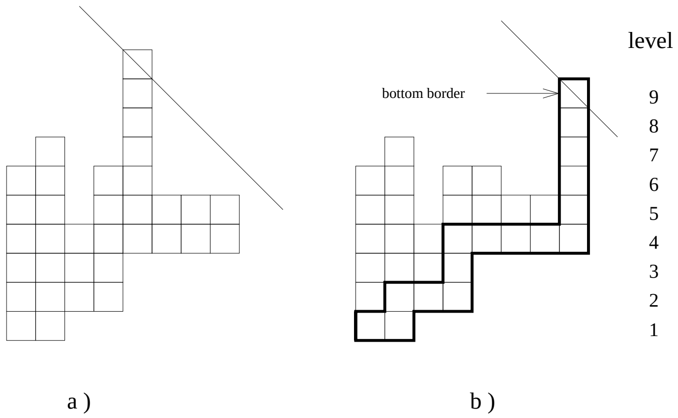

Figure 1: a) A directed column-convex polyomino. b) A polyomino of height 16, width 8, area 34, whose last column level is 9

2.2 Permutations

We denote the set of all permutations of the set $[n] = \{1, 2, \ldots, n\}$ by S_n.

We say that a permutation $\pi \in S_n$ *contains a subsequence of type* $\tau \in S_k$ if there exists a sequence of indices $1 \leq i_1 \leq i_2 \leq \ldots \leq i_k \leq n$ such that $\pi(i_1)\pi(i_2)\ldots\pi(i_k)$ is ordered as τ. If no such sequence exists, then the permutation π is said to be τ-avoiding. It is well known that 321-avoiding permutations are counted by the Catalan numbers (see [5], p. 133).

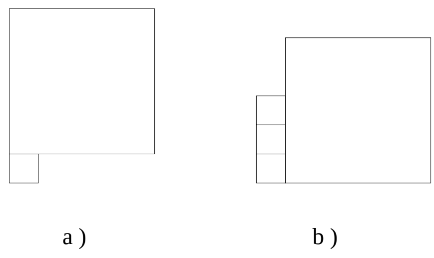

Figure 2: Decomposition of the deco polyominoes

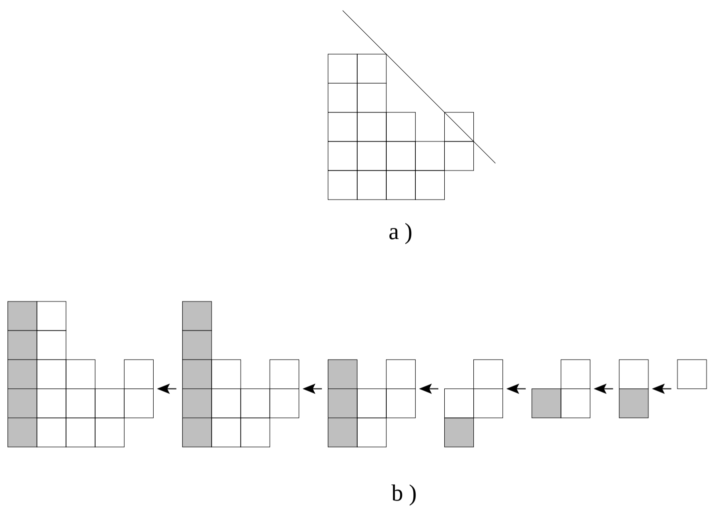

Figure 3: a) A deco polyomino. b) The step-by-step construction of the polyomino in a)

EXAMPLE 2.1 The permutation 351264 is 321-avoiding.

To each ordered list of n distinct positive integers we associate a permutation of $[n]$ in a natural way: we relabel the smallest number in s as 1, the second smallest number as 2, and so on, relabelling the largest number in s as n. We call this permutation the *reduction* of the sequence s, denoted $red(s)$.

EXAMPLE 2.2 Let $s = 572396$ then $red(s) = 351264$.

For a permutation $\pi = \pi_1\pi_2 \dots \pi_n$, we define the *reverse* of π as the permutation $\pi^r = \pi_n\pi_{n-1} \dots \pi_1$ and the *complement* of π as the permutation having entries $n+1-\pi_i$, $i = 1, \dots, n$.

Let $\pi = \pi_1\pi_2 \dots \pi_n$ be a permutation. We say that i is a *descent* of π if $\pi_i > \pi_{i+1}$. If π has $k-1$ descents, then π is the union of k increasing subsequences of consecutive entries. These are called the *ascending runs* of π. In the same way one defines the concepts of *ascent* and *descending runs*. An entry in a permutation which is smaller than all the entries that follow it is called *right-to-left minimum* (see [5], p. 98). Clearly, the right-to-left minima form an increasing sequence when read from left to right.

EXAMPLE 2.3 The permutation 2371546 has descents $3, 5$, ascents $1, 2, 4, 6$, three ascending runs $237, 15, 46$, five descending runs $2, 3, 71, 54, 6$ and right-to-left minima $6, 4, 1$.

In a permutation $\pi = \pi_1\pi_2 \dots \pi_n$, an *inversion* is a pair $i < j$ such that $\pi_i > \pi_j$. The number of inversions of π will be denoted by $inv(\pi)$.

If c_i is the number of $j > i$ with $\pi_j < \pi_i$, then $(c_1, c_2, \dots, c_n)$ is called the *right inversion vector* of π ([8] p. 188, where the terms inversion vector and inversion table are used). Clearly, $0 \leq c_i \leq n - i$. The right inversion vector is sometimes called code ([4] p. 357; [9] p. 9).

Clearly, $inv(\pi) = \Sigma_{i=1}^{n} c_i$.

It is known that the right inversion vector determines uniquely the permutation ([8] p. 188).

EXAMPLE 2.4 If $\pi = 53728146$, then the right inversion vector is $(4, 2, 4, 1, 3, 0, 0, 0)$, and $inv(\pi) = 14$.

Carlitz [6] (see also [10]) defines the statistics inv_c on S_n as follows: express $\pi \in S_n$ in standard cycle form (i.e. in each cycle the smallest element is in the first position and then the cycles are taken in increasing order of their first elements); then remove the parentheses and count the inversions in the obtained permutation.

