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Wilf equivalence for generalized factor orders modulo k

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Abstract. Kitaev, Liese, Remmel, and Sagan recently defined generalized factor order on words comprised of letters from a partially ordered set, and also defined a related notion of Wilf equivalence under this order when the poset in question is the positive integers with the usual total order. We adapt these ideas to the family of posets $\{\mathcal{P}_k\}$, $k \geq 2$, where for each k, $\mathcal{P}_k$ is defined as the positive integers with the ordering $i \leq_k j$ if $i \leq j$ and $i \equiv j \bmod k$. We make an extensive study of these posets, providing many analogues of results of Kitaev, Liese, Remmel, and Sagan. We also give an explicit formula for a weight generating function in this setting that generalizes a result of Langley, Liese, and Remmel.

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1 Introduction and definitions

In [2], Kitaev, Liese, Remmel, and Sagan introduced the generalized factor order on words comprised of letters from a partially ordered set (poset). That is, let $\mathcal{P} = (P, \leq_P)$ be a poset and let P^* be the Kleene closure of P so that

$$P^* = \{w_1 w_2 \dots w_n : n \geq 0 \text{ and } w_i \in P \text{ for all } i\}.$$

For $w = w_1 \dots w_n \in P^*$, let $|w| = n$ denote the length of w. Then for any $u, w \in P^*$, we say that u is less than or equal to w in the *generalized factor order* relative to $\mathcal{P}$, written $u \leq_{\mathcal{P}} w$, if there is a string x of $|u|$ consecutive characters in w such that the i-th character of x is greater than or equal to the i-th character of u under $\leq_P$ for each i, $1 \leq i \leq |u|$. If $u \leq_{\mathcal{P}} w$, we will also say that w *embeds* u, and if x begins at the j-th character of w, we will call x an *embedding of u into w* with *embedding index j*. For example, if $\mathcal{P}_1 = (\mathbb{P}, \leq)$ where $\mathbb{P}$ is the set of positive integers and $\leq$ is the usual total order on $\mathbb{P}$, then $u = 321 \leq_{\mathcal{P}_1} w = 142322$, and 423 and 322 are embeddings of u into w with embedding indices 2 and 4, respectively.

Let

$$
\begin{aligned}
\mathcal{F}_{\mathcal{P}}(u) &= \{w \in P^* \mid u \leq_{\mathcal{P}} w\}, \\
\mathcal{W}_{\mathcal{P}}(u) &= \{w \in P^* \mid u \leq_{\mathcal{P}} w \text{ and } |w| = |u|\}, \\
\mathcal{S}_{\mathcal{P}}(u) &= \{w \in P^* \mid u \leq_{\mathcal{P}} w \text{ and the last } |u| \text{ characters of } w \text{ is the only} \\
&\qquad \text{embedding of } u \text{ into } w\}, \text{ and} \\
\mathcal{A}_{\mathcal{P}}(u) &= \{w \in P^* \mid u \not\leq_{\mathcal{P}} w\}.
\end{aligned}
$$

Then

$$\mathcal{F}_{\mathcal{P}}(u) = \mathcal{S}_{\mathcal{P}}(u) P^* \tag{1}$$

and

$$\mathcal{A}_{\mathcal{P}}(u) = P^* - \mathcal{F}_{\mathcal{P}}(u). \tag{2}$$

If $P \subseteq \mathbb{P}$ and $w = w_1 \dots w_n \in P^*$, let $\Sigma(w) = \sum_{i=1}^n w_i$ and define the *weight* of w to be $\mathrm{wt}(w) = t^{|w|} x^{\Sigma(w)}$. We then define the weight generating functions

$$
\begin{aligned}
F_{\mathcal{P}}(u; t, x) &= \sum_{w \in \mathcal{F}_{\mathcal{P}}(u)} t^{|w|} x^{\Sigma(w)}, \\
W_{\mathcal{P}}(u; t, x) &= \sum_{w \in \mathcal{W}_{\mathcal{P}}(u)} t^{|w|} x^{\Sigma(w)}, \\
S_{\mathcal{P}}(u; t, x) &= \sum_{w \in \mathcal{S}_{\mathcal{P}}(u)} t^{|w|} x^{\Sigma(w)}, \text{ and} \\
A_{\mathcal{P}}(u; t, x) &= \sum_{w \in \mathcal{A}_{\mathcal{P}}(u)} t^{|w|} x^{\Sigma(w)}.
\end{aligned}
$$

One of the main results of [2] is that $F_{\mathcal{P}}(u; t, x)$, $A_{\mathcal{P}}(u; t, x)$, and $S_{\mathcal{P}}(u; t, x)$ are rational for all finite posets and certain classes of infinite posets such as the poset $\mathcal{P}_1 = (\mathbb{P}, \leq)$. For each such poset $\mathcal{P}$, we can define a natural notion of Wilf equivalence by defining $u \backsim_{\mathcal{P}} v$ if and only if $F_{\mathcal{P}}(u; t, x) = F_{\mathcal{P}}(v; t, x)$. Kitaev, Liese, Remmel, and Sagan [2] observed that in the special case where the poset is an antichain, then an embedding reduces to the special case of consecutive pattern matching. That is, in such a case, u embeds into w if and only if $w = xuy$ for some words x and y. Thus, in such a case, our notion of Wilf equivalence is the natural one for words relative to consecutive pattern matching. However, for general posets, our notion of Wilf equivalence is different from Wilf equivalence as defined in pattern matching since embeddings are not pattern matches. More information about Wilf equivalence in the pattern avoidance context is contained in the survey article by Wilf [3].

Kitaev, Liese, Remmel, and Sagan also considered the notion of strong Wilf equivalence for the poset $(\mathbb{P}, \leq)$. We can similarly define strong Wilf equivalence for any poset $\mathcal{P} = (P, \leq_P)$ with $P \subseteq \mathbb{P}$, as follows. Given u and w in P^*, define

$$\mathrm{Em}_{\mathcal{P}}(u, w) = \{i : i \text{ is an embedding index of } u \text{ into } w \text{ relative to } \mathcal{P}\}.$$

Then $u, v \in P^*$ are *strongly Wilf equivalent*, denoted $u \backsim_{\mathcal{P},s} v$, if there is a weight-preserving bijection $f : P^* \to P^*$ such that

$$\mathrm{Em}_{\mathcal{P}}(u, w) = \mathrm{Em}_{\mathcal{P}}(v, f(w))$$

for all $w \in P^*$. In this case f is said to *witness* the strong equivalence $u \backsim_{\mathcal{P},s} v$. Kitaev, Liese, Remmel, and Sagan proved a number of results about strong Wilf equivalence relative to $(\mathbb{P}, \leq)$. In particular, they showed that there are words $u, v \in \mathbb{P}^*$ such that u and v are Wilf equivalent but not strongly Wilf equivalent relative to $(\mathbb{P}, \leq)$.

In this paper, we shall focus on the family of posets $\mathcal{P}_k = (\mathbb{P}, \leq_k)$, $k \geq 2$, where we define $x \leq_k y$ if and only if $x \leq y$ and $x \equiv y \bmod k$. For example, the Hasse diagram of $\mathcal{P}_3$ consists of the three chains $1 \leq_3 4 \leq_3 7 \leq_3 \cdots$, $2 \leq_3 5 \leq_3 8 \leq_3 \cdots$, and $3 \leq_3 6 \leq_3 9 \leq_3 \cdots$. For any $k \geq 1$, let $\mathcal{F}_k(u) = \mathcal{F}_{\mathcal{P}_k}(u)$, $\mathcal{W}_k(u) = \mathcal{W}_{\mathcal{P}_k}(u)$, $\mathcal{A}_k(u) = \mathcal{A}_{\mathcal{P}_k}(u)$, $\mathcal{S}_k(u) = \mathcal{S}_{\mathcal{P}_k}(u)$, $F_k(u;t,x) = F_{\mathcal{P}_k}(u;t,x)$, $W_k(u;t,x) = W_{\mathcal{P}_k}(u;t,x)$, $A_k(u;t,x) = A_{\mathcal{P}_k}(u;t,x)$, and $S_k(u;t,x) = S_{\mathcal{P}_k}(u;t,x)$. We shall also write $u \backsim_k v$ for $u \backsim_{\mathcal{P}_k} v$ and $u \backsim_{k,s} v$ for $u \backsim_{\mathcal{P}_k,s} v$.

A key observation in [2] is that if any one of $F_1(u;t,x)$, $A_1(u;t,x)$ or $S_1(u;t,x)$ is rational, then so are the other two. This extends easily to all values of k since the weight generating function for all words in $\mathbb{P}^*$ is

$$\begin{aligned}\sum_{w \in \mathbb{P}^*} \mathrm{wt}(w) &= \frac{1}{1 - \sum_{n \geq 1} t x^n} \\ &= \frac{1}{1 - tx/(1-x)} \\ &= \frac{1-x}{1-x-tx},\end{aligned}$$

and therefore by (1) and (2) we have

$$F_k(u;t,x) = S_k(u;t,x)\frac{1-x}{1-x-tx}$$

and

$$F_k(u;t,x) = \frac{1-x}{1-x-tx} - A_k(u;t,x).$$

It follows from Theorem 8.2 of [2] that $S_k(u;t,x)$ is rational for all $u \in \mathbb{P}^*$ and $k \geq 1$, so indeed all three of $F_k(u;t,x)$, $A_k(u;t,x)$, and $S_k(u;t,x)$ are rational. Note that

$$W_k(u;t,x) = \frac{t^{|u|} x^{\Sigma(u)}}{(1-x^k)^{|u|}},$$

so $W_k(u;t,x)$ is rational for all $k \geq 1$ as well.

In [4], Langley, Liese, and Remmel gave an explicit formula for $S_1(u;t,x)$ in the case that u factors into a weakly increasing word followed by a weakly decreasing word. Specifically, for any word u, let u_{inc} be the longest prefix of u which is weakly increasing. If $u = u_{inc}v$ and v is weakly decreasing, then we shall say that u has an *increasing/decreasing factorization* and denote v as u_{dec}. Thus if $u = u_1u_2 \ldots u_n$ has an increasing/decreasing factorization, then either $u_1 \leq \cdots \leq u_n$, in which case $u_{inc} = u$ and u_{dec} is the empty string ε, or there is a $k < n$ such that $u_1 \leq \cdots \leq u_k > u_{k+1} \geq \cdots \geq u_n$, in which case $u_{inc} = u_1 \ldots u_k$ and $u_{dec} = u_{k+1} \ldots u_n$. For the theorem that follows, we define

$$D^{(i)}(u) = \{n - i + j : 1 \leq j \leq i \text{ and } u_j > u_{n-i+j}\}$$

and $d_i(u) = \sum_{n-i+j \in D^{(i)}(u)} (u_j - u_{n-i+j})$. For example, if $u = 1\ 2\ 3\ 4\ 4\ 3\ 1\ 1$ and $i = 5$, then by considering the diagram

$$\begin{matrix} 1 & 2 & 3 & 4 & 4 & 3 & 1 & 1 \\ & & & 1 & 2 & 3 & \underline{4} & \underline{4} \end{matrix}$$

we see that $D^{(5)}(u) = \{7, 8\}$ and $d_5(u) = (4-1) + (4-1) = 6$. Langley, Liese, and Remmel [4] proved the following.

Theorem 1.1 *Let $u = u_1u_2 \ldots u_n \in \mathbb{P}^*$ have an increasing/decreasing factorization. For $1 \leq i \leq n-1$, let $s_i = u_{i+1}u_{i+2} \ldots u_n$ and $d_i = d_i(u)$. Also let $s_n = \varepsilon$ and $d_n = 0$. Then*

$$S_1(u;t,x) = \frac{t^n x^{\Sigma(u)}}{t^n x^{\Sigma(u)} + (1 - x - tx) \sum_{i=1}^{n} t^{n-i} x^{d_i + \Sigma(s_i)} (1-x)^{i-1}}.$$

We shall prove an analogue of Theorem 1.1 for $k \geq 2$ for a rich class of words that generalizes the set of words with increasing/decreasing factorizations. We will use this result to help classify the Wilf equivalence classes of words of length 3 relative to $\mathcal{P}_k$ for $k \geq 2$, generalizing a similar result for $k = 1$ in [4]. We shall also see that there are considerable differences between Wilf equivalence for $k = 1$ and for $k \geq 2$. For example, Kitaev, Liese, Remmel, and Sagan [2] conjectured that if $u \backsim_1 v$, then u must be a rearrangement of v. We will call this conjecture the *weak rearrangement conjecture.* We will show that this conjecture fails in general for all $\mathcal{P}_k$ with $k \geq 2$. For example, our analogue of Theorem 1.1 will show that $1\ 4\ 2 \backsim_k 2\ 2\ 3$ for all $k \geq 2$. However, we do believe that the conjecture holds for words in which all characters belong to the same equivalence class mod k, and, in fact, we believe that a stronger conjecture is true. Specifically, in [4], Langley, Liese, and Remmel propose the following conjecture: if $u \backsim_1 v$, then there is a weight-preserving bijection $f : \mathbb{P}^* \to \mathbb{P}^*$ such that for all $w \in \mathbb{P}^*$, $f(w)$ is a rearrangement of w and $w \in \mathcal{F}_1(u) \iff f(w) \in \mathcal{F}_1(v)$. We will call such a bijection f a *rearrangement map* that *witnesses* the equivalence of u and v and refer to this conjecture as the *strong rearrangement conjecture.* Most of the Wilf equivalences in [2], including those which we list in Theorem 1.2 below, were proved by constructing a rearrangement map that witnessed a given Wilf equivalence. Let $[m] = \{1, 2, \ldots, m\}$. We will propose a method to test for a rearrangement map witnessing $u \backsim_k v$ by considering the family of finite posets $([m]^*, \leq_k)$, generalizing a similar result for $k = 1$ in [4].

We will also study analogues of results in [2] on Wilf equivalence and strong Wilf equivalences relative to $\mathcal{P}_1$ for the posets $\mathcal{P}_k$. In particular Kitaev, Liese, Remmel, and Sagan proved the following.

Theorem 1.2 (Kitaev, Liese, Remmel, Sagan) *Let $u, v \in \mathbb{P}^*$.*

1. $u \backsim_1 u^r$ *where* u^r *denotes the reverse of* u.
2. *If* $u \backsim_1 v$ *then* $1u \backsim_1 1v$.
3. *If* $u \backsim_1 v$ *then* $u^+ \backsim_1 v^+$ *where* $+$ *is the operation which increases each letter in a word by 1.*
4. *If* $y, y' \in \mathbb{P}^*$ *are weakly increasing and* $z, z' \in \mathbb{P}^*$ *are weakly decreasing such that* yz *is a rearrangement of* $y'z'$, *and* $m \geq \max\{y, z\} - 1$ *and* $u \backsim_1 v$, *then*
$$yu^{+m}z \backsim_1 y'v^{+m}z'$$
where $+m$ *denotes applying the* $+$ *operation* m *times (adding* m *to each character).*
5. *If* x, y, z *are in* $\{1, \ldots, m\}^*$ *and* $n > m$, *then* $x\,m\,y\,n\,z \backsim_1 x\,n\,y\,m\,z$.

We will show that statements 1, 3, 4, and 5 have natural analogues in the mod k setting and discuss several other extensions of results in [2].

The outline of this paper is as follows. In section 2, we prove the analogue of Theorem 1.1 for $\mathcal{P}_k$ for $k \geq 2$, and as an application, classify the Wilf equivalence classes of permutations of $\{1, 2, \ldots, n\}$ with $n \leq 2k$ for $k \geq 2$. In section 3, we classify the Wilf equivalence classes of all words of length 3 for $k \geq 2$. We discuss the rearrangement conjectures in section 4, and conclude in section 5 with analogues of results in [2] on Wilf equivalence and strong Wilf equivalence relative to $\mathcal{P}_1$ for the posets $\mathcal{P}_k$ with $k \geq 2$.

2 Words such that $S_k(u; t, x) = \frac{x^s t^r}{P(u;t,x)}$ where $P(u; t, x)$ is a polynomial

The key to constructing an analogue of Theorem 1.1 for $k \geq 2$ is to develop a generalization of the condition in Theorem 1.1 that the word u have an increasing/decreasing factorization. As we will see, the main difference between the case with $k \geq 2$ and the case with $k = 1$ is a restriction on how embeddings can overlap. That is, with $k = 1$, embeddings of a word u into a word w can overlap in any number of characters up to $|u| - 1$. For example, the word $w = 2\ 2\ 4\ 3\ 1$ embeds $u = 1\ 2\ 3$ beginning at positions 1 and 2, and these embeddings overlap in two characters. Overlapping of embeddings is often restricted with $k \geq 2$ since equivalence classes mod k must align in u and w. The most extreme case is when two embeddings cannot overlap at all, leading to the following definition.

Definition 2.1 A word $u = u_1 \ldots u_n$ has the **mod k-nonoverlapping property** if there is no i, $1 \leq i \leq n-1$, such that $u_{n-i+j} \equiv u_j \bmod k$ for all $j = 1, \ldots, i$. That is, there is no i, $1 \leq i \leq n-1$, such that the equivalences classes mod k of the first i letters of u match those of the last i letters of u.

