PU. M. A. Vol. **23** (2012), No.2, pp. 155 – 160

Power hyper-sums enumerate quasi-monotone functions

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(Received: August 23, 2012, and in revised form October 7, 2012)

Abstract. We show that the sequences obtained by taking repeated partial sums of regular powers, falling factorials, and rising factorials enumerate certain classes of what we term quasi-monotone functions. In the latter two cases, a q-analogue is also provided.

Mathematics Subject Classification(2010). 05A05, 05A19.

Keywords: hyper-sum, q-analogue, combinatorial proof.

1 Introduction

In what follows, $\mathbb{P}$ denotes the set of positive integers, and $[n] = \{1, \ldots, n\}$ for all $n \in \mathbb{P}$. If E is any finite set, then $|E|$ denotes the cardinality of E. If $\alpha : \mathbb{P} \to \mathbb{R}$, and $m \in \mathbb{P}$, the m-th degree hyper-sum $S_m^\alpha(n)$ is defined inductively by

$$S_1^\alpha(n) = \alpha(1) + \cdots + \alpha(n), \quad \text{and} \tag{1}$$

$$S_{m+1}^\alpha(n) = S_m^\alpha(n) + \cdots + S_m^\alpha(n) \quad \text{for all } m \in \mathbb{P}. \tag{2}$$

Since, for all $m \in \mathbb{P}$, the ordinary generating functions of the sequences $\{\alpha(n)\}_{n\geq1}$ and $\{S_m^\alpha(n)\}_{n\geq1}$ are clearly related by the equation

$$(1-x)^{-m} \sum_{n\geq1} \alpha(n)x^n = \sum_{n\geq1} S_m^\alpha(n)x^n, \tag{3}$$

it follows immediately that

$$S_m^\alpha(n) = \sum_{j=1}^{n} \alpha(j) \binom{n-j+m-1}{m-1}. \tag{4}$$

Let $r \in \mathbb{P}$. In what follows, we consider the special cases of the above given by (i) $\alpha(j) = j^r$, (ii) $\alpha(j) = j^{\underline{r}} := j(j-1)\cdots(j-r+1)$, and (iii) $\alpha(j) = j^{\overline{r}} := j(j+1)\cdots(j+r-1)$, denoting S_m^α in these three cases, respectively, by S_m^r, $S_m^{\underline{r}}$, and $S_m^{\overline{r}}$.

2 Quasi-monotone functions

If $r, m, n \in \mathbb{P}$, a function $f : [r+m] \to [n+m]$ is $(r, m, n)-$*quasi-monotone* if

$$f(i) < f(r+1) < f(r+2) < \cdots < f(r+m), \quad \text{for } i = 1, \ldots, r. \tag{5}$$

As shown below, the quantities $S^r_m(n)$ and $S^{\underline{r}}_m(n)$ each enumerate a certain class of $(r, m, n)-$quasi-monotone functions, and thus admit of simpler expressions than those furnished by formula (4). A slight variation on the notion of quasi-monotonicity facilitates a similar simplification of (4) in the case of $S^{\overline{r}}_m(n)$. Our analysis is based on three results from elementary combinatorics, namely, (i) $j^r = |\{f : [r] \to [j]\}|$, (ii) $j^{\underline{r}} = |\{f : [r] \to [j]$ such that f is injective$\}|$, and (iii) $j^{\overline{r}} =$ the number of distributions of balls labeled $1, \ldots, r$ among *contents-ordered* boxes labeled $1, \ldots, j$ [1, pp. 19–23].

Theorem 2.1 *For all $r, m, n \in \mathbb{P}$,*

$$S^r_m(n) = \sum_{j=1}^{n} j^r \binom{n-j+m-1}{m-1} = \sum_{k=1}^{r} \sigma(r,k) \binom{n+m}{k+m}, \tag{6}$$

where $\sigma(r,k) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^r$ is the number of surjective functions $f : [r] \to [k]$.

Proof. The n-fold sum in (6), which follows from (4), enumerates the set of $(r, m, n)-$quasi-monotone functions $f : [r+m] \to [n+m]$, the j-th term of this sum enumerating those f for which $f(r+1) = j+1$. In the r-fold sum, the k-th term enumerates those f for which $|\text{range}(f)| = k+m$. □

When $m = 1$, (6) reduces to the well-known power sum formula

$$\sum_{j=1}^{n} j^r = \sum_{k=1}^{r} \sigma(r,k) \binom{n+1}{k+1}; \tag{7}$$

see, e.g., [4, 6]. Various q-analogues have been developed for power sums; see, e.g., [2]. We remark that (7) often appears in the variant form,

$$\sum_{j=1}^{n} j^r = \sum_{k=1}^{r} \frac{\left\{ {r \atop k} \right\}}{k+1} (n+1)^{\underline{k+1}}, \tag{8}$$

where $\left\{ {r \atop k} \right\} = \frac{\sigma(r,k)}{k!}$ is the Stirling number of the second kind.