EXAMPLE 2.5 If $\pi = 2357146 = (1235)(476)$, then $inv_c(\pi) = 2$, the number of inversions in the permutation 1235476.

Clearly, the mapping defined above, which associates to any permutation the permutation obtained by removing the parentheses, is not a bijection (all images start with the entry 1).

3 Bijection No. 1

This bijection, say Φ_1, has been introduced in [3]. It is defined recursively in the following manner.

To the permutation 1 of S_1 there corresponds the single cycle-cell polyomino. Let $\pi = \pi_1\pi_2 \dots \pi_n$, $n \geq 2$. If $\pi_1 = n$, then the image of π is defined to be the polyomino obtained by attaching the polyomino corresponding to $\pi_2\pi_3\ldots\pi_n$ to the North side of the single-cell polyomino. If $\pi_1 = k < n$, then the image of π is defined to be the polyomino obtained by attaching the polyomino corresponding to the permutation $red(\pi_2, \pi_3, \ldots, \pi_n)$ to the East side of a column of k cells (see Figure 4).

It is straightforward to describe the inverse mapping of this bijection.

Figure 5 shows this bijection for $n = 1, 2, 3, 4$.

From the definition of this bijection it follows that the number of cells in the last column of the deco polyomino $\Phi_1(\pi)$ is equal to the length of the last descending run of π, the reverse of the corresponding permutation.

4 Bijection No. 2

We present here a simple bijection, say Φ_2, between permutations in S_n and deco polyominoes of height n, based on the coding we have derived from the step by step construction of a deco polyomino at the end of Section 2.1.

We start with $\Phi_2^{-1} : D_n \to S_n$. Let δ be a given deco polyomino and let $(b_n, b_{n-1}, \ldots, b_2, b_1)$ be its code. It is convenient to insert these numbers in the appropriate cells: the 0's in the new cells that produce the elevation and the k's ($k > 0$) in the bottom cells of the pasted columns (see Figure 6).

Since, as we have seen, $0 \le b_j \le j-1$, if we denote $c_j = b_{n+1-j}$, then $0 \le c_j \le n-j$. Consequently, $b_n, b_{n-1}, \ldots, b_2, b_1$ $(= c_1, c_2, \ldots, c_n)$ can be viewed as a right inversion vector. To it there corresponds a unique permutation $\pi = \Phi_2^{-1}(\delta)$.

Example 4.1 The deco polyomino of Figure 6 is coded by the sequence $(5, 0, 2,$ 0,4$, 2, 0, 0, 0)$ and to this right inversion vector there corresponds the permutation 614297358.