For example, if $k = 3$, then $u = 1\ 2\ 2$ has the mod 3-nonoverlapping property. Also, any permutation of $\{1, 2, \ldots, k\}$ has the mod k-nonoverlapping property. Words with the mod k-nonoverlapping property satisfy the following simple analogue of Theorem 1.1.

Theorem 2.2 *If* $u = u_1 \ldots u_n \in \mathbb{P}^*$ *has the mod* k*-nonoverlapping property, then*
$$S_k(u; t, x) = \frac{t^n x^{\Sigma(u)}}{t^n x^{\Sigma(u)} + [k]_x(1 - x - tx)(1 - x^k)^{n-1}}$$
where $[k]_x = \frac{1-x^k}{1-x} = 1 + x + \cdots + x^{k-1}$.

Proof. Suppose $u = u_1 \dots u_n$ has the mod k-nonoverlapping property. Then if $v \in \mathcal{A}_k(u)$ and $w = w_1 \dots w_n \in \mathcal{W}_k(u)$, then $vw \in \mathcal{S}_k(u)$ because the mod k-nonoverlapping property ensures that any embedding of u other than w must be disjoint from w. It follows that

$$\mathcal{S}_k(u) = \mathcal{A}_k(u)\mathcal{W}_k(u) \tag{3}$$

and

$$\begin{aligned} S_k(u;t,x) &= A_k(u;t,x)W_k(u;t,x) \\ &= \frac{(1-x)}{(1-x-tx)}(1-S_k(u;t,x))\frac{t^n x^{\Sigma(u)}}{(1-x^k)^n}. \end{aligned}$$

Solving for $S_k(u,t,x)$ gives the result. □

Theorem 2.2 has a number of simple consequences which show how different Wilf equivalence is relative to $\mathcal{P}_1$ versus $\mathcal{P}_k$ for $k \geq 2$. For example, the following is a simple corollary.

COROLLARY 2.3 *For any $k \geq 2$, if $u = u_1 \dots u_n$ and $v = v_1 \dots v_n$ have the mod k-nonoverlapping property, then $u \backsim_k v$ if and only if $\Sigma(u) = \Sigma(v)$.*

Thus, for example, any two permutations in the symmetric group S_k are Wilf equivalent relative to $\mathcal{P}_k$. Moreover, as we stated in the introduction, it is not the case that $u \backsim_k v$ implies u and v are rearrangements, since, for example, Corollary 2.3 implies $1\ 4\ 2 \backsim_k 2\ 2\ 3$ for all $k \geq 2$ or $1\ 4 \backsim_k 2\ 3$ for all $k \geq 4$.

We now consider the more general case where embeddings of u can overlap. If $u = u_1 \dots u_n$, we let $\mathcal{C}_k(u)$ be the set of i, $1 \leq i \leq n-1$, such that the equivalences classes mod k of the first i letters of u match those of the last i letters of u. So embeddings of u can overlap in i characters a word w if and only if $i \in \mathcal{C}_k(u)$. For example, if $k = 3$ and $u = 1\ 5\ 4\ 3\ 2\ 4\ 2\ 1$, then $\mathcal{C}_3(u) = \{1,3\}$ so that embeddings of u can overlap in either 1 or 3 characters.

For $i \in \mathcal{C}_k(u)$, let

$$D_k^{(i)}(u) = \{n-i+j : 1 \leq j \leq i \text{ and } u_j >_k u_{n-i+j}\}$$

and $d_{i,k}(u) = \sum_{n-i+j \in D_k^{(i)}(u)} (u_j - u_{n-i+j})$. For example, again with $k = 3$ and $u = 1\ 5\ 4\ 3\ 2\ 4\ 2\ 1$, we have $D_3^{(1)}(u) = \emptyset$, $d_{1,3}(u) = 0$, $D_3^{(3)}(u) = \{7,8\}$, and $d_{3,3}(u) = (5-2)+(4-1) = 6$.

Now, if $w = w_1 \dots w_m \in \mathcal{S}_k(u)$, then $w_1 \dots w_{m-n} \in \mathcal{A}_k(u)$ and $u \leq_k w_{m-n+1} \dots w_m$. However if $v \in \mathcal{A}_k(u)$ and $z = z_1 \dots z_n$ is such that $u \leq_k z$, then it may not be the case that $w = vz \in \mathcal{S}_k(u)$ as there might be another embedding of u in the last $2n-1$ letters of w, starting in v and ending in z. Of course, there can be no embedding of u which starts to the left of the last $2n-1$ letters of w since $v \in \mathcal{A}_k(u)$.

To find an analogue of (3), then, we let $\mathcal{S}_k^{(i)}(u)$ be the set of all words $w = w_1 \dots w_m$ such that

(i) $u \leq_k w_{m-n+1} \dots w_m$ (so u embeds into the suffix of length n of w under $\leq_k$), and

(ii) the left-most embedding of u into w relative to $\leq_k$ starts at position $m-2n+i$, where $1 \leq i \leq n$.

We also let

$$S_k^{(i)}(u;t,x) = \sum_{w \in \mathcal{S}_k^{(i)}(u)} x^{\Sigma(w)} t^{|w|}.$$

Thus the desired analogue of (3) is

$$\mathcal{S}_k(u) = \mathcal{A}_k(u)\mathcal{W}_k(u) - \left(\bigcup_{i\in\mathcal{C}_k(u)} \mathcal{S}_k^{(i)}(u)\right),$$

and

$$\begin{aligned} S_k(u;t,x) &= \frac{(1-x)}{(1-x-tx)}(1-(S_k(u;t,x))\frac{t^n x^{\Sigma(u)}}{(1-x^k)^n} \\ &\quad - \sum_{i\in\mathcal{C}_k(u)} S_k^{(i)}(u;t,x). \end{aligned} \tag{4}$$

Now let $\bar{\mathcal{S}}_k^{(i)}(u)$ denote the set of all words $w = w_1 \dots w_m \in \mathcal{S}_k(u)$ such that $w_{m-i+j} \geq_k u_j$ for $j = 1,\dots,i$, that is, the set of all words in $\mathcal{S}_k(u)$ whose last i characters embed the first i characters of u. Then $\bar{\mathcal{S}}_k^{(i)}(u)$ is non-empty if and only if $i \in \mathcal{C}_k(u)$. If $i \in \mathcal{C}_k(u)$, we let

$$\bar{S}_k^{(i)}(u;t,x) = \sum_{w\in\bar{\mathcal{S}}_k^{(i)}(u)} x^{\Sigma(w)} t^{|w|}.$$

Also let $s_i = u_{i+1}\dots u_n$. Then

$$W_k(s_i;t,x) = \frac{t^{n-i}x^{\Sigma(s_i)}}{(1-x^k)^{n-i}},$$

and if $i \in \mathcal{C}_k(u)$, then

$$\mathcal{S}_k^{(i)}(u) = \bar{\mathcal{S}}_k^{(i)}(u)\mathcal{W}_k(s_i).$$

Hence

$$S_k^{(i)}(u;t,x) = \bar{S}_k^{(i)}(u;t,x)\frac{t^{n-i}x^{\Sigma(s_i)}}{(1-x^k)^{n-i}}. \tag{5}$$

To proceed, we need a condition on u that will allow us to write $\bar{S}_k^{(i)}(u;t,x)$ in terms of $S_k(u;t,x)$. We can then substitute the resulting expression for $S_k^{(i)}(u;t,x)$ into (4) and solve for $S_k(u;t,x)$. We call the necessary condition the *mod k-comparison condition.*

DEFINITION 2.4 A word $u = u_1 \dots u_n \in \mathbb{P}^*$ has the **mod k-comparison condition** if whenever $i, s \in \mathcal{C}_k(u)$, $n-i < s$, and $u_j >_k u_{n-i+j}$ where $1 \leq j \leq i$, then $u_{n-s+n-i+j} \leq_k u_{n-i+j}$ if $n-s+n-i+j \leq n$.

Schematically, this can be pictured as in Figure 1. Here the overlapping lines are restricted to cases where the corresponding letters are in the same equivalence classes mod k.

The motivation for this definition will be found in the proof of the following lemma, which gives the necessary relationship between $\bar{S}_k^{(i)}(u;t,x)$ and $S_k(u;t,x)$.

LEMMA 2.5 *If u has the mod k-comparison condition, then*

$$\bar{S}_k^{(i)}(u;t,x) = x^{d_{i,k}(u)} S_k(u;t,x). \tag{6}$$

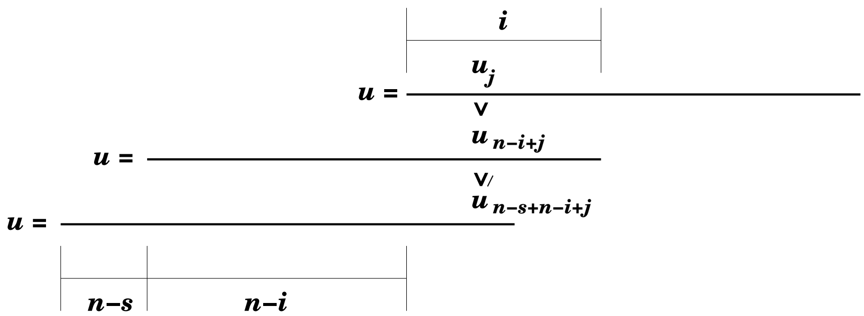

Figure 1: Schematic for the mod k-comparison condition

Proof. We can view the right hand side of (6) as the weight generating function of the set $T(u)$ of words formed by taking each word $\tilde{v} = \tilde{v}_1 \dots \tilde{v}_p$ in $\mathcal{S}_k(u)$ and increasing $\tilde{v}_{n-i+j}$ by $u_j - u_{n-i+j}$ if $n-i+j \in D_k^{(i)}(u)$ and leaving all other characters unchanged. So it suffices to show that $\bar{\mathcal{S}}_k^{(i)}(u) = T(u)$.

First suppose $v = v_1 \dots v_p \in \bar{\mathcal{S}}_k^{(i)}(u)$. Then let $\tilde{v} = \tilde{v}_1 \dots \tilde{v}_p$ be the word that results from v by decreasing v_{p-i+j} by $u_j - u_{n-i+j}$ if $n-i+j \in D_k^{(i)}(u)$ and leaving all other characters unchanged. If $n-i+j \in D_k^{(i)}(u)$, then $v_{p-i+j} \geq_k u_j$, and hence $\tilde{v}_{p-i+j} \geq_k u_{n-i+j}$. So $\tilde{v} \in \mathcal{S}_k(u)$, and therefore $v \in T(u)$.

Now suppose $v \in T(u)$, where v is formed from a word $\tilde{v} = \tilde{v}_1 \dots \tilde{v}_p$ in $\mathcal{S}_k(u)$ by increasing $\tilde{v}_{p-i+j}$ by $u_j - u_{n-i+j}$ if $n-i+j \in D^{(i)}(u)$ and leaving all other letters the same. By construction, the last i characters of v embed the first i characters of u. The only question that remains is whether or not v is still in $\mathcal{S}_k(u)$. That is, by increasing some letters in $\tilde{v}$ to get v, we might have created a new embedding of u which starts to the left of position $p-n+1$. However, the mod k-comparison condition prevents this from happening. If such an embedding were to be formed, this new embedding and the embedding beginning at position $p-n+1$ must overlap in s characters for some $s \in \mathcal{C}_k(u)$ such that $s > n-i$ (since only the last $n-i$ characters are potentially changed when forming v from $\tilde{v}$). Further, this new embedding must contain at least one position of the form $p-i+j$ where $n-i+j \in D_k^{(i)}(u)$, so that $v_{p-i+j} \geq_k u_{n-s+n-i+j}$. However, for any such j, we have $\tilde{v}_{p-i+j} \geq_k u_{n-i+j} \geq_k u_{n-s+n-i+j}$ by the mod k-comparison condition, so any new embedding that exists in v would have to have already existed in $\tilde{v}$, a contradiction since $\tilde{v} \in \mathcal{S}_k(u)$. Therefore $v \in \bar{\mathcal{S}}_k^{(i)}(u)$, completing the proof.

□

To illustrate the ideas in the proof of Lemma 2.5, let $k = 3$ and

$$u = 1\ 10\ 9\ 2\ 4\ 7\ 6\ 8\ 7\ 4\ 6\ 5\ 4\ 1.$$

To determine how embeddings of u can overlap, it is useful to reduce the entries of u mod k. That is, for any word u, let $u^{(k)}$ be the word in $\{1, 2, \dots, k\}^*$ whose entries are equivalent mod k to the entries of u. In this case, we have

$$u^{(3)} = 1\ 1\ 3\ 2 \mid 1\ 1\ 3\ 2 \mid 1\ 1\ 3\ 2 \mid 1\ 1,$$

where we have added dividers to emphasize the repeating pattern. So $\mathcal{C}_3(u) = \{1, 2, 6, 10\}$. This leads to three possibilities that we must consider to determine if the mod 3-comparison condition holds. We list them here, with corresponding diagrams as in Figure 1.

Case 1: $i = s = 10$.

$$\begin{array}{cccccccccccccccccccccc}
&&&&&&&& 1 & \mathbf{10} & \mathbf{9} & 2 & 4 & \mathbf{7} & 6 & 8 & 7 & 4 & 6 & 5 & 4 & 1 \\
&&&& 1 & 10 & 9 & 2 & 4 & \mathbf{7} & \mathbf{6} & 8 & 7 & \mathbf{4} & 6 & 5 & 4 & 1 \\
1 & 10 & 9 & 2 & 4 & 7 & 6 & 8 & 7 & \mathbf{4} & \mathbf{6} & 5 & 4 & \mathbf{1}
\end{array}$$

Case 2: $i = 10, s = 6$.

$$\begin{array}{cccccccccccccccccccccccccc}
&&&&&&&&&&&& 1 & \mathbf{10} & 9 & 2 & 4 & 7 & 6 & 8 & 7 & 4 & 6 & 5 & 4 & 1 \\
&&&&&&&& 1 & 10 & 9 & 2 & 4 & \mathbf{7} & 6 & 8 & 7 & 4 & 6 & 5 & 4 & 1 \\
1 & 10 & 9 & 2 & 4 & 7 & 6 & 8 & 7 & 4 & 4 & 5 & 4 & \mathbf{1}
\end{array}$$

Case 3: $i = 6, s = 10$.

$$\begin{array}{cccccccccccccccccccccccccc}
&&&&&&&&&&&& 1 & \mathbf{10} & 9 & 2 & 4 & 7 & 6 & 8 & 7 & 4 & 6 & 5 & 4 & 1 \\
&&&&& 1 & 10 & 9 & 2 & 4 & 7 & 6 & 8 & 7 & \mathbf{4} & 6 & 5 & 4 & 1 \\
1 & 10 & 9 & 2 & 4 & 7 & 6 & 8 & 7 & 4 & 6 & 5 & 4 & \mathbf{1}
\end{array}$$

In each case, the mod 3-comparison condition is satisfied since whenever there is a strict decrease in a column of size 3 from the first row to the second row, there is a weak decrease from the second row to the third row, as indicated by the columns in bold.

Now, for example, let $i = 10$. Then an element v of $\bar{\mathcal{S}}_k^{(10)}(u)$ must end in an embedding of u in the final 14 characters and an embedding of the first 10 characters of u in the final 10 characters, as shown:

$$\begin{array}{cccccccccccccccc}
v = & \cdots & \bullet & \bullet & \bullet & \bullet & \bullet & \star & \star & \bullet & \bullet & \star & \star & \star & \star & \star \\
&&&&&& 1 & \mathbf{10} & \mathbf{9} & 2 & 4 & \mathbf{7} & \mathbf{6} & \mathbf{8} & \mathbf{7} & \mathbf{4} \\
&& 1 & 10 & 9 & 2 & 4 & \mathbf{7} & \mathbf{6} & 8 & 7 & \mathbf{4} & \mathbf{6} & \mathbf{5} & \mathbf{4} & \mathbf{1}
\end{array}.$$

The stars indicate the positions in a word $\tilde{v} \in \mathcal{S}_k(u)$ that must be increased to form $v \in \bar{\mathcal{S}}_k^{(i)}$ as in the proof. If a new embedding of u is created in this process, this embedding must end either five or nine characters from the end of v, as shown in Cases 1 and 2 above, and reiterated in the following diagrams:

$$\begin{array}{cccccccccccccccccccc}
v = & \cdots & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \star & \star & \bullet & \bullet & \star & \star & \star & \star & \star \\
&&&&&&&&&& 1 & \mathbf{10} & \mathbf{9} & 2 & 4 & \mathbf{7} & \mathbf{6} & \mathbf{8} & \mathbf{7} & \mathbf{4} \\
&&&&&& 1 & 10 & 9 & 2 & 4 & \mathbf{7} & \mathbf{6} & 8 & 7 & \mathbf{4} & \mathbf{6} & \mathbf{5} & \mathbf{4} & \mathbf{1} \\
&& 1 & 10 & 9 & 2 & 4 & 7 & 6 & 8 & 7 & \mathbf{4} & \mathbf{6} & 5 & 4 & \mathbf{1}
\end{array}$$

$$\begin{array}{cccccccccccccccccccccccc}
v = & \cdots & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \star & \star & \bullet & \bullet & \star & \star & \star & \star & \star \\
&&&&&&&&&&&&&& 1 & \mathbf{10} & \mathbf{9} & 2 & 4 & \mathbf{7} & \mathbf{6} & \mathbf{8} & \mathbf{7} & \mathbf{4} \\
&&&&&&&&&& 1 & 10 & 9 & 2 & 4 & \mathbf{7} & \mathbf{6} & 8 & 7 & \mathbf{4} & \mathbf{6} & \mathbf{5} & \mathbf{4} & \mathbf{1} \\
&& 1 & 10 & 9 & 2 & 4 & 7 & 6 & 8 & 7 & 4 & 6 & 5 & 4 & \mathbf{1}
\end{array}.$$

Because the starred columns are weakly decreasing from top to bottom, any embedding of u into v that begins to the left of the final n characters of v would have already existed in $\tilde{v}$.