Remark 2.2 The n-fold sum in (6) may also be reduced to the r-fold sum by a more involved algebraic argument, using the fact that

$$j^r = \sum_{k=1}^{r} \sigma(r,k) \binom{j}{k}, \tag{9}$$

along with the binomial coefficient identity (see [3])

$$\sum_{j=1}^{n} \binom{n-j+m-1}{m-1} \binom{j}{k} = \binom{n+m}{k+m}. \tag{10}$$

We next consider the case when $\alpha(j) = j^{\underline{r}}$.

THEOREM 2.3 *For all $r, m, n \in \mathbb{P}$,*

$$S^{\underline{r}}_m(n) = \sum_{j=1}^{n} j^{\underline{r}} \binom{n-j+m-1}{m-1} = \frac{(n+m)^{\underline{r+m}}}{(r+m)^{\underline{m}}}. \tag{11}$$

Proof. The n-fold sum in (11), which follows from (4), enumerates the set of *injective* $(r,m,n)-$quasi-monotone functions $f : [r+m] \to [n+m]$, where, as above, the j-th term in this sum counts those f for which $f(r+1) = j+1$. This sum may be simplified as indicated in (11) by showing that

$$(n+m)^{\underline{r+m}} = (r+m)^{\underline{m}} S^{\underline{r}}_m(n). \tag{12}$$

Let $F = \{f : [r+m] \to [n+m]$ such that f is injective$\}$ and $G = \{g : [r+m] \to [n+m]$ such that g is (r,m,n) − quasi-monotone and injective$\}$. In what follows, we regard members of F as distributions of balls labeled $1, \ldots, r+m$ among boxes labeled $1, \ldots, n+m$, with at most one ball per box, and members of G as distributions of the aforementioned type such that (a) ball $r+i$ occupies a box with a smaller label than that of the box occupied by ball $r+i+1$, for $i = 1, \ldots, m-1$, and (b) each of the balls $1, \ldots, r$ occupies a box with smaller label than that of the box occupied by ball $r+1$. Now consider the map $\psi : F \to G$ defined as follows: Given a distribution $f \in F$, let $\psi(f) = g$, where (i) g has the same set $E \subseteq [n+m]$ of empty boxes as f, (ii) balls $1, \ldots, r$ are placed in the r boxes of $[n+m] - E$ with the smallest labels, and in the same order in which they appear in a left-to-right scan of the distribution f, and (iii) balls $r+1, \ldots, r+m$ are placed in the remaining boxes in their natural order.

$$f = \square_1 \boxed{5}_2 \square_3 \boxed{7}_4 \boxed{2}_5 \boxed{3}_6 \boxed{1}_7 \boxed{4}_8 \boxed{6}_9$$
$$\psi(f) = \square_1 \boxed{2}_2 \square_3 \boxed{3}_4 \boxed{1}_5 \boxed{4}_6 \boxed{5}_7 \boxed{6}_8 \boxed{7}_9$$

Figure 1: An illustration of the mapping ψ when $r = 3$, $m = 4$, and $n = 5$.

Clearly, each distribution in G has $(r+m)^{\underline{m}}$ pre-images in F under ψ. □

THEOREM 2.4 *For all $r, m, n \in \mathbb{P}$,*

$$S^{\overline{r}}_m(n) = \sum_{j=1}^{n} j^{\overline{r}} \binom{n-j+m-1}{m-1} = \frac{(r+m+n-1)^{\underline{r+m}}}{(r+m)^{\underline{m}}} = \frac{n^{\overline{r+m}}}{(r+1)^{\overline{m}}}. \tag{13}$$

Proof. The n-fold sum in (13), which follows from (4), enumerates the distributions of balls labeled $1, \ldots, r+m$ among contents-ordered boxes labeled $1, \ldots, n+m$ such that (a) each of the balls $r+1, \ldots, r+m$ is the sole occupant of its box, (b) ball $r+i$ occupies a box with smaller label than that of the box occupied by ball $r+i+1$, for $i = 1, \ldots, m-1$, and (c) each of the balls $1, \ldots, r$ occupies a box with smaller label than that of the box occupied by ball $r+1$. The j-th term in this sum enumerates those distributions in which ball $r+1$ occupies box $j+1$. This sum may be simplified as indicated in (13) by the following argument.

