PU. M. A. Vol. **23** (2012), No.3, pp. 317 – 333

Generalizations of the major index

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(Received: February 13, 2012, and in revised form February 17, 2013)

Abstract. In this paper, we give a general method for computing generating functions for the joint distribution of various analogues of the classical permutation statistics of the number of descents, the number of inversions, and the major index for various classes of words, permutations, colored permutations, and directed column-convex polyominoes. Our method allows us to give a uniform derivation of many known generating functions due to Garsia, Gessel, Mendes, Remmel, Reiner, and others as well as prove a number of new results.

Mathematics Subject Classification(2010). 05A15, 05E05.

Keywords: permutation, word, generating function, descent, major index, inversion.

1 Introduction

A permutation statistic is a function mapping permutations to nonnegative integers. The modern analysis of such objects began in the early twentieth century with the work of MacMahon [14]. He popularized the "classic" notions of the descent, inversion, and major index statistics. If A is a statement, then let $\chi(A)$ be 1 if A is true and 0 otherwise. Given a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in S_n$, we then define the descent number, the major index, and the inversion number as follows[†]:

- $\mathrm{des}(\sigma) = |\{i : \sigma_i > \sigma_{i+1}\}|$,
- $\mathrm{maj}(\sigma) = \sum_{i=1}^{n-1} i \cdot \chi(\sigma_i > \sigma_{i+1})$, and
- $\mathrm{inv}(\sigma) = |\{(i,j) : i < j, \sigma_i > \sigma_j\}|$.

[†] In addition to permutations, these definitions hold for any finite sequence of numbers.

MacMahon showed that the inversion number and the major index have the same distribution in S_n.

Gessel gave a generating function for descents, major index, and inversions both in his thesis [11] and in a paper coauthored with Garsia [10]. Define the q-shifted factorial of n, denoted $(a;q)_n$, to be $(1-a)(1-aq)\cdots(1-aq^{n-1})$ and the q-exponential function, denoted $e_q(z)$, to be $\sum_{n\geq 0}\frac{z^n}{(q;q)_n}$. Then Garsia and Gessel proved the following theorem.

THEOREM 1.1 (Garsia and Gessel)

$$\sum_{n\geq 0}\sum_{\sigma\in S_n}\frac{x^{\mathrm{des}(\sigma)}u^{\mathrm{maj}(\sigma)}q^{\mathrm{inv}(\sigma)}z^n}{(x;u)_{n+1}(q;q)_n}=\sum_{k\geq 0}x^k e_q(zu^k)e_q(zu^{k-1})\cdots e_q(z). \tag{1}$$

Analogous to the previous definitions, we define the ascent number, the co-major index, and the co-inversion number as follows:

- $\mathrm{asc}(\sigma)=|\{i:\sigma_i<\sigma_{i+1}\}|$,
- $\mathrm{comaj}(\sigma)=\sum_{i=1}^{n-1} i\cdot\chi(\sigma_i<\sigma_{i+1})$, and
- $\mathrm{coinv}(\sigma)=|\{(i,j):i<j,\sigma_i<\sigma_j\}|$.

When considering words instead of permutations, we instead define an ascent to be two consecutive letters that do not form a descent and define the co-major index and co-inversion number accordingly. Let $(a,b;p,q)_n=(a-b)(ap-bq)\cdots(ap^{n-1}-bq^{n-1})$ be the p,q-shifted factorial of n. Then it is easy to see that

$$\frac{p^{n+\mathrm{coinv}(\sigma)}q^{\mathrm{inv}(\sigma)}}{(p,q;p,q)_n}=\frac{(q/p)^{\mathrm{inv}(\sigma)}}{(q/p;q/p)_n}$$

for all $\sigma\in S_n$. Therefore, substituting q/p for q and z/p for z in (1) gives the p,q-exponential generating function that includes co-inversions. We can make similar substitutions to include the co-major index and ascent number. For clarity and brevity, we thus omit these statistics from the results in this paper while noting that it is straightforward to include them.

For an alphabet X, let X^n be the set of words of length n and X^* be the set of words of any length. In particular, the word of length 0 will be denoted ϵ. Many results in this paper will be stated in terms of the algebra of formal linear combinations of words in X^* over a ring, typically a ring of formal power series in one or more variables. In such cases, the product of two words will be given by their concatenation.