Now, to complete the analogue of Theorem 1.1 for $k \geq 2$, we substitute (6) into (5) to obtain

$$S_k^{(i)}(u;t,x) = S_k(u;t,x)\frac{t^{n-i}x^{\Sigma(s_i)+d_{i,k}(u)}}{(1-x^k)^{n-i}}. \tag{7}$$

Substituting (7) into (4) and solving for $S_k(u;t,x)$ yields the following.

THEOREM 2.6 *If $k \geq 2$ and $u = u_1 \dots u_n \in \mathbb{P}^*$ has the mod k-comparison condition, then*

$$S_k(u;t,x) = \frac{t^n x^{\Sigma(u)}}{t^n x^{\Sigma(u)} + [k]_x(1-x-tx)((1-x^k)^{n-1} + \sum_{i\in\mathcal{C}_k(u)} t^{n-i}x^{d_{i,k}+\Sigma(s_i)}(1-x^k)^{i-1})}.$$

There are a number of ways in which one can ensure that w satisfies the mod k-comparison condition. Two of these are immediate, and demonstrate the richness of the set of such words.

1. If u has the mod k-nonoverlapping property, then u satisfies the mod k-comparison condition since $\mathcal{C}_k(u)$ is empty. In this case Theorem 2.6 reduces to Theorem 2.2.

2. If u is such that $\max(\mathcal{C}_k(u)) \leq |u|/2$, then it is easy to see from Figure 1 that a violation of the mod k-comparison condition would require that $n-s+n-i \in \mathcal{C}_k(u)$ where $n-s+n-i > |u|/2$. So words u such that embeddings of u can overlap in at most one half of u satisfy the mod k-comparison condition.

So it remains to characterize those words u with the mod-k comparison condition such that embeddings of u can overlap in more than half of u. A further analysis of the preceding example will lead us to an alternate characterization of the mod k-comparison condition that completely describes the set of such words, and generalizes the idea of an increasing/decreasing factorization for words in $\mathcal{P}_k$.

To this end, again setting

$$u = 1\ 10\ 9\ 2 \mid 4\ 7\ 6\ 8 \mid 7\ 4\ 6\ 5 \mid 4\ 1$$

and

$$u^{(3)} = 1\ 1\ 3\ 2 \mid 1\ 1\ 3\ 2 \mid 1\ 1\ 3\ 2 \mid 1\ 1,$$

we let $a_1 = 1\ 10\ 9\ 2$, $a_2 = 4\ 7\ 6\ 8$, $a_3 = 7\ 4\ 6\ 5$, and $b = 4\ 1$ so that

$$u = a_1a_2a_3b,$$

$a_i^{(3)} = a_j^{(3)}$ for all i and j, and $b^{(3)}$ is a prefix of each $a_i^{(3)}$. Then for $1 \leq j \leq 4$, let $u[j]$ be the word consisting of the j-th characters of each a_i, and the j-th character of b if $j \leq 2$. So $u[1] = 1\ 4\ 7\ 4$, $u[2] = 10\ 7\ 4\ 1$, $u[3] = 9\ 6\ 6$ and $u[4] = 2\ 8\ 5$. Then in the mod k-comparison condition, characters of $u[j]$ are only compared with other characters of $u[j]$ for all j. The inequalities in the condition will be violated if there is ever a strict decrease in a $u[j]$ that is followed later by a strict increase, that is, if some $u[j]$ does not have an increasing/decreasing factorization. For example, suppose we modify our example so that $u[2] = 7\ 1\ 1\ 4$:

$$u = 1\ \mathbf{7}\ 9\ 2\ 4\ \mathbf{1}\ 6\ 8\ 7\ \mathbf{1}\ 6\ 5\ 4\ \mathbf{4}.$$

Then because $u[2]_1 > u[2]_2 < u[2]_4$ and $u[2]_1 > u[2]_3 < u[2]_4$, the mod 3-comparison condition can be seen to be violated in both of the following diagrams:

Case 2: $i = 10, s = 6, n - s = 8$.

$$\begin{array}{cccccccccccccccccccccccccc}
 & & & & & & & & & & & & 1 & \mathbf{7} & 9 & 2 & 4 & 1 & 6 & 8 & 7 & 1 & 6 & 5 & 4 & 4 \\
 & & & & & & & & 1 & 7 & 9 & 2 & 4 & \mathbf{1} & 6 & 8 & 7 & 1 & 6 & 5 & 4 & 4 & & & & \\
1 & 7 & 9 & 2 & 4 & 1 & 6 & 8 & 7 & 1 & 4 & 5 & 4 & \mathbf{4} & & & & & & & & & & & &
\end{array}$$

Case 3: $i = 6, s = 10, n - s = 4$.

$$\begin{array}{cccccccccccccccccccccccccc}
 & & & & & & & & & & & & 1 & \mathbf{7} & 9 & 2 & 4 & 1 & 6 & 8 & 7 & 1 & 6 & 5 & 4 & 4 \\
 & & & & 1 & 7 & 9 & 2 & 4 & 1 & 6 & 8 & 7 & \mathbf{1} & 6 & 5 & 4 & 4 & & & & & & & & \\
1 & 7 & 9 & 2 & 4 & 1 & 6 & 8 & 7 & 1 & 6 & 5 & 4 & \mathbf{4} & & & & & & & & & & & &
\end{array}$$

So if we factor an arbitrary u in the right way, the mod k-comparison condition is equivalent to each $u[j]$ having an increasing/decreasing factorization. The following lemma illustrates the factorization, after which we will prove the preceding statement.

LEMMA 2.7 *Let $u = u_1 \dots u_n \in \mathbb{P}^*$, $l = n - \max(\mathcal{C}_k(u))$ and suppose $n = ml + r$ with $0 \le l < r$. Define $a_i = u_{(i-1)l+1} \dots u_{il}$ for $1 \le i \le m$ and $b = u_{ml+1} \dots u_n$, so that $|a_i| = l$ for all i, $|b| = r$ and*

$$u = a_1 \dots a_m b.$$

Then

1. *$a_i^{(k)} = a_j^{(k)}$ for all i, $1 \le i \le m$,*
2. *$b^{(k)}$ is a prefix of $a_1^{(k)}$,*
3. *$\mathcal{C}_k(u) = \mathcal{C}_k(a_m b) \cup \{r, r+l, r+2l, \dots, r+(m-1)l\}$.*

Proof. First suppose $\max(\mathcal{C}_k(u)) < |u|/2$. Then $m = 1$ and $u = a_1 b$ so that statements 1 and 3 follow immediately. Also, $r = \max(\mathcal{C}_k(u))$, so embeddings of u can overlap in r characters, and therefore the first r characters of u must be equivalent mod k to the last r characters, which is equivalent to statement 2.

Now suppose $\max(\mathcal{C}_k(u)) \ge |u|/2$. Then embeddings of u can overlap in all but l characters, as shown:

$$\begin{array}{ccccccc}
a_1 & a_2 & a_3 & \cdots & a_m & b & \\
 & a_1 & a_2 & \cdots & a_{m-1} & a_m & b
\end{array}.$$

Since all the a_i's are the same length, this forces the first two conditions of the lemma, and shows that

$$\{r, r+l, r+2l, \dots, r+(m-1)l\} \subseteq \mathcal{C}_k(u).$$

Also, since $a_m^{(k)} = a_1^{(k)}$, we have $\mathcal{C}_k(a_m b) \subseteq \mathcal{C}_k(u)$. So

$$\mathcal{C}_k(a_m b) \cup \{r, r+l, r+2l, \dots, r+(m-1)l\} \subseteq \mathcal{C}_k(u).$$

All that remains is to show equality. If the containment is proper, then embeddings of u must be able to overlap in such a way that embeddings of the a_i's are offset, so that the overlap begins with the embedding of a nonempty suffix of some a_i also embedding a nonempty prefix of a_1. But because conditions 1 and 2 are satisfied, we would be able to shift this overlap to create overlapping embeddings of u beginning in a nonempty suffix of a_1, violating the condition that $l = n - \max(\mathcal{C}_k(u))$. □

With this lemma in hand, we can give our alternate characterization of the mod k-comparison condition. With u factored as above, again define $u[j]$ to be the string consisting of the j-th characters of each a_i and the j-th character of b if it exists. Statement 3 of Lemma 2.7 says that embeddings of u can only overlap in a substring of b (including all of b) or with an embedding of a_i overlapping an embedding of a_1 for some i, as pictured:

$$\begin{array}{cccccccccc} a_1 & a_2 & \cdots & a_i & a_{i+1} & \cdots & a_m & b & & \\ & & & a_1 & a_2 & \cdots & a_{m-i+1} & a_{m-i+2} & \cdots & a_m. \end{array}$$

So in particular, in any overlapping embeddings, characters of $u[j]$ must be lined up only with characters of $u[j]$, for all j, unless the overlap is only in b. This leads to the following.

THEOREM 2.8 *Let $u \in \mathbb{P}^*$. Then u has the mod k-comparison condition if and only if each $u[j]$ has an increasing/decreasing factorization.*

Proof. Since $|b| < |u|/2$, any embeddings that overlap in b do not affect the mod k-comparison condition. If each $u[j]$ has an increasing/decreasing factorization, it is easy to see that the mod k-comparison condition is satisfied. For the converse, suppose $u[j']$ does not have an increasing/decreasing factorization for some j'. Then since

$$u[j'] = u_{j'}u_{j'+l}\dots u_{j'+(m-1)l}u_{j'+ml}$$

(where $u_{j'+ml}$ is the empty string if $j' > |b|$), there exist $0 \le q < r < t \le m$ (or $m-1$ if $j' > |b|$) such that

$$u_{j'+ql} > u_{j'+rl} < u_{j'+tl}.$$

To see that the mod k comparison condition is violated, set $s = n - (t-r)l$, $i = n - (r-q)l$ and $j = j' + ql$. Then both s and i are in $\mathcal{C}_k(u)$, and

$$n - i = (r-q)l \le rl < rl + (n - tl) = n - (t-r)l = s.$$

But then

$$j = j' + ql < n - rl + ql = n - (r-q)l = i$$

and

$$u_j = u_{j'+ql} > u_{j'+rl} = u_{n-i+j},$$

but

$$u_{(n-s)+(n-i)+j} = u_{j'+tl} > u_{j'+rl} = u_{n-i+j},$$

violating the condition. □

We note here that if u has the mod k-nonoverlapping property, then $m = 1$ and $b = \varepsilon$, so that all $u[j]$'s consist of a single character, and therefore trivially have increasing/decreasing factorizations.

Also, if $\max(\mathcal{C}_k(u)) \leq |u|/2$, then either $u = a_1a_2$ or $u = a_1b$. In each case, the $u[j]$'s consist of at most two characters, and so again trivially have increasing/decreasing factorizations. Finally, if all the entries of u come from the same equivalence class mod k, then embeddings of u can overlap in any number of characters, so each a_i is just the i-th character of u, and $u[1] = u$. So in this case the mod k-comparison condition is equivalent to u having an increasing/decreasing factorization.

The methods used to prove Theorem 2.6 fail when u does not have the mod k-comparison condition, in particular Lemma 2.5. In [4], Langley, Liese, and Remmel conjecture that $S_1(u;t,x)$ has the form $t^r x^s/P(u;t,x)$ with $P(u;t,x)$ a polynomial if and only if u has in increasing/decreasing factorization. We conjecture the analogous statement here.

Conjecture 2.9 For $u \in \mathbb{P}^*$, $S_k(u;t,x)$ has the form $t^r x^s/P(u;t,x)$ with $P(u;t,x)$ a polynomial if and only if u has the mod k-comparison condition, that is, each $u[j]$ has an increasing/decreasing factorization.

As an example of the utility of Theorem 2.6, we shall classify the Wilf equivalence classes relative to $\mathcal{P}_k$ of the permutations in the symmetric group S_{k+l} where $l \leq k$. Suppose $u = u_1 \dots u_{k+l} \in S_{k+l}$. If $\mathcal{C}_k(u) = \emptyset$, then define $i(u) = 0$ and $p_u^{(k)} = \varepsilon$. If $\mathcal{C}_k(u) \neq \emptyset$, define $i(u) = \max(\mathcal{C}_k(u))$ and $p_u^{(k)} = (u_1u_2 \dots u_{i(u)})^{(k)}$.

Theorem 2.10 *Suppose $u, v \in S_{k+l}$ where $l \leq k$. Then $u \backsim_k v$ if and only if $i(u) = i(v)$ and $\Sigma(p_u^{(k)}) = \Sigma(p_v^{(k)})$.*

Proof. First, since $l \leq k$, there are at most two entries in any permutation of $\{1, 2, \dots, k+l\}$ in each equivalence class mod k. Now suppose that $u = u_1 \dots u_{k+l} \in S_{k+l}$ and $\mathcal{C}_k(u) \neq \emptyset$. Then we claim that $i(u) \leq l$ and $\mathcal{C}_k(u) = \{i(u)\}$. That is, by definition of $i(u)$, $u_j \equiv u_{k+l-i(u)+j} \bmod k$ for all j, $1 \leq j \leq i(u)$. So since there are at most two entries in u that are equivalent mod k, we must have $i(u) \leq l$. Similarly, if $i, j \in \mathcal{C}_k(u)$ and $i \neq j$, then $u_1 \equiv u_{n-i+1}$ and $u_1 \equiv u_{n-j+1}$, which would mean that there are at least three characters of u which have the same equivalence class mod k.

Now, since $i(u) \leq l \leq (l+k)/2$, u has the mod k-comparison condition and we can therefore apply Theorem 2.6 to compare any $u, v \in S_{k+l}$. First, since $u, v \in S_{k+l}$ implies that $t^{|u|}x^{\Sigma(u)} = t^{|v|}x^{\Sigma(v)}$, the only way that $S_k(u;t,x) = S_k(v;t,x)$ is if

$$\sum_{i \in \mathcal{C}_k(u)} t^{n-i} x^{d_{i,k}(u)+\Sigma(s_i(u))}(1-x^k)^{i-1} = \sum_{i \in \mathcal{C}_k(v)} t^{n-i} x^{d_{i,k}(v)+\Sigma(s_i(v))}(1-x^k)^{i-1}.$$

Thus if $i(u) \neq i(v)$, then $u \not\backsim_k v$.

Now there are two cases. First, if $i(u) = i(v) = 0$, then the above sums are empty and therefore $u \backsim_k v$ (u and v have the mod k-nonoverlapping property). Second, if $i(u) = i(v) = i$, then the above sums are equal if and only if $d_{i,k}(u) + \Sigma(s_i(u)) = d_{i,k}(v) + \Sigma(s_i(v))$. So it remains to show that $d_{i,k}(u) + \Sigma(s_i(u)) = d_{i,k}(v) + \Sigma(s_i(v))$ is equivalent to $\Sigma(p_u^{(k)}) = \Sigma(p_v^{(k)})$. Now, as mentioned above, if $p_u = u_1 \dots u_i$, we must have $u_i = u_{k+l-i+j} \bmod k$ for $1 \leq j \leq i$. Let r be the number of j such that u_j is greater than $u_{k+l-i+j}$. Then since there are at most two characters in each permutation in any one equivalence class, and they differ by k, we have $d_{i,k}(u) = rk$. Therefore

$$\Sigma(p_u) = \Sigma(p_u^{(k)}) + rk = \Sigma(p_u^{(k)}) + d_{i,k}(u),$$

which implies

$$\Sigma(p_u^{(k)}) = \Sigma(p_u) - d_{i,k}(u) = \Sigma(u) - \Sigma(s_i(u)) - d_{i,k}(u).$$

Similarly,

$$\Sigma(p_v^{(k)}) = \Sigma(p_v) - d_{i,k}(v) = \Sigma(v) - \Sigma(s_i(v)) - d_{i,k}(v).$$

Since $\Sigma(u) = \Sigma(v)$, we have $\Sigma(p_u^{(k)}) = \Sigma(p_v^{(k)})$ if, and only if, $\Sigma(s_i(u)) + d_{i,k}(u) = \Sigma(s_i(v)) + d_{i,k}(v)$, as required. □

As an example, suppose $k = 5$ and $l = 4$ so that we are dealing with permutations of $\{1, 2, \ldots, 9\}$ and working mod 5. Then the equivalence classes of permutations of $\{1, 2, \ldots, 9\}$ are as follows. We will organize these by the size of $p_u = u_1 \ldots u_{i(u)}$.