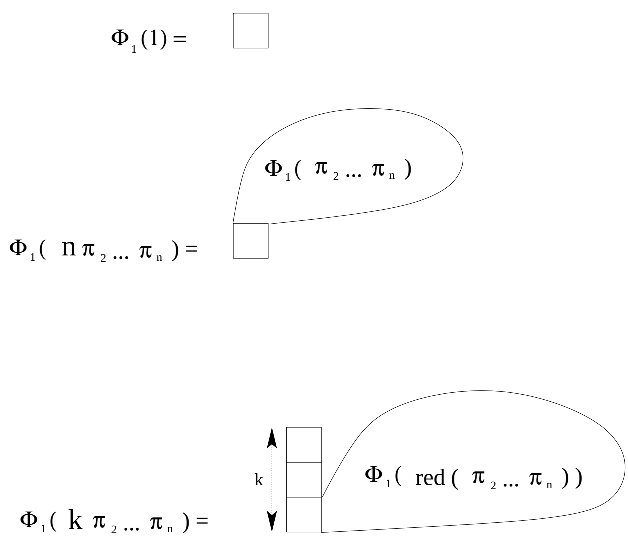

Figure 4: A graphical representation of bijection Φ_1

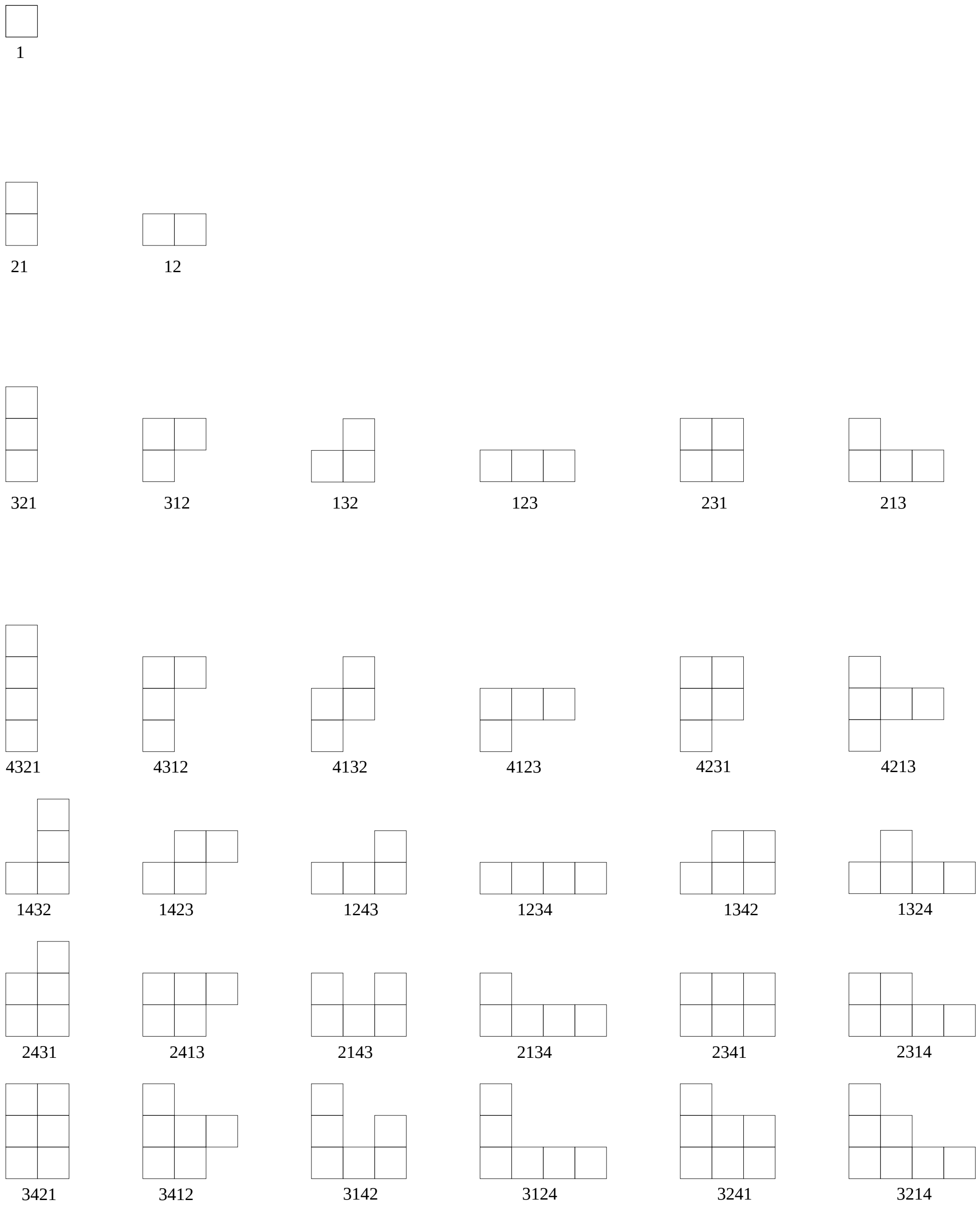

Figure 5: Bijection Φ_1 for $n = 1, 2, 3, 4$

It is straightforward to obtain the deco polyomino $\Phi_2(\pi)$ from a given permutation π: find the right inversion vector $(c_1, c_2, \ldots, c_n)$ of π and, viewing this as a code, perform the step-by-step construction of the corresponding deco polyomino.

Figure 7 shows this bijection for $n = 1, 2, 3, 4$.

The following relations between a permutation $\pi \in S_n$ and its corresponding deco polyomino $\Phi_2(\pi) \in D_n$ are immediate:

- the level of the last column of $\Phi_2(\pi)$ is equal to the number of right-to-left minima of π;
- the number of cells in the last column of $\Phi_2(\pi)$ is equal to the length of the last ascending run of π;
- the area of $\Phi_2(\pi)$ is equal to $inv(\pi)+$ number of right-to- left minima of π;
- if $m_1, m_2, \ldots, m_r$ are the positions of the right-to-left minima of π, then the lengths of the rows of the bottom border of $\Phi_2(\pi)$, starting from the lowest one, are $m_1, m_2 - m_1, \ldots, m_r - m_{r-1}$, respectively.

We also note that to the identity permutation in S_n there corresponds the deco polyomino that consists of a single column of n cells; clearly, the step-by-step construction of this polyomino involves only elevations. On the other hand, for an arbitrary permutation $\pi \in S_n$, $inv(\pi)$ is equal to the number of those cells in $\Phi_2(\pi)$ that have been added by column pasting.

THEOREM 4.2 *A permutation π is 321-avoiding if and only if $\Phi_2(\pi)$ is a parallelogram polyomino.*

Proof. Taking into account the interpretation of the b_k's in the step-by-step construction of a deco polyomino, it follows that we obtain a parallelogram polyomino if and only if the number of 0's between two consecutive nonzero b_i and b_j is at least $b_i - b_j$ (otherwise, the top of the column corresponding to b_i is higher than the top of the column corresponding to b_j). But this is exactly the necessary and sufficient condition for a permutation to be 321-avoiding, given by Billey, Jockush, and Stanley ([4], Theorem 2.1). □

5 Bijection No. 3

We now define a new bijection, say $\Phi_3 : S_n \to D_n$, recursively. We describe Φ_3^{-1}.