Let Λ denote the set of distributions of balls labeled $1, \ldots, r+m$ among contents-ordered boxes labeled $1, \ldots, n+m$ in which boxes $n+1, \ldots, n+m$ remain empty. By an earlier observation, $|\Lambda| = n^{\overline{r+m}}$. Given $\lambda \in \Lambda$, let x be the *right-most* ball, in the sense that there are no balls in boxes with a greater label than that of the box occupied by x and, if there is more than one ball in the box containing x, then x occupies the right-most position in its box. We first move x to the right by m boxes (so, for example, if x occupied box n in the distribution λ, it would now occupy box $n+m$). We then move the second right-most ball y of λ to the right by $m-1$ boxes (so if y belonged to the same box as x, necessarily preceding x directly in that box, y would now occupy the box directly preceding the one now containing x). Continuing in this fashion, move the m right-most balls of λ such that the i-th right-most ball is moved to the right by $m-i+1$ boxes, for each $i \in [m]$.

Let λ^* denote the configuration (now allowing for any of the $n+m$ boxes to be occupied by balls) which arises after applying the above procedure to λ. It may be verified that the map $\lambda \mapsto \lambda^*$ is a bijection from Λ to $\Lambda^* :=$ the set of distributions of balls labeled $1, \ldots, r+m$ among contents-ordered boxes labeled $1, \ldots, n+m$ in which the m right-most balls occupy distinct boxes. So also $|\Lambda^*| = n^{\overline{r+m}}$. But here we are interested only in those λ^* for which the m right-most balls are precisely $r+1, r+2, \ldots, r+m$, occurring in that order from left to right. Now the probability that a λ^* randomly chosen from Λ^* has this property is

$$\frac{r!}{(r+m)!} = \frac{1}{(r+m)(r+m-1)\cdots(r+1)} = \frac{1}{(r+1)^{\overline{m}}}.$$

This can be seen by fixing the number of elements that occupy each box, and then assigning the $r+m$ balls to the $r+m$ slots within the boxes to be occupied by at least one ball. It follows that

$$S_m^{\overline{r}}(n) = \frac{|\Lambda^*|}{(r+1)^{\overline{m}}} = \frac{n^{\overline{r+m}}}{(r+1)^{\overline{m}}}.$$

□

3 q-analogues

In this section, we consider q-analogues of the last two results. Given an indeterminate q, let $[j]_q = 1+q+\cdots+q^{j-1}$ if $j \in \mathbb{P}$, with $[0]_q = 0$. Let $[n]_q! = [n]_q[n-1]_q \cdots [1]_q$ if $n \in \mathbb{P}$, with $[0]_q! = 1$, denote the *q-factorial* and let $\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{[n]_q!}{[m]_q![n-m]_q!}$ denote the *q-binomial coefficient*, where $0 \leq m \leq n$. Given positive integers n and m, let $[n]_q^{\underline{m}} = [n]_q[n-1]_q \cdots [n-m+1]_q$ and $[n]_q^{\overline{m}} = [n]_q[n+1]_q \cdots [n+m-1]_q$, with $[n]_q^{\underline{0}} = [n]_q^{\overline{0}} = 1$.

Recall that the *number of inversions* in a word $w = w_1 w_2 \cdots w_n$ over some alphabet of non-negative integers is the cardinality of the set $\{(i,j) : 1 \leq i < j \leq n \text{ with } w_i > w_j\}$, which is often denoted by $\mathrm{inv}(w)$. We'll make use of the fact that the q-binomial coefficient $\begin{bmatrix} n \\ m \end{bmatrix}_q$ is the generating function for the statistic that records the number of inversions in binary words of length n containing exactly m 1's (see [5, Prop. 1.3.17]).

We have the following q-generalization of the second identity in Theorem 2.3 above.

THEOREM 3.1 *For all $r, m, n \in \mathbb{P}$,*

$$\sum_{j=1}^{n} q^{m(j-r)}[j]_q^{\underline{r}} \begin{bmatrix} n-j+m-1 \\ m-1 \end{bmatrix}_q = \frac{[n+m]_q^{\underline{r+m}}}{[r+m]_q^{\underline{m}}}. \tag{14}$$

Proof. Note that the lower index of the sum on the left-hand side of (14) may be started from $j = r$ since $[j]_q^{\underline{r}} = 0$ if $j < r$. Let us assume further $n \geq r$, for otherwise both sides of (14) are zero. We provide a combinatorial proof of (14), rewritten in the form

$$[n+m]_q^{\underline{r+m}} = [r+m]_q^{\underline{m}} \sum_{j=r}^{n} q^{m(j-r)}[j]_q^{\underline{r}} \begin{bmatrix} n-j+m-1 \\ m-1 \end{bmatrix}_q. \tag{15}$$