- $|p_u| = 0$. All words that fit this criterion are Wilf equivalent mod 5. This includes, but is not limited to, any word that begins or ends with 5 since there is no other entry equivalent to 5 mod k.

- $|p_u| = 1$. The equivalences here are of the form $i * j$ where $i = j$ mod 5 and $*$ represents all other characters (such that $|p_u| = 1$). So there are four equivalence classes in this case.

- $|p_u| = 2$. The equivalences here are organized by the sum of the first two entries, when these entries are reduced mod k. The smallest of these sums is 3, the largest 7 (remember that 5 can not be in p_u). So there are five equivalence classes in this case. The entries in p_u do not have to come from the same equivalence classes mod k, for example, $14 * 67 \backsim_5 23 * 78$ where $*$ is such that $|p_u| = 2$.

- $|p_u| = 3$. The equivalences here are organized by the sum of the first three entries, when these entries are reduced mod k. The smallest of these sums is 6, the largest 9. So there are four equivalence classes in this case.

- $|p_u| = 4$. The equivalences here are organized by the sum of the first four entries, when these entries are reduced mod k. As 5 can not be part of the sum, there is only one equivalence class of this form.

It is interesting to note, as this example shows, that the number of equivalence classes depends on l (the number of entries repeated mod k) and not k.

We can generalize this example to the following corollary.

COROLLARY 2.11 *Let $l \leq k$ be positive integers. Then there are $(l+1)(l^2 - l + 6)/6$ equivalence classes of permutations of $\{1, 2, \ldots k + l\}$ under Wilf equivalence mod k.*

Proof. As a result of Theorem 2.10, and as illustrated in the preceding example, for each i it suffices to count the possible values of $|\Sigma(p_u)|$ with $|p_u| = i$. But this is simply

$$\left(il - \binom{i}{2}\right) - \binom{i+1}{2} + 1 = il - i^2 + 1.$$

Summing over $0 \leq i \leq l$ gives the result. □

3 Wilf equivalence in $\mathcal{P}_k$ for words of length 3

In this section we use Theorem 2.6 as an aid to completely classify the Wilf equivalence classes of words of length 3 for any $k \geq 2$. Suppose we start with a word abc of length 3 and let $h, i, j \in \{1, \ldots, k\}$ such that $a \equiv h \bmod k$, $b \equiv i \bmod k$, and $c \equiv j \bmod k$. We note that if $u = u_1 \ldots u_n$ and $v = v_1 \ldots v_n$ are such that $u \backsim_k v$, then $\Sigma(u) = \Sigma(v)$ since $t^n x^{\Sigma(u)}$ and $t^n x^{\Sigma(v)}$ are the lowest degree terms in $F(u; t, x)$ and $F(v; t, x)$, respectively. We then have the following cases.

Case 1. abc has the mod k-nonoverlapping property.

It is easy to see that abc has the mod k-nonverlapping property if and only if $h \neq j$.
By Theorem 2.2,

$$S_k(abc; t, x) = \frac{t^3 x^{a+b+c}}{t^3 x^{a+b+c} + [k]_x (1 - x - tx)(1 - x^k)^2}. \tag{8}$$

As observed in Corollary 2.3, another word $a'b'c'$ with the mod k-noverlapping property is such that $abc \backsim_k a'b'c'$ if and only if $a + b + c = a' + b' + c'$.

Case 2 $h = j \neq i$.

In this case, $\mathcal{C}_k(abc) = \{1\}$. As noted in Theorem 1.2, Kitaev, Liese, Remmel, and Sagan [2] showed that $u \backsim_1 u^r$ for any u where u^r denotes the reverse of u. This also holds for any k, since the map $w \to w^r$ is a weight-preserving bijection from $\mathcal{F}_k(u)$ to $\mathcal{F}_k(v)$. So $abc \backsim_k cba$, and therefore $S_k(abc; t, x) = S_k(cba; t, x)$. Thus there is no loss in generality in assuming that $a \leq_k c$, so that $d_{1,k}(abc) = 0$ and $s_1(abc) = bc$. Since $\max\{\mathcal{C}_k(abc)\} < |abc|/2$, abc has the mod k-comparison condition, so it follows from Theorem 2.6 that

$$S_k(abc; t, x) = \frac{t^3 x^{a+b+c}}{t^3 x^{a+b+c} + [k]_x (1 - x - tx)((1 - x^k)^2 + t^2 x^{b+c})}. \tag{9}$$

It now follows from the fact that the denominators of (8) and (9) are different that no word in Case 2 can be Wilf equivalent relative to $\mathcal{P}_k$ to any word in Case 1. Moreover, it follows from (9) that if $a'b'c'$ is in Case 2 and $a' \leq c'$, then $abc \backsim_k a'b'c'$ if and only if $x^{b+c} = x^{b'+c'}$. Thus if $a \leq c$ and $a' \leq c'$, the set of words $\{abc, cba\}$ and $\{a'b'c', c'b'a'\}$ lie in the same Wilf equivalence class relative to $\mathcal{P}_k$ if and only if $a = a'$ and $b + c = b' + c'$.

Case 3 $h = i = j$.

In this case, we shall consider two subcases.

Case 3A. abc has an increasing/decreasing factorization.

Since abc has an increasing/decreasing factorization, it has the mod k-comparison condition. Also, $\mathcal{C}_k(abc) = \{1, 2\}$, so by Theorem 2.6,

$$S_k(abc; t, x) = \frac{t^3 x^{a+b+c}}{t^3 x^{a+b+c} + [k]_x (1 - x - tx)((1 - x^k)^2 + t x^{d_{2,k}+c}(1 - x^k) + t^2 x^{d_{1,k}+b+c})}. \tag{10}$$

Again, it is easy to see that the denominators in equations (8), (9), and (10) can never be equal so that any word abc that falls in Case 3A cannot be Wilf equivalent relative to $\mathcal{P}_k$ to any word that falls in Case 1 or Case 2.

It is routine to verify that if $a'b'c'$ is a rearrangement of abc and both abc and $a'b'c'$ have increasing/decreasing factorizations, then $d_{i,k}(abc) + s_i(abc) = d_{i,k}(a'b'c') + s_i(a'b'c')$ for $i = 1, 2$, and hence $S_k(abc; t, x) = S_k(a'b'c'; t, x)$. Thus if abc falls in Case 3A, any rearrangement of abc which has an increasing/decreasing factorization will be Wilf equivalent to abc relative to $\mathcal{P}_k$.

Finally, suppose abc and $a'b'c'$ are in Case 3A, $a \le b \le c$, $a' \le b' \le c'$ and $abc \backsim_k a'b'c'$. Then we know $a + b + c = a' + b' + c'$ and $d_{i,k}(abc) = d_{i,k}(a'b'c') = 0$ for $i = 1, 2$. Thus

$$\begin{aligned} S_k(abc; t, x) &= \frac{t^3 x^{a+b+c}}{t^3 x^{a+b+c} + [k]_x (1 - x - tx)((1 - x^k)^2 + tx^c(1 - x^k) + t^2 x^{b+c})} \text{ and} \\ S_k(a'b'c'; t, x) &= \frac{t^3 x^{a'+b'+c'}}{t^3 x^{a'+b'+c'} + [k]_x (1 - x - tx)((1 - x^k)^2 + tx^{c'}(1 - x^k) + t^2 x^{b'+c'})}. \end{aligned}$$

Hence the fact that $S_k(abc; t, x) = S_k(a'b'c'; t, x)$ implies

$$(1 - x^k)^2 + tx^c(1 - x^k) + t^2 x^{b+c} = (1 - x^k)^2 + tx^{c'}(1 - x^k) + t^2 x^{b'+c'}.$$

Thus we must have $c = c'$, $b + c = b' + c'$, and $a + b + c = a' + b' + c'$ which forces $abc = a'b'c'$. So if abc and $a'b'c'$ are in Case 3A, $abc \backsim_k a'b'c'$ if and only if $a'b'c'$ is a rearrangement of abc.

Case 3B. w is a word of length 3 which is in Case 3 but not in Case 3A.

Since any word w is Wilf equivalent to its reverse relative to $\mathcal{P}_k$, we can assume that w is of the form $w = bac$ where $a < b \le c$. We start with the following expression analogous to the argument leading to Theorem 2.6, but modified to account for the fact that bac does not have an increasing/decreasing factorization:

$$\begin{aligned} S_k(bac; t, x) &= A_k(bac; t, x)\, t^3 \frac{x^{a+b+c}}{(1 - x^k)^3} - S_k(bac; t, x)\, t^2 \frac{x^{a+c}}{(1 - x^k)^2} - S_k(bac; t, x)\, t \frac{x^c}{1 - x^k} \\ &\quad + A_k(bac; t, x)\, t^4 \frac{x^{b+2c}}{(1 - x^k)^3} (x^a + x^{a+k} + \cdots + x^{b-k}) \\ &\quad - S_k(bac; t, x)\, t^3 \frac{x^{2c}}{(1 - x^k)^2} (x^a + x^{a+k} + \cdots + x^{b-k}). \end{aligned} \tag{11}$$

The first term,

$$A_k(bac; t, x)\, t^3 \frac{x^{a+b+c}}{(1 - x^k)^3},$$

is the generating function for words consisting of an embedding of bac appended to a word that avoids bac. This includes the words in $\mathcal{S}_k(bac)$, but also includes words that end in overlapping embeddings of bac, either in the last four or five characters (and do not embed bac prior to those embeddings). So we need to remove the terms associated with these words. First consider words w that end in overlapping embeddings in the last five characters, as shown:

$$\begin{array}{ccccccc} w = & \cdots & b^{+k} & a^{+k} & c^{+k} & a^{+k} & c^{+k} \\ \hline & & b & a & c & & \\ & & & & b & a & c \end{array},$$

where for a positive integer m, m^{+k} represents any integer greater than or equal to m and equivalent to m mod k. Since $b \leq c$, these words can be formed by appending an embedding of ac to words in $\mathcal{S}_k(bac)$. So the second term on the right hand side,

$$S_k(bac; t, x)\, t^2 \frac{x^{a+c}}{(1 - x^k)^2},$$

accounts for these words.

Now consider words that end in overlapping embeddings in the final four characters only:

$$\begin{array}{cccccc} w = & \cdots & b^{+k} & b^{+k} & c^{+k} & c^{+k} \\ \hline & & b & a & c & \\ & & & b & a & c \end{array}.$$

The third term,

$$S_k(bac; t, x)\, t \frac{x^c}{1 - x^k},$$

removes the terms associated with these words by appending an embedding of c to words in $\mathcal{S}_k(bac)$. However, because $a < b$, this also includes terms associated with words of the form

$$\begin{array}{cccccc} w = & \cdots & b^{+k} & [a,b)_k & c^{+k} & c^{+k} \\ \hline & & b & a & c & \\ & & & b & a & c \end{array},$$

where $[a, b)_k = \{a + lk \mid l \geq 0 \text{ and } lk < b\}$, that is, words that first embed bac starting at the fourth character from the end, and end in an embedding of $bacc$ but not $bbcc$. To correct for these terms, consider the fourth term on the right hand side:

$$A_k(bac; t, x)\, t^4 \frac{x^{b+2c}}{(1 - x^k)^3}(x^a + x^{a+k} + \cdots + x^{b-k}).$$

This is the generating function for those words that end in an embedding of $bacc$ but not $bbcc$ (hence the $x^a + x^{a+k} + \cdots + x^{b-k}$ term), appended to words that avoid bac. So this includes the words that we want, but also may include words that first embed bac beginning at the fifth or sixth character from the end. However, the first of these situations is impossible, as shown,

$$\begin{array}{ccccccc} w = & \cdots & \bullet & b^{+k} & [a,b)_k & c^{+k} & c^{+k} \\ \hline & & b & a & c & & \\ & & & b & a & c & \\ & & & & b & a & c \end{array},$$

since a character in $[a,b)$ cannot embed c. The final term,

$$S_k(bac;t,x)\, t^3 \frac{x^{2c}}{(1-x^k)^2}(x^a+\cdots+x^{b-k}),$$

accounts for the second possibility,

$$\begin{array}{ccccccc} w = \;\cdots & b^{+k} & a^{+k} & c^{+k} & [a,b)_k & c^{+k} & c^{+k} \\ \hline & b & a & c & & & \\ & & & b & a & c & \\ & & & & b & a & c \end{array},$$

by appending an embedding of acc, whose first character is in $[a,b)$, to words in $\mathcal{S}_k(bac)$.

Now let $x^{[a,b)_k} = x^a + x^{a+k} + \cdots + x^{b-k}$, $S_k = S_k(bac;t,x)$, and $A_k = A_k(bac;t,x)$. Then we can rewrite (11) as

$$\begin{aligned} S_k &= A_k\left(\frac{t^3x^{a+b+c}}{(1-x^k)^3} + \frac{t^4x^{b+2c}x^{[a,b)_k}}{(1-x^k)^3}\right) \\ &\quad - S_k\left(\frac{tx^c}{(1-x^k)} + \frac{t^2x^{a+c}}{(1-x^k)^2} + \frac{t^3x^{2c}x^{[a,b)_k}}{(1-x^k)^2}\right). \end{aligned} \tag{12}$$

Using the fact that $A_k = \frac{(1-x)}{(1-x-tx)}(1-S_k)$ and multiplying both side of (12) by $(1-x^k)^3(1-x-tx)$, we have

$$\begin{aligned} &(1-x^k)^3(1-x-tx)S_k = \\ &(1-S_k)(1-x)(t^3x^{a+b+c} + t^4x^{b+2c}x^{[a,b)_k}) \\ &-S_k(1-x-tx)(tx^c(1-x^k)^2 + t^2x^{a+c}(1-x^k) + t^3x^{2c}x^{[a,b)_k}(1-x^k)). \end{aligned} \tag{13}$$

Dividing (13) by $(1-x)$ and bringing all the terms involving S_k to the left hand side, we obtain

$$S_k(t^3x^{a+b+c} + t^4x^{b+2c}x^{[a,b)_k} + [k]_x(1-x-tx)\psi_k(t,x)) = t^3x^{a+b+c} + t^4x^{b+2c}x^{[a,b)_k}$$

where

$$\psi_k(bac;t,x) = (1-x^k)^2 + tx^c(1-x^k) + t^2x^{a+c} + t^3x^{2c}x^{[a,b)_k}.$$

So

$$S_k(bac;t,x) = \frac{t^3x^{a+b+c}(1+tx^{c-a}x^{[a,b)_k})}{t^3x^{a+b+c} + t^4x^{b+2c}x^{[a,b)_k} + [k]_x(1-x-tx)\psi_k(bac;t,x)}. \tag{14}$$

One can easily check that when $x \neq 1$ the numerator of (14) divides (in fact is equal to) $t^3x^{a+b+c} + t^4x^{b+2c}x^{[a,b)_k}$ from the denominator, but does not divide the term $[k]_x(1-x-tx)\psi_k(bac;t,x)$ from the denominator. Thus, we can not write $S_k(bac;t,x)$ in the form of $\dfrac{t^rx^s}{P(t,x)}$ where $P(t,x)$ is a polynomial. Therefore no word in Case 3B can be Wilf equivalent to a word that is in Case 1, Case 2, or Case 3A relative to $\mathcal{P}_k$.

Thus to finish our classification of the Wilf equivalence classes relative to $\mathcal{P}_k$ of words of length 3, we shall show that if $a < b \le c$ and $a' < b' \le c'$, and $S_k(bac;t,x) = S_k(b'a'c';t,x)$, then $bac = b'a'c'$. Now, if $S_k(bac;t,x) = S_k(b'a'c';t,x)$, then

$$\begin{aligned}&(t^3x^{a+b+c} + t^4x^{b+2c}x^{[a,b)_k})(t^3x^{a'+b'+c'} + t^4x^{b'+2c'}x^{[a',b')_k} + [k]_x(1-x-tx)\psi_k(b'a'c';t,x)) = \\ &(t^3x^{a'+b'+c'} + t^4x^{b'+2c'}x^{[a',b')_k})(t^3x^{a+b+c} + t^4x^{b+2c}x^{[a,b)_k} + [k]_x(1-x-tx)\psi_k(bac;t,x)). \quad (15)\end{aligned}$$

First, since $bac \backsim_k b'a'c'$, we have that $a+b+c = a'+b'+c'$. Taking the coefficient of t^8 on both sides of (15), we see that

$$\begin{aligned}&x^{b+2c}x^{[a,b)_k}(x^{b'+2c'}x^{[a',b')_k} - x[k]_x x^{2c'}x^{[a',b')_k}) = \\ &x^{b'+2c'}x^{[a',b')_k}(x^{b+2c}x^{[a,b)_k} - x[k]_x x^{2c}x^{[a,b)_k}). \quad (16)\end{aligned}$$

Simplifying (16), we have

$$x^{b+2c}x^{[a,b)_k}x^{2c'}x^{[a',b')_k} = x^{b'+2c'}x^{[a',b')_k}x^{2c}x^{[a,b)_k},$$

which implies $x^b = x^{b'}$, and so $b = b'$. Finally, taking the coefficient of t^4 on both sides of (15), we see that

$$\begin{aligned}&x^{a+b+c}(-x[k]_x(1-x^k)^2) + x^{a+b+c}[k]_x(1-x)x^{c'}(1-x^k) \\ &+x^{b+2c}x^{[a,b)_k}[k]_x(1-x)(1-x^k)^2 = \\ &x^{a'+b'+c'}(-x[k]_x(1-x^k)^2) + x^{a'+b'+c'}[k]_x(1-x)x^{c}(1-x^k) \\ &+x^{b'+2c'}x^{[a',b')_k}[k]_x(1-x)(1-x^k)^2. \quad (17)\end{aligned}$$

Since $a+b+c = a'+b'+c'$, the first terms on each side of (17) are identical, so we can eliminate those terms and divide by $[k]_x(1-x)(1-x^k)$ to obtain

$$x^{a+b+c+c'} + x^{b+2c}x^{[a,b)_k}(1-x^k) = x^{a'+b'+c'+c} + x^{b'+2c'}x^{[a',b')_k}(1-x^k). \quad (18)$$

But $x^{[a,b)_k}(1-x^k) = x^a - x^b$ and $x^{[a',b')_k}(1-x^k) = x^{a'} - x^{b'}$ so that (18) reduces to

$$x^{a+b+c+c'} + x^{a+b+c+c} - x^{2c} = x^{a'+b'+c'+c} + x^{a'+b'+c'+c'} - x^{2c'}. \quad (19)$$

However, since $a+b+c = a'+b'+c'$, the sums of the first two terms on each side of (19) are identical, so we can conclude that $x^{2c} = x^{2c'}$. Hence $c = c'$, which in turn implies $a = a'$. Thus cab is the only word in Case 3B which is Wilf equivalent to bac relative to $\mathcal{P}_k$.