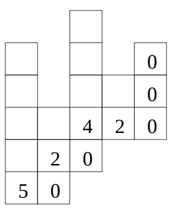

Figure 6: A deco polyomino and its code according to bijection Φ_2

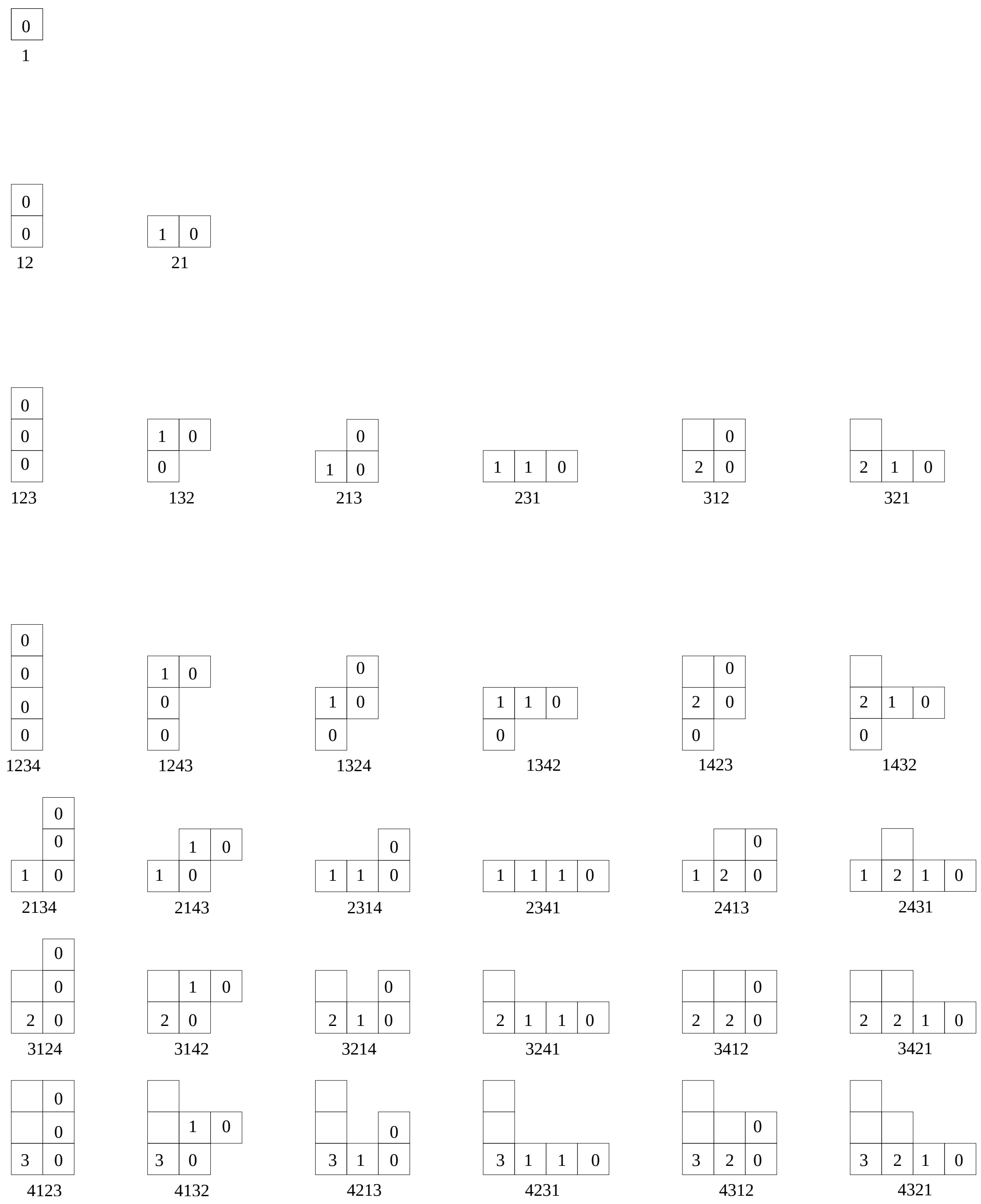

Figure 7: Bijection Φ_2 for $n = 1, 2, 3, 4$

To the deco polyomino of height 1 there corresponds the permutation 1. Let $\delta \in D_n$. If δ is obtained by elevation from $\delta' \in D_{n-1}$, then $\Phi_3^{-1}(\delta)$ is obtained by adding n as a new cycle to the cycle form of $\Phi_3^{-1}(\delta')$. If δ is obtained by column pasting from $\delta' \in D_{n-1}$, then $\Phi_3^{-1}(\delta)$ is obtained by inserting n into the cycle form of $\Phi_3^{-1}(\delta')$ on the immediate right of k, where k is the length of the pasted column (see Figure 8).

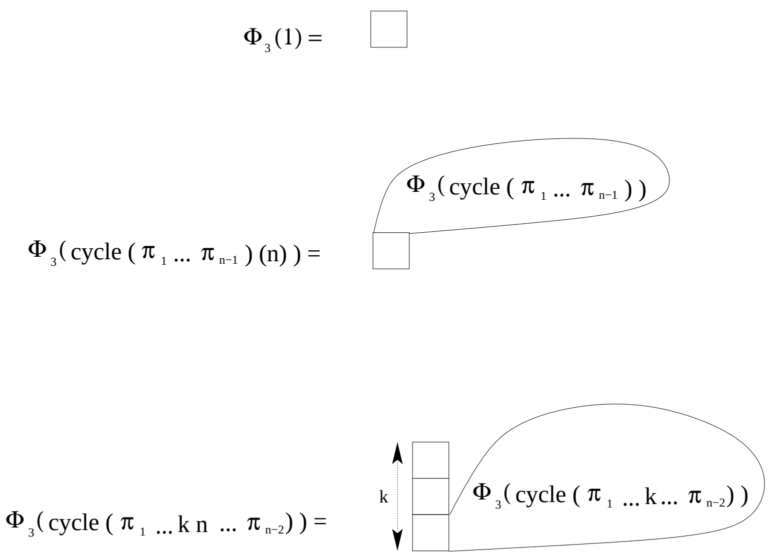

Figure 8: A graphical representation of bijection Φ_3

Again, it is straightforward to describe the inverse mapping.

Figure 9 shows this bijection for $n = 1, 2, 3, 4$.

It is easy to see that the number of cycles of a permutation π is equal to the level of the last column of $\Phi_3(\pi)$. Indeed, this is true for the permutation $1 \in S_1$ and new cycles are obtained only through elevation. Equivalently, the width of a deco polyomino of height n is equal to $n + 1 - s$, where s is the number of cycles of the corresponding permutation.

6 Bijection No. 4

We now define a new bijection $\Phi_4 : S_n \to D_n$, recursively. We describe Φ_4^{-1}. To the deco polyomino of height 1 there corresponds the permutation 1. Let $\delta \in D_n$. If δ is obtained by elevation from $\delta' \in D_{n-1}$ and $\Phi_4^{-1}(\delta') = \pi_1\pi_2 \ldots \pi_{n-1}$, then we define $\Phi_4^{-1}(\delta) = \pi_1\pi_2 \ldots \pi_{n-1}n$. If δ is obtained by column pasting from $\delta' \in D_{n-1}$, then we define $\Phi_4^{-1}(\delta) = \pi_1\pi_2 \ldots \pi_{n-1-k}n\pi_{n-k} \ldots \pi_{n-1}$, where k is the length of the pasted column (see Figure 10).

Again, it is straightforward to describe the inverse mapping.

Figure 11 shows this bijection for $n = 1, 2, 3, 4$.

It is easy to show by induction that for a permutation $\pi \in S_n$ and its corresponding deco polyomino $\Phi_4(\pi) \in D_n$ we have:

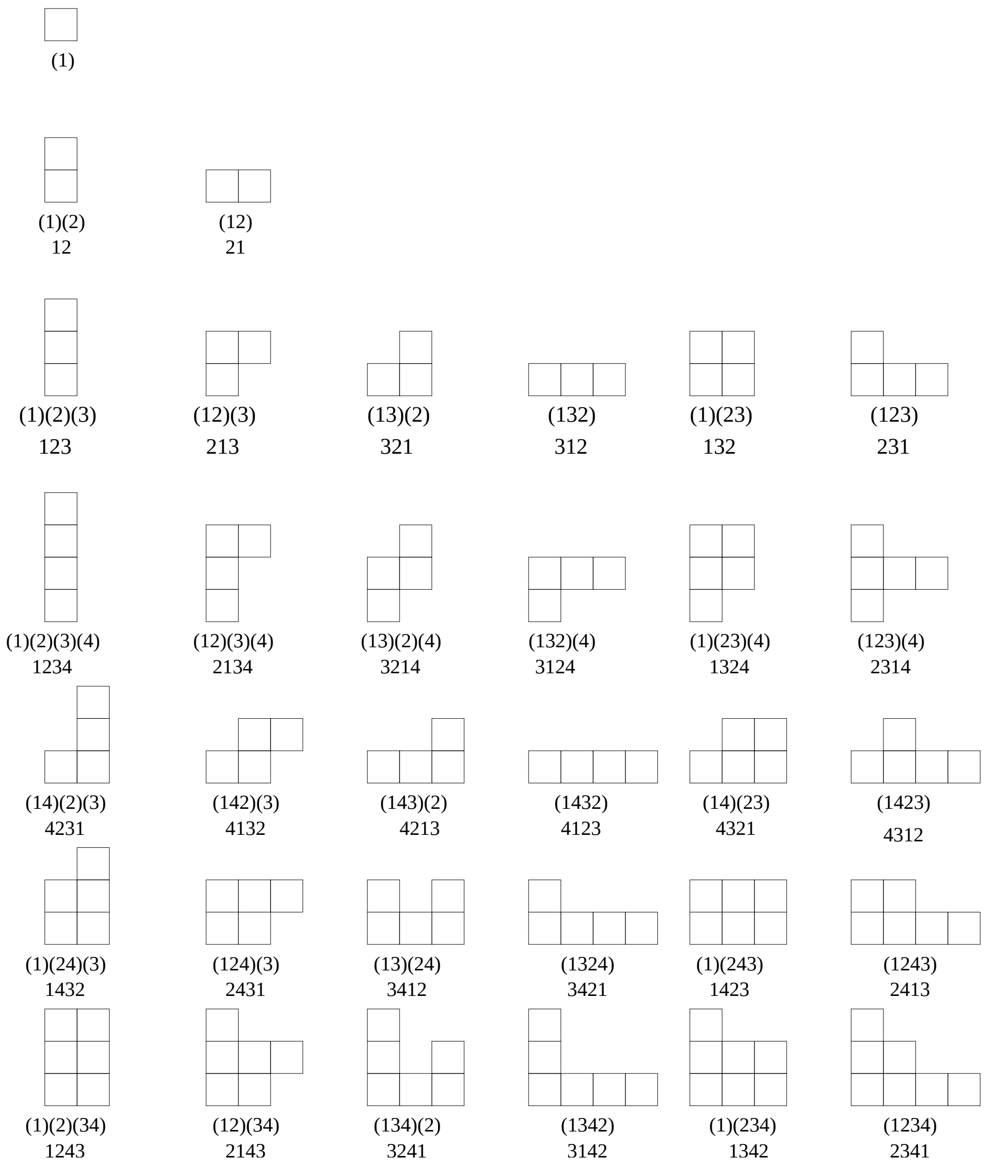

Figure 9: Bijection Φ_3 for $n = 1, 2, 3, 4$

- $inv(\pi) = area(\Phi_4(\pi)) -$ level of the last column of $\Phi_4(\pi)$;
  equivalently,
- $inv(\pi) = area(\Phi_4(\pi)) + width(\Phi_4(\pi)) - (n+1)$.

The proof of the next lemma, needed for the subsequent theorem, is left to the reader.

LEMMA 6.1 *Let δ be a parallelogram polyomino and let π be the corresponding permutation ($\pi = \Phi_4^{-1}(\delta)$). Then the length of the first column of δ is equal to the length of the last ascending run of π.*

THEOREM 6.2 *A permutation π is 321-avoiding if and only if $\Phi_4(\pi)$ is a parallelogram polyomino.*

Proof. Let δ be a deco polyomino of height n, let $\delta_1, \delta_2, \ldots, \delta_n = \delta$ be the deco polyominoes obtained successively by the step-by-step construction of δ, and let π_i be the permutation corresponding to δ_i, $i = 1, \ldots, n$ under the considered bijection. As long as δ_i consists of a single column, the permutation π_i is the identity permutation on S_i. As long as δ_i consists of two columns, δ_i is a parallelogram polyomino (because height is attained only in the last column) and π_i is 321-avoiding because it has exactly one descent. In the subsequent steps, if any, we obtain for the first time a deco polyomino δ_j that is not a parallelogram polyomino only following a pasting of a column of length greater than the first column of δ_{j-1}. But, due to the lemma, this is the only possibility for the 321-avoiding permutation π_{j-1} to go into a permutation π_j containing the pattern 321. □

7 Bijection No. 5

Parallel with the presentation of the next bijection, say $\Phi_5 : S_n \to D_n$, for an arbitrary permutation $\pi \in S_n$, we show its steps on the example $\pi = 372196458$. We write π in standard cycle form,