Let N be the alphabet $\{0,1,2,\ldots\}$. In this paper, we use the term composition to refer to a word in N^*. Likewise, a partition will be a composition $\lambda=\lambda_1\lambda_2\cdots\lambda_n$ such that $\lambda_i\leq\lambda_{i+1}$ for $1\leq i\leq n-1$. We will denote the set of partitions of length n by Λ_n. We will denote the set of partitions with parts of size at most n by $\Lambda^{\leq n}$. For a word w, we will define $\ell(w)$ to be the length of w, and if w is a composition, we will define $\mathrm{sum}(w)$ to be the sum of the letters in w. We will make repeated use of the well-known fact that $\frac{1}{(q;q)_n}$ is the generating function for Λ_n by sum and that $\frac{1}{(x;q)_{n+1}}$ is the generating function for $\Lambda^{\leq n}$ by length (counted by x) and sum (counted by q).

The key to using words to study questions about permutations is the inverse of Fédou's insertion-shift bijection as employed by Rawlings and Tiefenbruck [17]. For $\sigma\in S_n$ and $1\leq i\leq n$, let $\mathrm{inv}_i(\sigma)=|\{j:i<j\leq n,\sigma_i>\sigma_j\}|$. Then $\nabla_n:S_n\times\Lambda_n\to N^n$ is given by the rule $\nabla_n(\sigma,\lambda)=w$, where $w_i=\mathrm{inv}_i(\sigma)+\lambda_{\sigma_i}$. For example, $\nabla_6(2\,5\,6\,1\,4\,3,\ 2\,2\,4\,4\,4\,4)=3\,7\,7\,2\,5\,4$.

There are two key properties of the map ∇_n to note. First, if $\nabla_n(\sigma, \lambda) = w$, then

$$\mathrm{inv}(\sigma) + \mathrm{sum}(\lambda) = \mathrm{sum}(w).$$

Second, ∇_n roughly transfers the overall shape and patterns of σ to the corresponding w. That is, if $1 \leq i < m \leq n$, then

$$\sigma_i < \sigma_m \text{ if and only if } w_i \leq w_m + |\{j : i < j < m,\, \sigma_i > \sigma_j\}|.$$

In particular, ∇_n preserves descents. That is, if $\nabla_n(\sigma, \lambda) = w$, then for all $1 \leq i < n$, $\sigma_i > \sigma_{i+1}$ if and only if $w_i > w_{i+1}$ so that $\mathrm{des}(\sigma) = des(w)$ and $\mathrm{maj}(\sigma) = \mathrm{maj}(w)$.

Rawlings and Tiefenbruck go on to note that

$$\frac{q^{\mathrm{inv}(\sigma)}}{(q;q)_n} = \sum_{\lambda \in \Lambda_n} q^{\mathrm{inv}(\sigma)+\mathrm{sum}(\lambda)} = \sum_{w \in \nabla_n(\sigma, \Lambda_n)} q^{\mathrm{sum}(w)} \tag{2}$$

for all $\sigma \in S_n$. Therefore, the q-exponential generating function for many permutation statistics and inversions can be obtained from the corresponding ordinary generating function for compositions by sum.

Several authors have extended the Garsia-Gessel formula (1) to other groups. For example, Reiner [18] gave a B_n version of the Garsia-Gessel formula where B_n is the hyperoctahedral group, and Mendes and Remmel [15] gave versions of the Garsia-Gessel formula for groups that are the wreath product of a cyclic group C_k and the symmetric group.

Fuller and Remmel [8] studied analogues of Theorem 1.1 in compositions. Given a composition $w = w_1 \cdots w_n$, let z^w be the monomial $z_{w_1} \cdots z_{w_n}$. Since compositions can have repeated entries, it is natural to have analogues of des and maj where we replace $>$ by $\geq$ or $=$ in the definition of des and maj. That is, we let $Des(w) = \{i : w_i > w_{i+1}\}$, $WDes(w) = \{i : w_i \geq w_{i+1}\}$, and $Lev(w) = \{i : w_i = w_{i+1}\}$. Then we define

- $\mathrm{des}(w) = |Des(w)|$ and $\mathrm{maj}(w) = \sum_{i \in Des(w)} i$,
- $\mathrm{wdes}(w) = |WDes(w)|$ and $\mathrm{wmaj}(w) = \sum_{i \in WDes(w)} i$, and
- $\mathrm{lev}(w) = |Lev(w)|$ and $\mathrm{levmaj}(w) = \sum_{i \in Lev(w)} i$.