4 The rearrangement conjectures

In this section we discuss the weak and strong rearrangement conjectures for $k \ge 2$. We begin with a further discussion of equivalent words that need not be rearrangements. As noted following Corollary 2.3, the weak rearrangement conjecture does not hold in general for $k \ge 2$, and we gave examples of words with the mod k-nonoverlapping property that are equivalent mod k for $k \ge 2$ but not rearrangements. It is worth noting that words that do not have the mod k-nonoverlapping property may also be equivalent mod k but not rearrangements. In particular, for words for which embeddings may overlap in at most half of the word, that is, words u with $\max\{\mathcal{C}_k(u)\} \le |u|/2$, as a corollary of Theorem 2.6 we can state the following generalization of Corollary 2.3.

COROLLARY 4.1 *If u and v are such that $u^{(k)} = v^{(k)}$, then $uxv \backsim_k uyv$ if* $\max\{\mathcal{C}_k(uxv)\} = \max\{\mathcal{C}_k(uyv)\} = |u|$ *and* $\Sigma(x) = \Sigma(y)$.

For example, with $k = 3$, $u = 1\ 2\ 3$, $v = 4\ 2\ 6$, $x = 2\ 2$, and $y = 1\ 3$, we have $1\ 2\ 3\ \mathbf{2}\ \mathbf{2}\ 4\ 2\ 6 \backsim_3 1\ 2\ 3\ \mathbf{1}\ \mathbf{3}\ 4\ 2\ 6$.

Words u with $\max\{\mathcal{C}_k(u)\} \geq |u|/2$ may also be equivalent without being rearrangements. For example, with $k = 3$, consider $u = 1\ 1\ 3\ 1\ 1\ 3\ 1\ 1\ 3$ and $v = 1\ 2\ 2\ 1\ 2\ 2\ 1\ 2\ 2$. Then $\max\{\mathcal{C}_k(u)\} = \max\{\mathcal{C}_k(v)\} = 6$, but u and v both have the mod k-comparison condition and are easily seen to be equivalent by Theorem 2.6.

So although the analogues of the rearrangement conjectures for $k = 1$ do not hold in general for $k \geq 2$, the only counterexamples that we have found involve words with entries from at least two equivalence classes mod k. If all the entries of two equivalent words are from the same equivalence class mod k, we do believe the analogue of the rearrangement conjecture holds.

CONJECTURE 4.2 If $u \backsim_k v$ and all characters of both u and v belong to the same equivalence class mod k, then u is a rearrangement of v.

We can state the analogue of the strong rearrangement conjecture a little more generally.

CONJECTURE 4.3 If $u \backsim_k v$ and u is a rearrangement of v, then there is a rearrangement map witnessing $u \backsim_k v$.

Langley, Liese, and Remmel [4] gave a way to test for a rearrangement map if $u \backsim_1 v$ by restricting to a finite poset. This has a direct analogue here. In particular, let $[m] = \{1, 2, \ldots, m\}$ and for a word w in $[m]^*$ and $i \in [m]$, let $c_i(w)$ equal the number of occurrences of i in w. Then we introduce variables $x_1, x_2, \ldots, x_m$, and define the weight of w, $W_{[m]}(w)$, to be $W_{[m]}(w) = \prod_{i=1}^m x_i^{c_i(w)}$. Also define

$$\begin{aligned}
F_k(u; x_1, \ldots, x_m) &= \sum_{w \in \mathcal{F}_k(u) \cap [m]^*} W_{[m]}(w) \\
W_k(u; x_1, \ldots, x_m) &= \sum_{w \in \mathcal{W}_k(u) \cap [m]^*} W_{[m]}(w) \\
S_k(u; x_1, \ldots, x_m) &= \sum_{w \in \mathcal{S}_k(u) \cap [m]^*} W_{[m]}(w) \\
A_k(u; x_1, \ldots, x_m) &= \sum_{w \in \mathcal{A}_k(u) \cap [m]^*} W_{[m]}(w).
\end{aligned}$$

Note that we have dropped the t dependence in these generating functions since the length is recorded by the number of variables in a monomial. In this case,

$$\sum_{w \in [m]^*} W_{[m]}(w) = \frac{1}{1 - \sum_{i=1}^m x_i}. \tag{20}$$

Thus since $\mathcal{F}_k(u) \cap [m]^* = (\mathcal{S}_k(u) \cap [m]^*)[m]^*$ and $A_k(u) \cap [m]^* = [m]^* - (\mathcal{F}_k(u) \cap [m]^*)$, we have that

$$F_k(u; x_1, \ldots, x_m) = S_k(u; x_1, \ldots, x_m) \frac{1}{1 - \sum_{i=1}^m x_i} \text{ and} \tag{21}$$

$$A_k(u; x_1, \ldots, x_m) = \frac{1}{1 - \sum_{i=1}^m x_i} - F_k(u; x_1, \ldots, x_m) \tag{22}$$

so that if any one of $F_k(u;x_1,\ldots,x_m)$, $A_k(u;x_1,\ldots,x_m)$, or $S_k(u;x_1,\ldots,x_m)$ is rational, then they all are rational. It follows from Theorem 8.2 of [2] that $S_k(u;x_1,\ldots,x_m)$ is rational for all $m\geq 1$. Thus $F_k(u;x_1,\ldots,x_m)$, $A_k(u;x_1,\ldots,x_m)$, and $S_k(u;x_1,\ldots,x_m)$ are rational for all $m\geq 1$. Also note that if $u=u_1\ldots u_n$, then

$$W_k(u;x_1,\ldots,x_m)=\prod_{r=1}^{n}\sum_{j\geq 0}x_{u_r+jk},$$

where the sum is taken to terminate at the largest value of j such that $u_r+jk\leq m$. So $W_k(u;x_1,\ldots,x_m)$ is rational.

In this context two words u and v are Wilf equivalent, denoted $u\backsim_{[m],k}v$, if $F_k(u;x_1,\ldots,x_m)=F_k(v;x_1,\ldots,x_m)$, or equivalently, $S_k(u;x_1,\ldots,x_m)=S_k(v;x_1,\ldots,x_m)$. The following shows that this notion of Wilf equivalence is equivalent to the existence of a rearrangement map witnessing the equivalence relative to $\mathcal{P}_k$.

Theorem 4.4 *Suppose $u,v\in[m]^*$ for some positive integer m. Then $u\backsim_{[m],k}v$ if and only if there exists a rearrangement map $f:\mathbb{P}^*\to\mathbb{P}^*$ that witnesses the Wilf equivalence $u\backsim_k v$.*

Proof. First note that if there is a rearrangement map $f:\mathbb{P}^*\to\mathbb{P}^*$ that witnesses the Wilf equivalence $u\backsim_k v$, then the restriction of f to $[m]^*$ is a $W_{[m]}$-preserving bijection that shows $u\backsim_{[m],k}v$.

For the converse, suppose $u,v\in[m]^*$ and $u\backsim_{[m],k}v$. Thus it must be the case that $F(u;x_1,\ldots,x_m)=F(v;x_1,\ldots,x_m)$ and there is a $W_{[m]}$-preserving bijection

$$g:\mathcal{F}_k(u)\cap[m]^*\to\mathcal{F}_k(v)\cap[m]^*$$

which is necessarily a rearrangement map on $[m]^*$. This bijection can then be lifted to the desired rearrangement map f, as follows. Suppose $w=w_1\cdots w_n\in\mathbb{P}^*$. For each $j=1,\ldots,k$, let m_j be the largest element of $[m]$ which is equivalent to j mod k. Suppose $w=w_1\cdots w_n\in\mathbb{P}^*$. For each $j=1,\ldots,k$, let $1\leq i_1^j<\cdots<i_{l_j}^j\leq n$ be the sequence of indices i such that $w_i\geq m_j$ and $w_i\equiv j$ mod k. Then let $\overline{w}$ be the word in $[m]^*$ that results by replacing each $w_{i_x^j}$ with m_j for $j=1,\ldots,k$. Then $u\leq w$ if, and only if, $u\leq\overline{w}$. Now apply g to $\overline{w}$. Since $z=g(\overline{w})$ is a rearrangement of $\overline{w}$, for each j there is a sequence $1\leq a_1^j<\cdots<a_{l_j}^j\leq n$ consisting of all the indices r such that $z_r=m_j$. Let $f(w)$ be the result of replacing $z_{a_r^j}$ by $w_{i_r^j}$ for $r=1,\ldots,l_j$ and $j=1,\ldots,n$. □

In view of Theorem 4.4, we consider an analogue of Theorem 2.6 for the more refined generating functions $S_k(u;x_1,\ldots,x_n)$. It is still the case that

$$\mathcal{S}_k(u)\cap[m]^*=(\mathcal{A}_k(u)\cap[m]^*)(\mathcal{W}_k(u)\cap[m]^*)-\bigcup_{i=1}^{n-1}(\mathcal{S}_k^{(i)}(u)\cap[m]^*). \tag{23}$$

It is easy to see that

$$\begin{aligned}\sum_{w\in\mathcal{A}_k(u)\mathcal{W}_k(u)\cap[m]^*}W_{[m]}(w) &= A_k(u;x_1,\ldots,x_m)\prod_{i=1}^{n}\sum_{j\geq 0}^{m}x_{u_i+jk}\\ &= \frac{1}{1-\sum_{i=1}^{m}x_i}(1-S_k(u;x_1,\ldots,x_m))\prod_{r=1}^{n}\sum_{j\geq 0}x_{u_r+jk}.\end{aligned} \tag{24}$$

We also have that

$$\mathcal{S}_k^{(i)}(u) \cap [m]^* = \bar{\mathcal{S}}_k^{(i)}(u)\mathcal{W}_k(s_i(u)) \cap [m]^*.$$

Thus if

$$\begin{aligned} S_k^{(i)}(u, x_1, \ldots, x_m) &= \sum_{w \in \mathcal{S}_k^{(i)}(u) \cap [m]^*} W_{[m]}(w) \text{ and} \\ \bar{S}_k^{(i)}(u, x_1, \ldots, x_m) &= \sum_{w \in \bar{\mathcal{S}}_k^{(i)}(u) \cap [m]^*} W_{[m]}(w), \end{aligned}$$

then we will have

$$S_k^{(i)}(u, x_1, \ldots, x_m) = \bar{S}_k^{(i)}(u, x_1, \ldots, x_m) \prod_{r=i+1}^{n} \sum_{j \geq 0} x_{u_r+jk}.$$

The only step in our proof of Theorem 2.6 which does not have an analogue in this case is the fact that

$$\bar{S}_k^{(i)}(u; t, x) = x^{d_{i,k}(u)} S_k(u; t, x).$$

It will no longer be the case that $\bar{S}_k^{(i)}(u; x_1, \ldots, x_m)$ is a multiple of $S_k(u; x_1, \ldots, x_m)$ if $d_{i,k}(u) > 0$. However, if $d_{i,k}(u) = 0$, then it will be the case that $\bar{\mathcal{S}}_k^{(i)}(u) \cap [m]^* = \mathcal{S}_k^{(i)}(u) \cap [m]^*$ so that

$$\bar{S}_k^{(i)}(u, x_1, \ldots, x_m) = S_k(u, x_1, \ldots, x_m).$$

Thus if $d_{i,k}(u) = 0$ for all $i \in \mathcal{C}_k(u)$, we will have

$$S_k^{(i)}(u, x_1, \ldots, x_m) = S_k(u, x_1, \ldots, x_m) \prod_{r=i+1}^{n} \sum_{j \geq 0} x_{u_r+jk} \tag{25}$$

for all i. It is easy to see that $d_{i,k}(u) = 0$ for all $i \in \mathcal{C}_k(u)$ if and only if each $u[j]$ is nondecreasing when u is factored as in Lemma 2.7. In that case, it follows from (23), (24), and (25) that

$$\begin{aligned} S_k(u, x_1, \ldots, x_m) &= \frac{1}{1 - \sum_{i=1}^{m} x_i}(1 - S_k(u; x_1, \ldots, x_m)) \prod_{r=1}^{n} \sum_{j \geq 0} x_{u_r+jk} \\ &\quad - \sum_{i=1}^{n-1} S_k(u, x_1, \ldots, x_m) \prod_{r=i+1}^{n} \sum_{j \geq 0} x_{u_r+jk}. \end{aligned}$$

Solving for $S_k(u, x_1, \ldots, x_m)$ will then result in the following theorem.

THEOREM 4.5 *Suppose $u = u_1 \ldots u_n \in [m]^*$ satisfies the mod k-comparison condition and $u[j]$ is nondecreasing for each j, so that $D_k^{(i)} = \emptyset$ for all $i \in \mathcal{C}_k(u)$. Then*

$$\begin{aligned} S_k(u; x_1, \ldots, x_m) &= \sum_{w \in \mathcal{S}_k(u) \cap [m]^*} W_{[m]}(w) \\ &= \frac{\prod_{r=1}^{n} \sum_{j \geq 0} x_{u_r+jk}}{\left(1 + \sum_{i \in \mathcal{C}_k(u)} \prod_{r=i+1}^{n} \sum_{j \geq 0} x_{u_r+jk}\right)(1 - \sum_{i=1}^{m} x_i) + \prod_{r=1}^{n} \sum_{j \geq 0} x_{u_r+jk}}, \end{aligned}$$

where a sum such as $\sum_{j\geq 0} x_{u_r+jk}$ *is taken to terminate at the largest value of* u_r+jk *that is less than or equal to* m.

In particular, if $u \in [m]^*$ has the mod k-nonoverlapping property, then u satisfies the hypothesis of Theorem 4.5 so that we have the following corollary.

Corollary 4.6 *Suppose* $u = u_1 \ldots u_n \in [m]^*$ *has the mod* k*-nonoverlapping property. Then*

$$\begin{aligned} S_k(u; x_1, \ldots, x_m) &= \sum_{w \in \mathcal{S}_k(u) \cap [m]^*} W_{[m]}(w) \\ &= \frac{\prod_{r=1}^{n} \sum_{j\geq 0} x_{u_r+jk}}{1 - \sum_{i=1}^{m} x_i + \prod_{r=1}^{n} \sum_{j\geq 0} x_{u_r+jk}}. \end{aligned}$$

where a sum such as $\sum_{j\geq 0} x_{u_r+jk}$ *is taken to terminate at the largest value of* u_i+jk *that is less than or equal to* m.

Thus, for example, it follows from Corollary 4.6 and Theorem 4.4 that for any two permutations $\sigma, \tau \in S_k$, there is a rearrangement map witnessing $\sigma \backsim_k \tau$.

5 Wilf equivalence and strong Wilf equivalence relative to $\mathcal{P}_k$ with $k \geq 2$

In this section, we shall prove a number of results for Wilf equivalence and strong Wilf equivalence relative to $\mathcal{P}_k$ for $k \geq 2$ which are analogues of results proved in [2] for Wilf equivalence and strong Wilf equivalence relative to $\mathcal{P}_1$. As we have already seen, there are significant differences between Wilf equivalence relative to $\mathcal{P}_k$ for $k \geq 2$ and Wilf equivalence relative to $\mathcal{P}_1$. As another example, part 2 of Theorem 1.2 says that if $u \backsim_1 v$, then $1u \backsim_1 1v$. One might hope that if i is a minimal element in $\mathcal{P}_k$, then we would have that if $u \backsim_k v$, then $iu \backsim_k iv$. However, this is false. For example, we know that $12 \backsim_2 21$. However it follows from our classification of the Wilf equivalence classes of words of length 3 for $\mathcal{P}_2$ that $112 \not\backsim_2 121$ and $212 \not\backsim_2 221$.