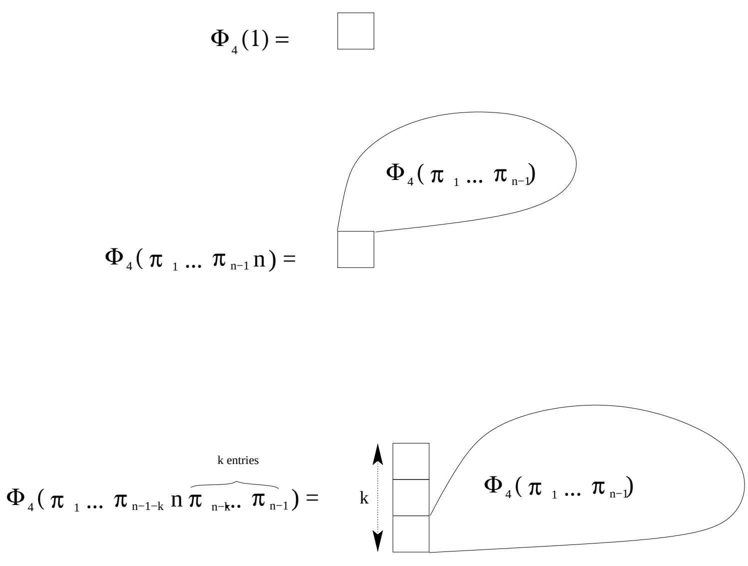

Figure 10: A graphical representation of bijection Φ_4

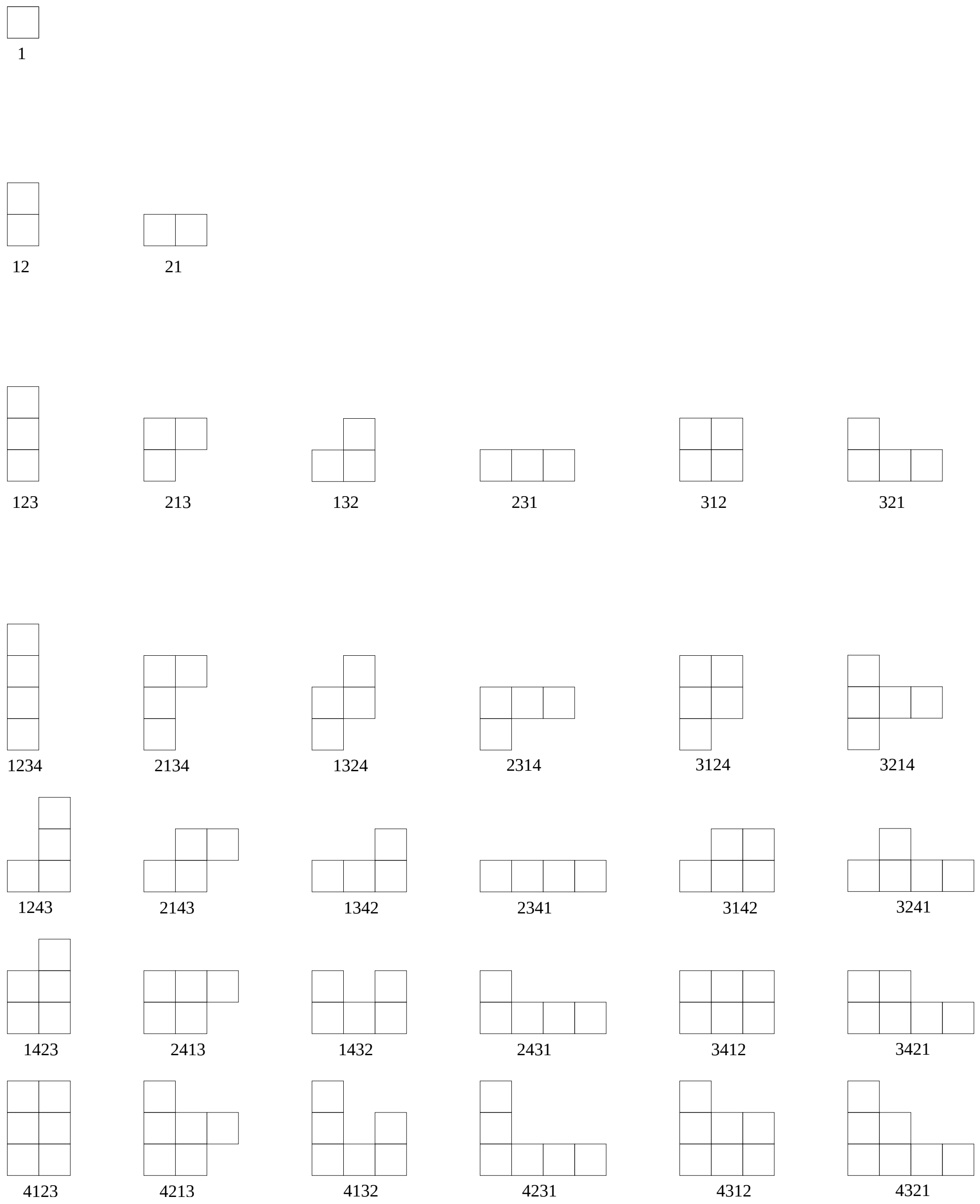

Figure 11: Bijection Φ_4 for $n = 1, 2, 3, 4$

$\pi = (13274)(598)(6)$. The length of these cycles (5, 3, and 1) will be the lengths of the rows of the bottom border of the corresponding deco polyomino (see Figure 12 a)). Removing the parentheses in

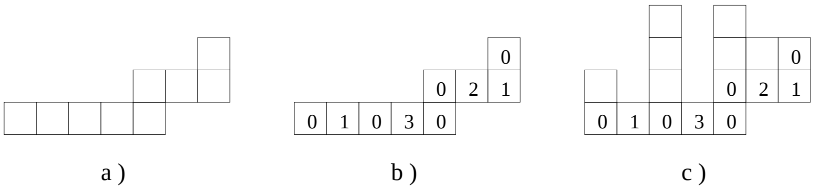

Figure 12: An example for bijection Φ_5

the last expression for π, we obtain the auxiliary permutation $\pi' = 132745986$. We consider its right inversion vector $(0, 1, 0, 3, 0, 0, 2, 1, 0)$. Clearly, the sum of these numbers (i.e. the inversion number of π') is the Carlitz inversion number of $inv_c(\pi)$, defined in Section 2.2. We place these numbers in the cells of the bottom border of the desired deco polyomino (see Figure 12 b)) and for each cell with a nonzero entry k, we place k cells on the top of its left neighbor. The obtained deco polyomino is the image $\Phi_5(\pi)$ of the given permutation (Figure 12 c)).

The inverse map is defined in the following manner. We take a deco polyomino of height n and we place a 0 in the first cell of each row of the bottom border. In the remaining cells of the bottom border we place the number of cells situated above its left neighbor (Figure 12 c)). We obtain the sequence $b = (b_1, b_2, \ldots, b_n)$. We do have $b_i \leq n - i$ for $i = 1, 2, \ldots, n$. Indeed, mark the first i cells along the bottom border of the deco polyomino and note that, due to the fact that height is attained only in the last column, the number of the cells that define b_i does not exceed the number of unmarked cells along this bottom border. Consequently, there is a unique permutation $\pi_1\pi_2 \ldots \pi_n \in S_n$ whose right inversion vector is b. If the lengths of the rows of the bottom border of the deco polyomino are $s_1, s_2, \ldots, s_r$, then its image under the inverse bijection, written in cycle form, is defined to be

$$(\pi_1\pi_2 \ldots \pi_{s_1})(\pi_{s_1+1}\pi_{s_1+2} \ldots \pi_{s_1+s_2}) \ldots (\pi_{s_1+s_2+\ldots+s_{r-1}+1} \ldots \pi_{s_1+s_2+\ldots+s_r})$$

Figure 13 shows this bijection for $n = 1, 2, 3, 4$.