First we extend $\mathbb{P}$ by adding the *infinity symbol* ∞, it being understood that $n < \infty$ for all $n \in \mathbb{P}$. Let $\mathcal{A}$ denote the set of words of length $n+m$ containing exactly $n-r$ infinity symbols and each member of $[r+m]$ once. Then $[n+m]_q^{\underline{r+m}}$ counts the members of $\mathcal{A}$ according to the number of inversions. To see this, first note that the $[n-r+1]_q$ factor accounts for the placement of the element $r+m$ amongst the $n-r$ infinity symbols, written in a row, since anywhere from 0 to $n-r$ inversions are created. Then $[n-r+2]_q$ accounts for the placement of the element $r+m-1$ once the position for $r+m$ has been determined, and, in general, $[n+m-i+1]_q$ accounts for the placement of the element i, $1 \leq i \leq r+m$, once the positions for all letters greater than i have been determined.

To show that the right-hand side of (15) also counts the members of $\mathcal{A}$ according to the number of inversions, we first describe a procedure for generating the members of $\mathcal{A}$. We start with a sequence ρ of length $n+m$ consisting of $n-r$ infinity symbols, $m-1$ zeros, and one occurrence of each element of $[r+1]$, where all the elements of $[r+1]$ occur to the left of all the zeros, the element $r+1$ occurs to the right of all the elements of $[r]$, and $r+1$ is in the $(j+1)$-st position for some $j \in [r,n] = \{r, r+1, \ldots, n\}$. We transform ρ into another sequence $\delta \in \mathcal{A}$ as follows: (i) Replace each letter in $[r+1]$ occurring in ρ with a zero, (ii) Replace m of the $r+m$ zeros in the word resulting from the first step with elements of $[r+1, r+m]$ so that each letter occurs once, and (iii) Replace the r remaining zeros with the elements of $[r]$ so that they occur in the same order in which they appear in a left-to-right scan of the word ρ. From this, we see that there are $(r+m)^{\underline{m}} \cdot j^{\underline{r}}\binom{n-j+m-1}{m-1}$ sequences $\delta \in \mathcal{A}$ in which the $(r+1)$-st left-most letter of δ that is not an infinity symbol occupies the $(j+1)$-st position, $r \leq j \leq n$.

Then the distribution of the inv statistic on the set consisting of such sequences $\delta \in \mathcal{A}$ is given by

$$[r+m]_q^{\underline{m}} \cdot q^{m(j-r)}[j]_q^{\underline{r}} \begin{bmatrix} n-j+m-1 \\ m-1 \end{bmatrix}_q,$$

whence (15) follows from summing over j. To see this, first note that the factor $[r+m]_q^{\underline{m}} = [r+m]_q[r+m-1]_q \cdots [r+1]_q$ accounts for both the choice of the positions for the members of $[r+1, r+m]$ relative to the positions of all the members of $[r+m]$ within δ and the inversions between two letters which aren't an ∞ in which at least one of the letters belongs to $[r+1, r+m]$. The factor $[j]_q^{\underline{r}} = [j]_q[j-1]_q \cdots [j-r+1]_q$ accounts for the choice of the positions for the left-most r members of $[r+m]$ within δ, the inversions between these members and infinity symbols, and inversions between two members of $[r]$ (note that the relative order of the members of $[r]$ did not change in the transformation from ρ to δ described above). The factor $q^{m(j-r)}$ accounts for the inversions between the left-most $j-r$ ∞'s and the right-most

m members of $[r+m]$ within δ. Finally, $\left[{n-j+m-1 \atop m-1}\right]_q$ accounts for the choice of the positions for the right-most $(n-r)-(j-r)=n-j$ ∞'s amongst the final $n+m-j-1$ positions of δ along with inversions involving these ∞'s. □

One may also generalize the second identity in Theorem 2.4 above.

THEOREM 3.2 *For all* $r, m, n \in \mathbb{P}$,

$$\sum_{j=1}^{n} q^{m(j-1)} [j]_q^{\overline{r}} \begin{bmatrix} n-j+m-1 \\ m-1 \end{bmatrix}_q = \frac{[n]_q^{\overline{r+m}}}{[r+1]_q^{\overline{m}}}. \tag{16}$$

Proof. A proof comparable to the one given for Theorem 3.1 above, the details of which we leave to the interested reader, may be given for (16), upon multiplying both sides by $[r+1]_q^{\overline{m}}$. Here, one would count sequences of length $r+m+n-1$ containing $n-1$ infinity symbols and each element of $[r+m]$ once according to the number of inversions. Note that in this case, if there are $j-1$ infinity symbols occurring to the left of the $(r+1)$-st left-most element of $[r+m]$ within such a sequence, then there are $m(j-1)$ inversions between these symbols and the m right-most elements of $[r+m]$ occurring in the sequence, whence the factor of $q^{m(j-1)}$. □

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