We start out this section with two results which allows us to transfer Wilf equivalences relative $\mathcal{P}_1$ to Wilf equivalences relative to $\mathcal{P}_k$ for $k \geq 2$ and to transfer Wilf equivalences relative $\mathcal{P}_k$ with $k \geq 2$ to Wilf equivalences relative to $\mathcal{P}_n$ for $k < n$.

Theorem 5.1 *For integers* $s \geq 0$ *and* $1 \leq j \leq k$*, define* $f_{j,k,s} : \mathbb{P} \rightarrow \{j + k(i-1) + ks : i \in \mathbb{P}\}$ *by* $f_{j,k,s}(i) = j + k(i-1) + ks$*. Extend this to* $\mathbb{P}^*$ *by defining*

$$f_{j,k,s}(u) = f_{j,k,s}(u_1 u_2 \ldots u_n) = f_{j,k,s}(u_1) f_{j,k,s}(u_2) \ldots f_{j,k,s}(u_n)$$

for any word $u = u_1 \ldots u_n \in \mathbb{P}^*$*. Then for any words* $u, v \in \mathbb{P}^*$*, we have* $u \backsim_1 v$ *if and only if* $f_{j,k,s}(u) \backsim_k f_{j,k,s}(v)$.

Proof. Suppose $u \backsim_1 v$. Then there is a weight-preserving bijection $g : \mathbb{P}^* \rightarrow \mathbb{P}^*$ such that for all $w \in \mathbb{P}^*$, $w \in \mathcal{F}(u) \iff g(w) \in \mathcal{F}(v)$. Define a weight-preserving bijection $h : \mathbb{P}^* \rightarrow \mathbb{P}^*$ in the following way. For $w \in \mathbb{P}^*$, let $h(w)$ be the word obtained from $w = w_1 \ldots w_t$ by replacing the subword

$\overline{w}$ of w consisting of each letter w_i of w which is equivalent to j mod k and greater than or equal to $j+ks$ by $(f_{j,k,s} \circ g \circ f_{j,k,s}^{-1})(\overline{w})$ and leaving all letters w_i such that either $w_i < j+ks$ or $w_i \not\equiv j$ mod k unchanged. It is then easy to see that $w \in \mathcal{F}_k(f_{j,k,s}(u)) \iff h(w) \in \mathcal{F}_k(f_{j,k,s}(v))$. Thus h witnesses that $f_{j,k,s}(u) \backsim_k f_{j,k,s}(v)$.

For the converse, suppose that there is a weight-preserving bijection $h : \mathbb{P}^* \to \mathbb{P}^*$ such that for all $w \in \mathbb{P}^*$, $w \in \mathcal{F}_k(f_{j,k,s}(u)) \iff h(w) \in \mathcal{F}_k(f_{j,k,s}(v))$. Let $U_{j,k,s} = \{j+k(i-1)+ks : i \in \mathbb{P}\}$. Our first goal is to construct a weight-preserving map $f : U_{j,k,s}^* \to U_{j,k,s}^*$ such that for all $w \in U_{j,k,s}^*$, $w \in \mathcal{F}_k(f_{j,k,s}(u)) \iff f(v) \in \mathcal{F}_k(f_{j,k,s}(v))$. Note that we cannot just let f equal h restricted to $U_{j,k,s}^*$ because there is no guarantee that h maps $U_{j,k,s}^*$ onto $U_{j,k,s}^*$. To construct our desired bijection f we shall use the involution principle due to Garsia and Milne [1].

Theorem 5.2 (Involution Principle) *Suppose that S and S' are finite sets of weighted signed objects. That is, we have sign maps $sgn : S \to \{1,-1\}$ and $sgn' : S' \to \{1,-1\}$ and we have weight maps $wt : S \to A$ and $wt' : S' \to A$ for some space A. Suppose that there is a sign and weight preserving bijection $f : S \to S'$, i.e. for all $s \in S$, $sgn(s)wt(s) = sgn'(f(s))wt'(f(s))$. In addition, suppose that $I : S \to S$ and $I' : S \to S'$ are sign-reversing weight-preserving involutions with positive fixed points. That is, for all $s \in S$ and $s' \in S'$,*

$$\begin{aligned} sgn(s)wt(s) &= -sgn(I(s))wt(I(s)) \text{ if } I(s) \neq s, \\ sgn(s')wt'(s') &= -sgn(I'(s'))wt'(I'(s')) \text{ if } I'(s') \neq s', \\ sgn(s) &= 1 \text{ if } I(s) = s, \text{ and} \\ sgn'(s') &= 1 \text{ if } I'(s') = s'. \end{aligned}$$

Let $F = \{s \in S : I(s) = s\}$ and $F' = \{s' \in S : I'(s') = s'\}$ be the set of fixed points of I and I', respectively. Then there is a weight-preserving bijection $g : F \to F'$ constructed canonically from I, I', and f. That is, for all $x \in F$, $wt(x) = wt'(g(x))$ and one can compute $g(x)$ for $x \in F$ by repeatedly applying f, then I', then f^{-1}, then I, until an application of f yields an element of F'. See Figure 2.

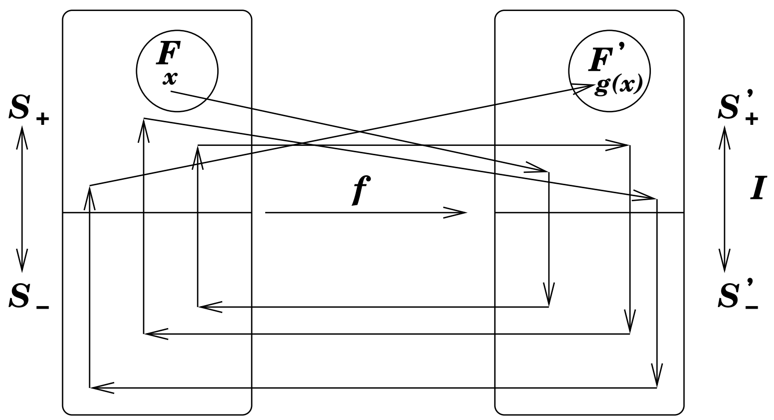

Figure 2: Schematic setup for the Involution Principle.

For any $n,m \in \mathbb{P}$, let $U_{n,m} = \{w \in \mathbb{P}^* : |w| = n \text{ and } \sum(w) = m\}$. Thus $U_{n,m}$ is a finite set for all $n,m \in \mathbb{P}$. For any word $w = w_1 \dots w_n \in U_{n,m}$, let $I(w)$ equal the set of all i such that either

$w_i < j+ks$ or w_i is not equivalent to j mod k. For any word $w \in U_{n,m}$ such that $I(w) \neq \emptyset$, let $m(w)$ be the smallest element in $I(w)$. For example, if $w = 1\ 3\ 4\ 4\ 5\ 4\ 7\ 3\ 1\ 2\ 2\ 4\ 3$ and $s = j = 1$ and $k = 2$ so that $j + ks = 3$, then $I(w) = \{1,3,4,6,9,10,11,12\}$ and $m(w) = 1$. Note that $I(w) = \emptyset$ if and only if $w \in U^*_{j,k,s}$.

Let $A_{n,m,j,k,s} = \{(w,S) : S \subseteq I(w)\}$. If $(w,S) \in A_{n,m,j,k,s}$, then define $sgn(w,S) = (-1)^{|S|}$, $wt(w,S) = \chi(f_{j,k,s}(u) \leq_k w)$, and $wt'(w,S) = \chi(f_{j,k,s}(v) \leq_k w)$ where for any statement A, $\chi(A) = 1$ if A is true and $\chi(A) = 0$ if A is false. Then define an involution $J : A_{n,m,j,k,s} \to A_{n,m,j,k,s}$ by let

$$J(w,S) = \begin{cases} (w, S \cup \{m(w)\}) & \text{if } I(w) \neq \emptyset \text{ and } m(w) \notin S \\ (w, S - \{m(w)\}) & \text{if } I(w) \neq \emptyset \text{ and } m(w) \in S \text{ and} \\ (w, \emptyset) & \text{if } I(w) = \emptyset. \end{cases}$$

It is easy to see that J is a sign-reversing weight-preserving involution whose fixed points are $(w,\emptyset)$ such that $I(w) = \emptyset$.

Next we define a bijection $f : A_{n,m,j,k,s} \to A_{n,m,j,k,s}$. Given any $(w,S) \in A_{n,m,j,k,s}$, we can uniquely factor $w = a_1 b_1 \ldots a_r b_r a_{r+1}$ where S consists of those positions which lie in the words $b_1, \ldots, b_r$. For example, suppose that $w = 1\ 3\ 4\ 4\ 5\ 4\ 7\ 3\ 1\ 2\ 2\ 4\ 3$ and $s = j = 1$ and $k = 2$ so that $I(w) = \{1,3,4,6,9,10,11,12\}$. Then for $(w, \{3,4,9,10,11\})$, $a_1 = 1\ 3$, $b_1 = 4\ 4$, $a_2 = 5\ 4\ 7\ 3$, $b_2 = 1\ 1\ 2$, and $a_3 = 4\ 3$. In such a situation $f_{j,k,s}(u) \leq_k w$ if and only if $f_{j,k,s}(u) \leq_k a_j$ for some $j = 1, \ldots, r$. We then define $\phi_S(w) = h(a_1)b_1h(a_2)b_2 \ldots h(a_r)b_rh(a_{r+1})$. Note that since $|h(a_j)| = |a_j|$ and $\sum(h(a_j)) = \sum(a_j)$ for all j, it follows that $\phi_S(w) \in U_{n,m}$. It is easy to see that ϕ_S is bijection since we are assuming that h is bijection. Moreover, since we did not change the values of the letters w_i for $i \in S$, we are guaranteed that $S \subseteq I(\phi_S(w))$. In such a situation, we let $f(w,S) = (\phi_S(w), S)$. It is easy to see that f is bijection such that for all $(w,S) \in A_{n,m,j,k,s}$, $f_{j,k,s}(u) \leq_k w \iff f_{j,k,s}(v) \leq_k \phi_S(w)$. Thus it follows that $wt(w,S) = wt'(f(w,S))$.

We are now in position to apply the involution principle where $S = S' = A_{n,m,j,k,s}$, $I = I' = J$ and $f : A_{n,m,j,k,s} \to A_{n,m,j,k,s}$ is the sign-preserving weight-preserving bijection described above. Note that in this case, $F = F' = \{(w,\emptyset) : w \in U_{n,m} \cap U^*_{j,k,s}\}$. Then let $\overline{g}_{n,m,j,k,s}$ be the map constructed by the involution principle for this choice of S, S', I, I', and f. Thus $\overline{g}_{n,m,j,k,s}$ is a weight-preserving bijection from $(U_{n,m} \cap U^*_{j,k,s}) \times \{\emptyset\}$ onto $(U_{n,m} \cap U^*_{j,k,s}) \times \{\emptyset\}$. Then let $g_{n,m,j,k,s} : U_{n,m} \cap U^*_{j,k,s} \to U_{n,m} \cap U^*_{j,k,s}$ be defined so that for all $(w,\emptyset) \in (U_{n,m} \cap U^*_{j,k,s}) \times \{\emptyset\}$, $\overline{g}_{n,m,j,k,s}((w,\emptyset)) = (g_{n,m,j,k,s}(w), \emptyset)$. Our definitions ensure that for all $w \in U_{n,m} \cap U^*_{j,k,s}$,

$$f_{j,k,s}(u) \leq_k w \iff f_{j,k,s}(v) \leq_k g_{n,m,j,k,s}(w).$$

Thus $g_{j,k,s} = \bigcup_{n,m \in \mathbb{P}} g_{n,mj,k,s}$ is a bijection from $U^*_{j,k,s} \to U^*_{j,k,s}$ such that for all $w \in U^*_{j,k,s}$, $|w| = |g_{j,k,s}(w)|$, $\sum(w) = \sum(g_{j,k,s}(w))$, and

$$f_{j,k,s}(u) \leq_k w \iff f_{j,k,s}(v) \leq_k g_{j,k,s}(w).$$

It then easy to see that $\psi = f^{-1}_{j,k,s} \circ g_{j,k,s} \circ f_{j,k,s}$ is a weight-preserving bijection $\psi : \mathbb{P}^* \to \mathbb{P}^*$ such that

$$\begin{aligned} u \leq_1 w &\iff f_{j,k,s}(u) \leq_k f_{j,k,s}(w) \iff f_{j,k,s}(v) \leq_k g_{j,k,s} \circ f_{j,k,s}(w) \\ &\iff v \leq_1 f^{-1}_{j,k,s} \circ g_{j,k,s} \circ f_{j,k,s}(w). \end{aligned}$$

Thus ψ shows that $u \backsim_1 v$. □

For example, let $k = 3$ and $s = 0$. Then for $j = 1$, $j = 2$, and $j = 3$, respectively, the equivalence $1342 \backsim_1 1432$ implies the following equivalences mod 3:

$$\begin{array}{ccc} 1\ 7\ 10\ 4 & \backsim_3 & 1\ 10\ 7\ 4 \\ 2\ 8\ 11\ 5 & \backsim_3 & 2\ 11\ 8\ 5 \\ 3\ 9\ 12\ 6 & \backsim_3 & 3\ 12\ 9\ 6 \end{array}$$

Our next result allows us to transfer Wilf equivalence relative to $\mathcal{P}_k$ where $k \geq 2$ to Wilf equivalence relative to $\mathcal{P}_n$ where $k < n$. We say that a map $\theta : \mathbb{P} \to \mathbb{P}$ is a (k,n)-*transfer map* if for all $i, j \in \mathbb{P}^*$,

$$i <_k j \iff \theta(i) <_n \theta(j).$$

For example suppose that $k < n$ and we are given $1 \leq i_1 < \cdots < i_k \leq n$. Then for $1 \leq j \leq k$ and $s \geq 0$, let $\theta(j + sk) = i_j + sn$. Then it is easy to see that θ is a (k,n)-transfer map. We can produce a (k,k)-transfer map by fixing some $m \geq 1$ and letting $\theta_m(j + sk) = m + j + sk$ for all $1 \leq j \leq k$ and $s \geq 0$. If we are given a (k,n)-transfer map $\theta : \mathbb{P} \to \mathbb{P}$, then we say that i is θ-reducible if there exists a $r \in \mathbb{P}$ such that $\theta(r) \leq_n i$. If i is θ-reducible, then we define the θ-collapse of i to be the maximum r such that $\theta(r) \leq_n i$ and write $r = \mathrm{clp}_\theta(i)$. We say that i is θ-irreducible if it is not θ-reducible.

We then have the following theorem.

THEOREM 5.3 *Let k and n be positive integers with $2 \leq k \leq n$ and let $\theta : \mathbb{P} \to \mathbb{P}$ be a (k,n)-transfer map. Then for all $u, v \in \mathbb{P}^*$, if there is a rearrangement map f which witnesses that $u \backsim_k v$, then there is a rearrangement map g which witnesses $\theta(u) \backsim_n \theta(v)$.*

Proof. Given $w = w_1 \dots w_t \in \mathbb{P}^*$, we can uniquely factor w as

$$w = y^{(1)}\, z^{(1)}\, y^{(2)}\, z^{(2)} \dots z^{(m-1)}\, y^{(m)}$$

such that the $y^{(j)}$s consist of all of the characters w_s of w that are θ-irreducible and $z^{(j)}$s consists of all the characters w_s of w that are θ-reducible. If $z^{(j)} = z_1^{(j)} \dots z_{\ell_j}^{(j)}$, then we define $\mathrm{clp}_\theta(z^{(j)}) = \mathrm{clp}_\theta(z_1^{(j)}) \dots \mathrm{clp}_\theta(z_{\ell_j}^{(j)})$. We claim that for each j

$$\theta(u) \leq_n z^{(j)} \iff u \leq_k \mathrm{clp}_\theta(z^{(j)}). \tag{26}$$

That is, suppose that $u = u_1 \dots u_m$ and there is an $s \geq 0$ such that $\theta(u_i) \leq_n z_{i+s}^{(j)}$ for $1 \leq i \leq m$. Then clearly by our definition of $\mathrm{clp}_\theta(z_{i+s}^{(j)})$, we must have

$$\theta(u_i) \leq_n \theta(\mathrm{clp}_\theta(z_{i+s}^{(j)})) \leq_n z_{i+s}^{(j)}.$$

Hence $u_i \leq_k \mathrm{clp}_\theta(z_{i+s}^{(j)})$ since θ is a (k,n)-transfer map. This shows that if $\theta(u) \leq_n z^{(j)}$, then $u \leq_k \mathrm{clp}_\theta(z^{(j)})$. On the other hand, if $u_i \leq_k \mathrm{clp}_\theta(z_{i+s}^{(j)})$ for $1 \leq j \leq m$, then $\theta(u) \leq_n \theta(\mathrm{clp}_\theta(z^{(j)}) \leq_n z^{(j)}$.