Example 7.1 Consider the deco polyomino of Figure 12 c). Then $b = (0, 1, 0, 3, 0, 0, 2, 1, 0)$. The unique permutation with this right inversion vector is $\pi' = 132745986$. Now, taking into account that the lengths of the rows of the bottom border of the given deco polyomino are 5, 3 and 1, we recapture the desired permutation $\pi = (13274)(598)(6) = 372196458$.

From the definition of this bijection we obtain the following relations regarding a permutation $\pi \in S_n$ and its corresponding deco polyomino $\Phi_5(\pi)$:

- the area of $\Phi_5(\pi)$ is equal to $n + inv_c(\pi)$;
- the level of the last column of $\Phi_5(\pi)$ is equal to the number of cycles of π;

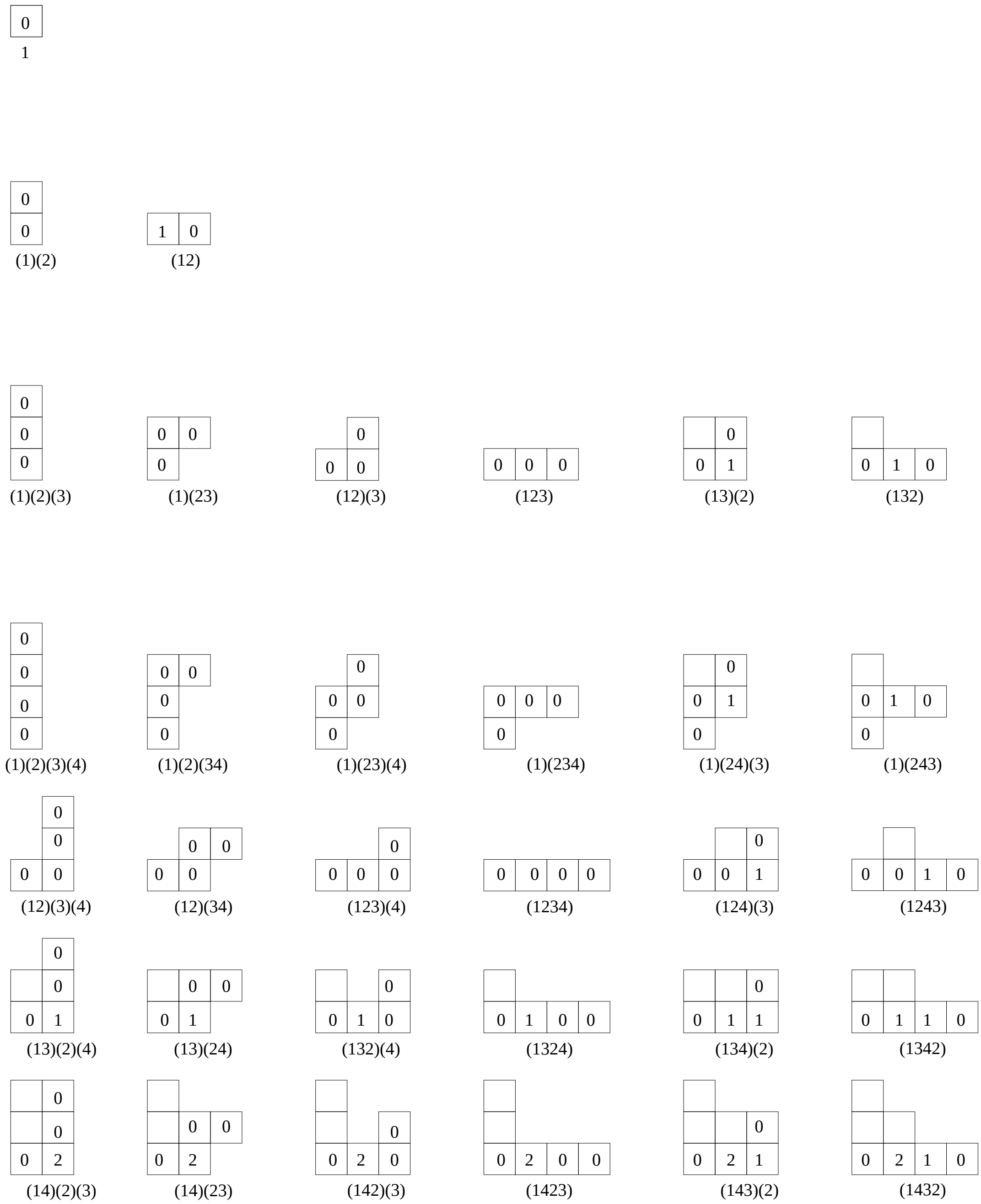

Figure 13: Bijection Φ_5 for $n = 1, 2, 3, 4$

- the lengths of the cycles of π, in the order they are listed in the standard cycle form, are equal, respectively, to the lengths of the rows of the bottom border of $\Phi_5(\pi)$, starting with the lowest one.

8 Bijection No. 6

We define now our last bijection $\Phi_6 : S_n \to D_n$. We start with Φ_6^{-1}. Parallel with its presentation for an arbitrary deco polyomino of height n, we exemplify its steps on the polyomino of Figure 14.

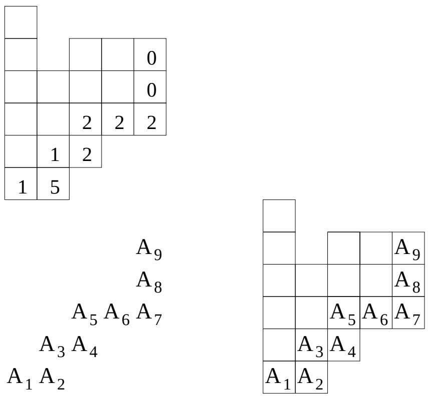

Figure 14: An example for bijection Φ_6

In each cell of the bottom border of the given deco polyomino we write a number according to the following rule. In the first cell of each row of the bottom border we write the number of cells on its right. For the polyomino of Figure 14 these are the numbers 1, 1, 2, 0 and 0. For each of the remaining cells of the bottom border we write the number of cells above the cell on the left. For the deco polyomino of Figure 14 these are the numbers 5, 2, 2 and 2. Next we consider the sequence $b_1, b_2, \ldots, b_n$ of the numbers placed in the cells of the bottom order $(1, 5, 1, 2, 2, 2, 2, 0, 0$ in the case of our example). As in Section 7, one can justify that $0 \le b_i \le n - i$ for $i = 1, \ldots, n$. Now we find the unique permutation $\pi = \pi_1\pi_2 \ldots \pi_n$ whose right inversion vector is $(b_1, b_2, \ldots, b_n)$. This permutation π is the image of the δ under our bijection. For our example, we find $\pi = 273568914$.

We explain the reverse map on this example, i.e. we take

$$\begin{aligned} \pi \;=\; & 2\ 7\ 3\ 5\ 6\ 8\ 9\ 1\ 4 \\ & 1\ 5\ 1\ 2\ 2\ 2\ 2\ 0\ 0 \end{aligned}$$

where below the entries π_i we have placed the terms b_i of the right inversion vector of π. To the right of the source cell we have $b_1 = 1$ cells. This allows us to determine the positions of the first 3 cells of the bottom border of the desired polyomino (cells A_1, A_2, A_3 on Figure 14). Since $b_3 = 1$, there

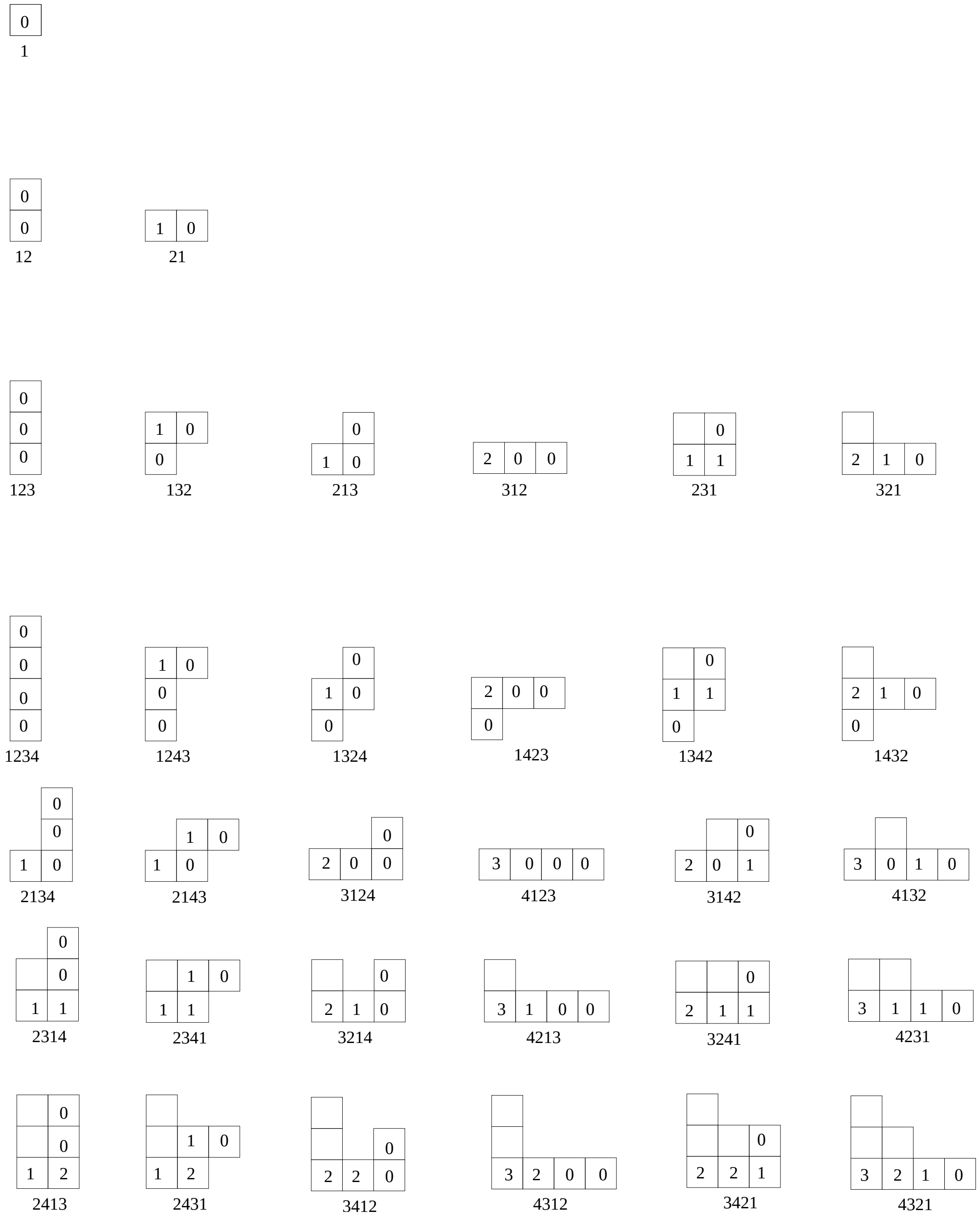

Figure 15: Bijection Φ_6 for $n = 1, 2, 3, 4$

is one more cell in the second row of the bottom border; this determines the positions of cell A_4 and A_5 of the bottom border. Since $b_5 = 2$, there are 2 more cells in the third row of the bottom border, determining the positions of the cells A_6, A_7, and A_8. Since $b_8 = 0$, there are no other cells in the fourth row of the bottom border, determining the position of cell A_9. Since $b_9 = 0$, there are no cells to the right of A_9. Finally, we place columns of lengths $b_2 = 5$, $b_4 = 2$, $b_6 = 2$, and $b_7 = 2$ over the cells situated at the left of the cells A_2, A_4, A_6, and A_7, respectively.

Figure 15 shows this bijection for $n = 1, 2, 3, 4$.

From the definition of this bijection we obtain the following relations regarding a permutation $\pi \in S_n$ and its corresponding deco polyomino $\Phi_6(\pi)$:

- the number of cells of the first row of $\Phi_6(\pi)$ is equal to π_1 (the first entry of the permutation);
- the area of $\Phi_6(\pi)$ − the level of the last column of $\Phi_6(\pi)$ is equal to $inv(\pi)$.

9 Acknowledgments

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