But then it follows that for each j,

$$\theta(u) \leq_n z^{(j)} \iff u \leq_k \mathrm{clp}_\theta(z^{(j)}) \iff v \leq_k f(\mathrm{clp}_\theta(z^{(j)}))$$

by (26) and the definition of f. For each letter a in $f(\mathrm{clp}_\theta(z^{(j)}))$, let $1 \leq p_1 < \cdots < p_{r_a} \leq \ell_j$ be the positions of a's in $f(\mathrm{clp}(z^{(j)}))$. Since f is a rearrangement map, there are r_a occurrences of a in $\mathrm{clp}(z^{(j)})$, say these are in positions $1 \leq q_1 < \cdots < q_{r_a} \leq \ell_j$ in $\mathrm{clp}(z^{(j)})$. Then we replace r_a occurrences of s in $f(\mathrm{clp}_\theta(z^{(j)})))$ by $z^{(j)}_{q_1}, \ldots, z^{(j)}_{q_{r_a}}$ reading from left to right. Doing this for each a which occurs in $f(\mathrm{clp}_\theta(z^{(j)}))$ results in $g(z^{(j)})$. It is easy to see that

$$v \leq_k f(\mathrm{clp}_\theta(z^{(j)})) \iff \theta(v) \leq_n \theta(f(\mathrm{clp}_\theta(z^{(j)}))) \iff \theta(v) \leq_n g(z^{(j)}).$$

Thus $g(z^{(j)})$ is a rearrangement of $z^{(j)}$, and $\mathrm{clp}_\theta(g(z^{(j)})) = f(\mathrm{clp}_\theta(z^{(j)}))$. Hence

$$\theta(v) \leq_k g(z^{(j)}) \iff v \leq_k \mathrm{clp}(g(z^{(j)})) = f(\mathrm{clp}(z^{(j)})) \iff \theta(u) \leq_k z^{(j)}.$$

Finally let

$$g(w) = y_1\, g(z_1)\, y_2\, g(z_2) \ldots g(z_{m-1}) y_m.$$

Then g is a rearrangement that witnesses $\theta(u) \backsim_k \theta(v)$. To show that g is a bijection, we can simply form g^{-1} from f^{-1} in the same manner that we constructed g from f. □

Our next result will show that parts 1 and 3 of Theorem 1.2 hold for $\mathcal{P}_k$ for all $k \geq 2$. In fact, we shall strengthen part 3 of Theorem 1.2 by showing that $u \backsim_1 v$ if and only if $u^+ \backsim_1 v^+$. For the following, as in Theorem 1.2, let u^r denote the reverse of u and u^+ the word obtained from u by adding 1 to each letter in w. Also if u has no occurrence of 1, then we let u^- be the word obtained from u by subtracting one from each letter in w.

Recall that for $k \geq 1$, $u, v \in \mathbb{P}_k$ are *strongly Wilf equivalent*, denoted $u \backsim_{s,k} w$, if there is a weight-preserving bijection $f : \mathbb{P}^* \to \mathbb{P}^*$ such that

$$\mathrm{Em}(u, w) = \mathrm{Em}(v, f(w))$$

for all $w \in \mathbb{P}^*$, that is, the embeddings of v into $f(w)$ occur in the same positions as the embeddings of u into w.

THEOREM 5.4 *For $u, v \in \mathbb{P}^*$, the following hold.*

1. $u \backsim_k u^r$.
2. $u \backsim_k v$ *if and only if* $u^+ \backsim_k v^+$.
3. $u \backsim_{s,k} v$ *if and only if* $u^+ \backsim_{s,k} v^+$.

Proof.

As we have previously noted, the map $w \to w^r$ is a weight-preserving bijection from $\mathbb{P}^*$ onto $\mathbb{P}^*$ which maps $w \in \mathcal{F}_k(u)$ onto $\mathcal{F}_k(u^r)$. Thus $u \backsim_k u^r$.

For part 2, suppose that $u \backsim_k v$ so that there exists a weight-preserving bijection $g : \mathbb{P}^* \to \mathbb{P}^*$ that maps $\mathcal{F}_k(u)$ onto $\mathcal{F}_k(v)$. Given w in $\mathbb{P}^*$, factor w as

$$w = y_1\, z_1\, y_2\, z_2 \cdots y_{l-1} z_{l-1}\, y_l$$

(with y_1, y_l possibly empty), where the y_i's consist of all of the 1s in w. Then $w \in \mathcal{F}_k(u^+)$ if and only if $z_i \in \mathcal{F}_k(u^+)$ for some i. This is equivalent to $z_i^- \in \mathcal{F}_k(u)$ (z_i^- exists since each z_i contains no 1s). So for $w \in \mathbb{P}^*$, define f by

$$f(w) = y_1\, g(z_1^-)^+\, y_2\, g(z_2^-)^+ \cdots g(z_{l-1}^-)^+\, y_l.$$

Then $f : \mathbb{P}^* \to \mathbb{P}^*$ is a weight-preserving bijection which maps $\mathcal{F}_k(u^+)$ onto $\mathcal{F}_k(v^+)$, and therefore $u^+ \backsim_k v^+$.

Note that in this case, if the map g is such that for all words $z \in \mathbb{P}^*$, $Em_{\mathcal{P}_k}(u, z) = Em_{\mathcal{P}_k}(v, g(z))$, then it will be the case that for each z_i,

$$Em_{\mathcal{P}_k}(u^+, z_i) = Em_{\mathcal{P}_k}(u, z_i^-) = Em_{\mathcal{P}_k}(v, g(z_i^-)) = Em_{\mathcal{P}_k}(v^+, g(z_i^-)^+)$$

so that if g witnesses that $u \backsim_{s,k} v$, then f will witness that $u^+ \backsim_{s,k} v^+$.

For the converse, suppose that $u^+ \backsim_k v^+$. Thus there is a weight preserving bijection $h : \mathbb{P}^* \to \mathbb{P}^*$ such that for all $w \in \mathbb{P}^*$, $w \in \mathcal{F}_k(u^+) \iff h(w) \in \mathcal{F}_k(v^+)$. Let $\mathbb{P}^+ = \{2, 3, \ldots\}$. Our first goal is to construct a weight-preserving map $f : (\mathbb{P}^+)^* \to (\mathbb{P}^+)^*$ such that for all $w \in (\mathbb{P}^+)^*$, $w \in \mathcal{F}_k(u^+) \iff f(w) \in \mathcal{F}_k(v^+)$. Note that we cannot just let f equal h restricted to $(\mathbb{P}^+)^*$ because there is no guarantee that h maps $(\mathbb{P}^+)^*$ onto $(\mathbb{P}^+)^*$. As in Theorem 5.1, we shall use the involution principle to construct f.

Recall that for any $n, m \in \mathbb{P}$, $U_{n,m} = \{w \in \mathbb{P}^* : |w| = n \text{ and } \sum(w) = m\}$. For any word $w = w_1 \ldots w_n \in U_{n,m}$, let $I(w)$ equal the set of all i such that $w_i = 1$ and, if $I(w) \neq \emptyset$, let $m(w)$ be the smallest element in $I(w)$. For example, if $w = 1\ 3\ 4\ 4\ 1\ 1\ 7\ 3\ 1\ 2\ 2\ 4\ 3$ then $I(w) = \{1, 5, 6, 9\}$ and $m(w) = 1$. Note that $I(w) = \emptyset$ if and only if $w \in (\mathbb{P}^+)^*$.

Let $B_{n,m} = \{(w, S) : S \subseteq I(w)\}$. If $(w, S) \in B_{n,m}$, then define $sgn(w, S) = (-1)^{|S|}$, $wt(w, S) = \chi(u^+ \leq_k w)$, and $wt'(w, S) = \chi(v^+ \leq_k w)$. Then define an involution $J : B_{n,m} \to B_{n,m}$ by letting

$$J(w, S) = \begin{cases} (w, S \cup \{m(w)\}) & \text{if } I(w) \neq \emptyset \text{ and } m(w) \notin S \\ (w, S - \{m(w)\}) & \text{if } I(w) \neq \emptyset \text{ and } m(w) \in S \text{ and} \\ (w, \emptyset) & \text{if } I(w) = \emptyset. \end{cases}$$

It is easy to see that J is a sign-reversing weight-preserving involution whose fixed points are $(w, \emptyset)$ such that $I(w) = \emptyset$.

Next we define a bijection $f : B_{n,m} \to B_{n,m}$. Given any $(w, S) \in B_{n,m}$, we can uniquely factor $w = a_1 b_1 \ldots a_r b_r a_{r+1}$ where S consists of those positions which lie in the words $b_1, \ldots, b_r$. For example, suppose that $w = 1\ 3\ 4\ 4\ 1\ 1\ 7\ 3\ 1\ 2\ 1\ 4\ 3$ and $I(w) = \{1, 5, 6, 9, 11\}$. Then for $(w, \{5, 6, 11\})$, $a_1 = 1\ 3\ 4\ 4$, $b_1 = 1\ 1$, $a_2 = 7\ 3\ 1\ 2$, $b_2 = 1$, and $a_3 = 4\ 3$. In such a situation $u^+ \leq_k w$ if and only if $u^+ \leq_k a_j$ for some $j = 1, \ldots, r+1$. We then define $\phi_S(w) = h(a_1) b_1 h(a_2) b_2 \ldots h(a_r) b_r h(a_{r+1})$. Note that since $|h(a_j)| = |a_j|$ and $\sum(h(a_j)) = \sum(a_j)$ for all j, it follows that $\phi_S(w) \in U_{n,m}$. It is easy to see that ϕ_S is bijection since we are assuming that h is bijection. Moreover, since we did not change the values of the letters w_i for $i \in S$, we are guaranteed that $S \subseteq I(\phi_S(w))$. In such a situation, we let $f(w, S) = (\phi_S(w), S)$. It is easy to see that f is bijection such that for all $(w, S) \in B_{n,m}$, $u^+ \leq_k w \iff v^+ \leq_k \phi_S(w)$. Thus it follows that $wt(w, S) = wt'(f(w, S))$.

We are now in position to apply the involution principle where $S = S' = B_{n,m}$, $I = I' = J$, and $f : B_{n,m} \to B_{n,m}$ is the sign-preserving weight-preserving bijection described above. Note that in this case, $F = F' = \{(w, \emptyset) : w \in U_{n,m} \cap (\mathbb{P}^+)^*\}$. Then let $\overline{g}_{n,m}$ be the map constructed by the involution principle for this choice of S, S', I, I', and f. Thus $\overline{g}_{n,m}$ is a weight-preserving bijection from $(U_{n,m} \cap (\mathbb{P}^+)^*) \times \{\emptyset\}$ onto $(U_{n,m} \cap (\mathbb{P}^+)^*) \times \{\emptyset\}$. Then let $g_{n,m} : U_{n,m} \cap (\mathbb{P}^+) \to U_{n,m} \cap (\mathbb{P}^+)$ be defined so that for all $(w, \emptyset) \in (U_{n,m} \cap (\mathbb{P}^+)^*) \times \{\emptyset\}$, $\overline{g}_{n,m}((w, \emptyset)) = (g_{n,m}(w), \emptyset)$. Then our definitions ensure that

$$u^+ \leq_k w \iff v^+ \leq_k g_{m,n}(w).$$

Thus $g = \bigcup_{n,m \in \mathbb{P}} g_{n,m}$ is a bijection from $(\mathbb{P}^+)^*$ onto $(\mathbb{P}^+)^*$ such that for all $w \in (\mathbb{P}^+)^*$, $|w| = |g(w)|$, $\Sigma(w) = \Sigma(g(w))$, and

$$u^+ \leq_k w \iff v^+ \leq_k g(w).$$

It is then easy to see that $\psi(w) = (g(w^+))^-$ is a weight-preserving bijection $\psi : \mathbb{P}^* \to \mathbb{P}^*$ such that

$$u \leq_k w \iff u^+ \leq_k w^+ \iff v^+ \leq_k g(w^+) \iff v \leq_k (g(w^+))^-.$$

Thus ψ shows that $u \backsim_k v$.

For part (3), assume that the strong equivalence $u^+ \backsim_{s,k} v^+$ is witnessed by the function h. That is, h has the additional property that for all words $a \in \mathbb{P}^*$, $Em_{\mathcal{P}_k}(u^+, a) = Em_{\mathcal{P}_k}(b^+, h(a))$. Then since 1 cannot be part of any embedding of u^+, it follows that given our factorization $w = a_1 b_1 \ldots a_r b_r a_{r+1}$ for (w, S) described above that

$$Em_{\mathcal{P}_k}(u^+, a_1 b_1 \ldots a_r b_r a_{r+1}) = Em_{\mathcal{P}_k}(v^+, h(a_1) b_1 \ldots h(a_r) b_r h(a_{r+1})).$$

Thus for all pairs $(w, S) \in B_{n,m}$, $Em_{\mathcal{P}_k}(u^+, w) = Em_{\mathcal{P}_k}(v^+, \phi_S(w))$. Then it is easy to see that the definition of $g_{n,m}$ via the involution principle will also preserve embeddings indices. That is for all $w \in U_{n,m} \cap (\mathbb{P}^+)^*$,

$$Em_{\mathcal{P}_k}(u^+, w) = Em_{\mathcal{P}_k}(v^+, g_{n,m}(w)).$$

Thus g has the property that for all words $w \in (\mathbb{P}^+)^*$,

$$Em_{\mathcal{P}_k}(u^+, w) = Em_{\mathcal{P}_k}(v^+, g(w)).$$

It then follows that ϕ witnesses that $u \backsim_{s,k} v$. □

We note that in general, it is not the case that $u \backsim_{s,k} u^r$. For example, Kitaev, Liese, Remmel, and Sagan [2] showed that $2143 \not\backsim_{s,1} 3412$. However if u has the mod k-nonoverlapping property, then it is the case that $u \backsim_{s,k} u^r$. That is, we have the following lemma.

Lemma 5.5 *Suppose that $k \geq 2$ and u has the mod k-nonoverlapping property. Then there is a rearrangement map g which witnesses that $u \backsim_{s,k} u^r$.*

Proof. First observe that since u has the mod k-nonoverlapping property, u^r also has the mod k-nonoverlapping property. In addition, for any word w, the embeddings of u into w cannot overlap. Thus we can write any word w in the form $y_0 v_1 y_1 v_2 y_2 \ldots v_k y_k$ where for all $1 \leq i \leq k$, $|v_i| = |u|$ and u embeds into v_i and , for $0 \leq i \leq k$, $y_i \in \mathcal{A}_k(u)$. We call such a factorization, the u-factorization of w. We then define $g(w) = y_0^r v_1^r y_1^r v_2^r y_2^r \ldots v_k^r y_k^r$. Note that for each i, $y_i^r \in \mathcal{A}_k(u^r)$ since $y_i \in \mathcal{A}_k$. It follows that the only embedding of u^r must be $v_1^r, \ldots, v_k^r$ since u^r has the the mod k-nonoverlapping property. Thus $y_0^r v_1^r y_1^r v_2^r y_2^r \ldots v_k^r y_k^r$ is the u^r-factorization of $g(w)$. It follows that g is invertible and, hence, g is a rearrangement map. Moreover, relative to $\mathcal{P}_k$, $Em(u,w) = Em(u^r, g(w))$. Thus g is a rearrangement map which witnesses that $u \backsim_{s,k} u^r$. □

Our next result is an analogue of statement 5 of Theorem 1.2, which states that if x, y, z are in $[m]^*$ and $n > m$, then

$$x\,m\,y\,n\,z \backsim_1 x\,n\,y\,m\,z.$$

The mod k analogue simply requires a unique maximum among all entries in a fixed equivalence class. First, we need some notation. For a positive integer m, $1 \le m \le k$, and for a word u, let n be the largest entry of u that is equivalent to m mod k. Set

$$\max_m(u) = \{i : u_i = n\}.$$

Then define an m-psuedo-embedding of a string u into a string w to be a factor w' of w such that $|w'| = |u|$, $w'_i \ge_k u_i$ for all $i \notin \max_m(u)$, and $w'_i = u_i$ mod k for all $i \in \max_m(u)$. So the only difference between an m-psuedo-embedding and an embedding is that we may have $w'_i <_k u_i$ for $i \in \max_m(u)$ in the m-psuedo-embedding. Then we have the following.

Theorem 5.6 *Let $m < n$ be positive integers such that $m = n$ mod k. Also let $x, y, z \in \mathbb{P}^*$ be such that all entries in x, y or z that are equivalent to m mod k are less than or equal to m. Then*

$$x\,m\,y\,n\,z \backsim_k x\,n\,y\,m\,z.$$

Proof. Let $u = x\,m\,y\,n\,z$ and $v = x\,n\,y\,m\,z$. We will construct a weight-preserving bijection between $\mathcal{A}_k(u)$ and $\mathcal{A}_k(v)$. This bijection will act as the identity map on words that avoid both u and v. We will bijectively map words in $\mathcal{A}_k(u) - \mathcal{A}_k(v)$ to $\mathcal{A}_k(v) - \mathcal{A}_k(u)$ as follows. Given $w \in \mathcal{A}_k(u) - \mathcal{A}_k(v)$, define

$$\eta(w) = \{i : \text{there is an embedding of } v \text{ into } w \text{ with the } n \text{ in position } i\}.$$

So for each $i \in \eta(w)$, we have $w_i = n$ mod k and $w_i \ge n$. Let $r = |y| + 1$. Then since w embeds v but not u, we must have $w_{i+r} = n$ mod k and $m \le w_{i+r} < n$. Also consider the set

$$\sigma(i) = \{i, i + r, i + 2r, \dots, i + l_i r\}$$

where l_i is the least nonnegative integer such that there is no m-pseudo-embedding of v into w with the n in position $i + l_i r$. Note that $l_i \ge 1$ for each i since there is an embedding of v into w with the n at position i. Also, for each $j \in \sigma(i)$ greater than i, we must have $w_j = n$ mod k and $m \le w_j < n$ by the same reasoning as for w_{i+r}. This implies that any two sets $\sigma(i)$ are disjoint, since $w_i \ge n$ for each $i \in \eta(w)$.

We now map w to a new string $\overline{w}$ by switching the values of w_i and $w_{i+l_i k}$ for every $i \in \eta(w)$. Since the sets $\sigma(i)$ are disjoint, this switching operation is well defined. So we need to show that $\overline{w}$ is in $\mathcal{A}_k(v) - \mathcal{A}_k(u)$, and that the process is invertible. First, to see that $\overline{w} \in \mathcal{A}_k(v)$, note that the switching operation removes each original embedding of v, since each $w_{i+l_i r}$ is less than n. Further, because there is no m-pseudo-embedding of u with the n in position $i + l_i r$, moving w_i to position $i + l_i r$ can not create a new embedding of v.

Next, to show $\overline{w} \notin \mathcal{A}_k(u)$, that is, to show $u \le_k \overline{w}$, we can actually prove the stronger statement that there is an embedding of u in $\overline{w}$ with the n in position $i + l_i r$ for each $i \in \eta(w)$, and that these are the only embeddings. These embeddings exist since there is an m-pseudo-embedding of v into w with the n in position $i+(l_i-1)r$, as shown:

$$\begin{array}{rccccccc} w = & \cdots & * & m^{+k} & * & m^{+k} & * & \cdots \\ v = & & x & n & y & m & z & \end{array},$$

where m^{+k} represents a character l with $l \geq_k m$, and the two m^{+k}'s are in positions $i+(l_i-1)r$ and $i+l_ir$ in w. When forming $\overline{w}$, the second m^{+k} changes to n^{+k}, resulting in an embedding of u:

$$\begin{array}{rccccccc} \overline{w} = & \cdots & * & m^{+k} & * & n^{+k} & * & \cdots \\ u = & & x & m & y & n & z & \end{array}.$$

These are the only embeddings of u in $\overline{w}$ since w avoids u, only characters greater than m in the mod k ordering move when forming $\overline{w}$, and the only characters larger than n that move are placed in positions $i+l_ir$.

Finally, to show that the switching process is invertible, we can modify the above construction by exchanging the roles of u and v and building the strings from right to left. ☐

To illustrate the ideas in the proof, suppose $k=2$, $u = 1\,4\,5\,4\,7\,6\,3$, $v = 1\,6\,5\,4\,7\,4\,3$, so that we are switching elements equivalent to 2 mod 2 in forming $\overline{w}$. The following chart shows a word $w \in \mathcal{A}_k(u) - \mathcal{A}_k(v)$, along with the embeddings of v, and a relevant m-pseudo-embedding.

position :	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
w :	1	1	1	8	7	6	9	4	7	4	9	4	7	8	1	6	5	4	11	4	7
			1	**6**	5	4	7	**4**	3						1	**6**	5	4	7	**4**	3
							1	⋆	5	4	7	**4**	3								
					1	**6**	5	4	7	**4**	3										

So there are three embeddings of v (and none of u) into w with the 6 in positions $\eta(w) = \{4,6,16\}$. For $i=4$, there are m-pseudo-embeddings of v with the 6 in positions 4 (the embedding at that position) and 8, but not in position 12. So $\sigma(4) = \{4,8,12\}$. Similarly, $\sigma(6) = \{6,10\}$ and $\sigma(16) = \{16,20\}$. So $\overline{w}$ is formed by switching w_4 and w_{12}, w_6 and w_{10}, and w_{16} and w_{20}, removing the embeddings of v and resulting in embeddings of u with the 6 in positions 10, 12 and 20, as shown (the characters moved in forming $\overline{w}$ are in bold):

position :	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
$\overline{w}$:	1	1	1	**4**	7	**4**	9	4	7	**6**	9	**8**	7	8	1	**4**	5	4	11	**6**	7
							1	4	5	4	7	6	3		1	4	5	4	7	6	3
					1	4	5	4	7	6	3										

In [2], the authors note that the existence of a unique maximum is necessary, citing as an example the inequivalence of 122 and 212 under the standard ordering.

Our next two results will allow us to generate a large number of strong Wilf equivalences for $\mathcal{P}_k$. Given any $k \geq 2$ and $A \subseteq \{1,\ldots,k\}$, let $\mathbb{P}_A = \{i \in \mathbb{P} : (\exists a \in A)(i \equiv a \bmod k)\}$. Let $u \in \mathbb{P}^*$. Then we can uniquely factor $u = a_1b_1a_2b_2\ldots a_rb_ra_{r+1}$ where $a_i \in \mathbb{P}_A^*$ for all i and $b_j \in (\mathbb{P}-\mathbb{P}_A)^*$ for all j. We shall call $a_1b_1a_2b_2\ldots a_rb_ra_{r+1}$, the *$(k,A)$-factorization of u*. We then let $u(A) = a_1a_2\ldots a_{r+1}$. Let v be a word of length $|u(A)|$ in $\mathbb{P}_A^*$. Then we can write $v = c_1c_2\ldots c_{r+1}$ such that for all i, $|a_i| = |c_i|$. In such a situation, we let $u|_{u(A)\to v}$ be the word $c_1b_1c_2b_2\dots c_rb_rc_{r+1}$. For example, if $k=4$, $A=\{1,2\}$, and $u=1\ 2\ 3\ 4\ 4\ 2\ 1\ 3\ 3\ 4\ 5\ 10\ 6$, then $u(A)=1\ 2\ 2\ 1\ 5\ 10\ 6$. Thus if $v=2\ 5\ 6\ 1\ 5\ 1\ 6$, then $u|_{u(A)\to v}=2\ 5\ 3\ 4\ 4\ 6\ 1\ 3\ 3\ 4\ 1\ 6$.

THEOREM 5.7 *Suppose that* $k\geq 2$ *and* A *be a nonempty subset of* $\{1,\dots,k\}$. *Then for all* $u,v\in\mathbb{P}^*$, (i) *if* $u(A)\backsim_{s,k} v$, *then* $u\backsim_{s,k} u|_{u(A)\to v}$ *and* (ii) *if* $u(A)\backsim_k v$, *then* $u\backsim_k u|_{u(A)\to v}$.

Proof. For (i), assume that $u(A)\backsim_{s,k} v$. Then there is a weight-preserving bijection $h:\mathbb{P}^*\to\mathbb{P}^*$ such that for all $w\in\mathbb{P}^*$, $Em_{\mathcal{P}_k}(u(A),w)=Em_{\mathcal{P}_k}(v,h(w))$. Our first task is to construct a map $g:\mathbb{P}^*_A\to\mathbb{P}^*_A$ such that for all $w\in\mathbb{P}^*_A$, $Em_{\mathcal{P}_k}(u(A),w)=Em_{\mathcal{P}_k}(v,g(w))$. Note that we cannot just restrict h to $\mathbb{P}^*_A$ because there is no guarantee that h maps $\mathbb{P}^*_A$ onto $\mathbb{P}^*_A$.

Recall that for any $n,m\in\mathbb{P}$, $U_{n,m}=\{w\in\mathbb{P}^*:|w|=n \text{ and } \sum(w)=m\}$. For any word $w=w_1\dots w_n\in U_{n,m}$, let $I(w)$ equal the set of all i such that $w_i\notin\mathbb{P}_A$ and, if $I(w)\neq\emptyset$, let $m(w)$ be the smallest element in $I(w)$. For example, if $k=4$ and $A=\{1,2\}$, and $w=1\ 3\ 4\ 4\ 1\ 1\ 7\ 3\ 1\ 6\ 11\ 4\ 9$ then $I(w)=\{2,3,4,7,8,11\}$ and $m(w)=2$. Note that $I(w)=\emptyset$ if and only if $w\in\mathbb{P}^*_A$.

Let $C_{n,m}=\{(w,S):S\subseteq I(w)\}$. If $(w,S)\in C_{n,m}$, then define $sgn(w,S)=(-1)^{|S|}$ and $wt(w,S)=\chi(u(A)\leq_k w)$ and $wt'(w,S)=\chi(v\leq_k w)$. Then define an involution $J:C_{n,m}\to C_{n,m}$ by letting

$$J(w,S)=\begin{cases}(w,S\cup\{m(w)\}) & \text{if } I(w)\neq\emptyset \text{ and } m(w)\notin S\\ (w,S-\{m(w)\}) & \text{if } I(w)\neq\emptyset \text{ and } m(w)\in S \text{ and}\\ (w,\emptyset) & \text{if } I(w)=\emptyset.\end{cases}$$

It is easy to see that J is sign-reversing weight-preserving involution whose fixed points are $(w,\emptyset)$ such that $I(w)=\emptyset$.

Next we define a bijection $f:C_{n,m}\to C_{n,m}$. Given any $(w,S)\in C_{n,m}$, we can uniquely factor $w=a_1b_1\dots a_rb_ra_{r+1}$ where S consists of those positions which lie in the words $b_1,\dots,b_r$. For example, if $k=4$ and $A=\{1,2\}$, and $w=8\ 3\ 4\ 4\ 1\ 1\ 7\ 3\ 1\ 6\ 11\ 4\ 9$ then $(w,\{1,2,3,4,7,8,\}$ $a_1=\epsilon$, $b_1=8\ 3\ 4\ 4$, $a_2=1\ 1$, $b_2=7\ 3$, $a_3=1\ 6\ 11\ 4\ 9$, . In such a situation $u(A)\leq_k w$ if and only if $u(A)\leq_k a_j$ for some $j=0,\dots,r$. We then define $\phi_S(w)=h(a_1)b_1h(a_2)b_2\dots h(a_r)b_rh(a_{r+1})$. Note that since $|h(a_j)|=|a_j|$ and $\sum(h(a_j))=\sum(a_j)$ for all j, it follows that $\phi_S(w)\in U_{n,m}$. It is easy to see that ϕ_S is bijection since we are assuming that h is bijection. Moreover, since we did not change the values of the letters w_i for $i\in S$, we are guaranteed that $S\subseteq I(\phi_S(w))$. In such a situation, we let $f(w,S)=(\phi_S(w),S)$. Since no letter in $\{1,\dots,k\}-A$ can be part of any embedding of $u(A)$ or v,

$$Em_{\mathcal{P}_k}(u(A),a_1b_1\dots a_rb_ra_{r+1})=Em_{\mathcal{P}_k}(v,h(a_1)b_1\dots h(a_r)b_rh(a_{r+1})).$$

Thus for all pairs $(w,S)\in B_{n,m}$, $Em_{\mathcal{P}_k}(u(A),w)=Em_{\mathcal{P}_k}(v,\phi_S(w))$.

We are now in position apply the involution principle where $S=S'=C_{n,m}$, $I=I'=J$ and $f:C_{n,m}\to C_{n,m}$ is the sign-preserving weight-preserving bijection described above. Note that in this case, $F=F'=\{(w,\emptyset):w\in U_{n,m}\cap\mathbb{P}^*_A\}$. Then let $\overline{g}_{n,m}$ be the map constructed by the involution principle for this choice of S, S', I, I', and f. Thus $\overline{g}_{n,m}$ is a weight-preserving bijection from $(U_{n,m}\cap\mathbb{P}^*_A)\times\{\emptyset\}$ onto $(U_{n,m}\cap\mathbb{P}^*_A)\times\{\emptyset\}$. Then let $g_{n,m}:U_{n,m}\cap\mathbb{P}^*_A\to U_{n,m}\cap\mathbb{P}^*_A$ be defined so that for all $(w,\emptyset)\in(U_{n,m}\cap\mathbb{P}^*_A)\times\{\emptyset\}$, $\overline{g}_{n,m}((w,\emptyset))=(g_{n,m}(w),\emptyset)$. Then our definitions ensure that for all $w\in U_{n,m}\cap\mathbb{P}^*_A$,

$$Em_{\mathcal{P}_k}(u(A),w)=Em_{\mathcal{P}_k}(v,g_{n,m}(w)).$$

Thus $g = \bigcup_{n,m\in\mathbb{P}} g_{n,m}$ is a weight-preserving bijection from $\mathbb{P}_A^*$ onto $\mathbb{P}_A^*$ such that for all $w \in \mathbb{P}_A^*$,

$$Em_{\mathcal{P}_k}(u(A), w) = Em_{\mathcal{P}_k}(v, g(w)). \tag{27}$$

Given g, define a bijection $\psi : \mathbb{P}^* \to \mathbb{P}^*$ as follows. Given $w = w_1 \dots w_t \in \mathbb{P}^*$, let $a_1b_1 \dots a_rb_ra_{r+1}$ be the (k, A)-factorization of w where $a_i \in \mathbb{P}_A^*$ for all i and $b_j \in (\mathbb{P} - \mathbb{P}_A)^*$ for all j. Then $w(A) = a_1 \dots a_{r+1} \in \mathbb{P}_A^*$ and we can write $g(w(A)) = c_1 \dots c_{r+1}$ where $|a_i| = |c_i|$ for all i. Then we define $\psi(w)$ to be $c_1b_1 \dots c_rb_rc_{r+1}$. We claim that

$$Em_{\mathcal{P}_k}(u, w) = Em_{\mathcal{P}_k}(u|_{u(A)\to v}, \psi(w)).$$

That is, suppose $|u| = n$, $|u(A)| = m$ and $w_{i+1} \dots w_{i+n}$ embeds u. Then let $d_1e_1 \dots d_se_sd_{s+1}$ be the (k, A)-factorization of $w_{i+1} \dots w_{i+n}$ where $d_i \in \mathbb{P}_A^*$ for all i and $e_j \in (\mathbb{P} - \mathbb{P}_A)^*$ for all j. Thus it must be the case $u(A) \leq_k d_1 \dots d_{s+1}$. Suppose that that $d_1 \dots d_{s+1}$ starts at position $x + 1$ in $w(A)$. Then we know that if $z_{s+1} \dots z_{s+m}$ is the factor of $g(w(A))$ starting at position $x+1$, then $v \leq_k z_{s+1} \dots z_{s+m}$ and we can write $z_{s+1} \dots z_{s+m} = c_1 \dots c_{s+1}$ where $c_i \in \mathbb{P}_A^*$ and $|d_j| = |c_j|$ for all j. But our definition of ψ ensures that the factor of length n which starts at i in $\psi(w)$ is $c_1e_1 \dots c_se_sc_{s+1}$ which embeds $u|_{u(A)\to v}$. Vice versa, if $c_1e_1 \dots c_se_sc_{s+1}$ embeds $u|_{u[A]\to v}$, then $w_{i+1} \dots w_{i+n}$ must embed u. Thus ψ shows that $u \backsim_{s,k} u|_{u(A)\to v}$.

For (ii), it is easy to see that if h just witnesses that $u(A) \backsim_k v$, then it will still be the case that ψ witnesses that $u \backsim_k u|_{u(A)\to v}$. □

For example, it follows from Lemma 5.5 that the strong Wilf equivalences $122 \backsim_{s,4} 221$ and $344 \backsim_{s,4} 443$ are witnessed by rearrangement maps. Thus using Theorem 5.7 with $k = 4$ and $A = \{3, 4\}$, we obtain that $122344 \backsim_{s,4} 122443$. But then using Theorem 5.7 with $k = 4$ and $A = \{1, 2\}$, we obtain that $122344 \backsim_{s,4} 221344$ and $122443 \backsim_{s,4} 221443$. Thus, we have that

$$122344 \backsim_{s,4} 122443 \backsim_{s,4} 221443 \backsim_{s,4} 221344.$$

Similarly, one can show that

$$132244 \backsim_{s,4} 142243 \backsim_{s,4} 242143 \backsim_{s,4} 232144.$$

Our final result is a direct analogue Theorem 5.3 in [2]. We include the proof here for completeness.

THEOREM 5.8 *Suppose $u_1 \dots u_n \backsim_{s,k} v_1 \dots v_n$. Then for any $m \in \mathbb{P}$,*

$$u_1^m \dots u_n^m \backsim_{s,k} v_1^m \dots v_n^m,$$

where for any word w is the concatenation of m copies of w.

Proof. Suppose $g : \mathbb{P}^* \to \mathbb{P}^*$ witnesses the strong equivalence $u_1 \dots u_n \backsim_{s,k} v_1 \dots v_n$. For $w \in \mathbb{P}^*$ and $i \leq l$, let $w[i] = w_iw_{i+m}w_{i+2m} \dots$. Then the embeddings of $u_1^m \dots u_n^m$ in w are completely determined by the embeddings of u in the $w[i]$ and vice-versa. So define $f : \mathbb{P}^* \to \mathbb{P}^*$ by setting $f(w)$ to be the word obtained from w by replacing each $w[i]$ with $g(w[i])$. Then f witnesses $u_1^m \dots u_n^m \backsim_{s,k} v_1^m \dots v_n^m$. □

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