Fuller and Remmel [8] proved the following results.

Theorem 1.2 (Fuller and Remmel)

$$\sum_{n \geq 0} \sum_{w \in N^n} \frac{x^{\mathrm{des}(w)} u^{\mathrm{maj}(w)} z^w}{(x;u)_{n+1}} = \sum_{k \geq 0} \frac{x^k}{\prod_{i \geq 0} (z_i;u)_{k+1}}.$$

Theorem 1.3 (Fuller and Remmel)

$$\sum_{n \geq 0} \sum_{w \in N^n} \frac{x^{\mathrm{wdes}(w)} u^{\mathrm{wmaj}(w)} z^w}{(x;u)_{n+1}} = \sum_{k \geq 0} x^k \prod_{i \geq 0} (-z_i;u)_{k+1}.$$

Theorem 1.4 (Fuller and Remmel)

$$\sum_{n\geq 0}\sum_{w\in N^n}\frac{x^{\mathrm{lev}(w)}u^{\mathrm{levmaj}(w)}z^w}{(x;u)_{n+1}}=\sum_{k\geq 0}\frac{x^k}{\prod_{j=0}^{k}(\sum_{n\geq 0}(-u^j)^n p_n)},$$

where $p_n = p_n(z_0, z_1, \ldots) = \sum_{i\geq 0} z_i^n$ *is the power symmetric function.*

The goal of this paper is to prove a common generalization of the results of Fuller and Remmel [8], and then we will show how we can use this result to not only recover the results of Garsia and Gessel [10], Reiner [18], Mendes and Remmel [15], and others but also to prove several new analogues of (1).

The outline of the paper is as follows. In Section 2, we shall state and prove our main theorem that generalizes (1) as well as Fuller and Remmel's results. In Section 3, we will present many extensions of this theorem and apply it to a variety of combinatorial objects and variations on the major index statistic. Finally, in Section 4, we will list a few open problems.

2 The Main Theorem

In this section, we will derive a general version of Garsia and Gessel's result. Let X be an alphabet, $A \subseteq X^2$, and $B = X^2 \setminus A$. We will refer to such pairs (A, B) as complementary pairs. We will typically think of the case $X = N$, $A = \{w_1w_2 : w_1 \leq w_2\}$, $B = \{w_1w_2 : w_1 > w_2\}$, but our results will hold in general. For $w \in X^n$, we can then define the descent number and major index with respect to (A, B):

- $\mathrm{des}_A(w) = |\{i : w_iw_{i+1} \in B\}|$
- $\mathrm{maj}_A(w) = \sum_{i=1}^{n-1} i \cdot \chi(w_iw_{i+1} \in B)$

Note that des_A and maj_A can be defined purely in terms of A, since $w_iw_{i+1} \in B$ if and only if $w_iw_{i+1} \notin A$. When the meaning is clear, we will drop the subscript A. Further, let $A_n = \{w \in X^n : w_iw_{i+1} \in A, 1 \leq i < n\} = \{w \in X^n : \mathrm{des}_A(w) = 0\}$, $a_n = \sum_{w\in A_n} w$ (taken as a formal sum), and $A(z) = a_0 + a_1z + a_2z^2 + \cdots$. We will similarly define B_n, b_n, and $B(z)$ for B, and we will use analogous notation for other complementary pairs. We can now state a general result.

Theorem 2.1

$$\sum_{n\geq 0}\sum_{w\in X^n}\frac{x^{\mathrm{des}_A(w)}u^{\mathrm{maj}_A(w)}z^n}{(x;u)_{n+1}}w=\sum_{k\geq 0}x^kA(zu^k)A(zu^{k-1})\cdots A(z) \tag{3}$$

Proof. Recall that $\frac{1}{(x;u)_{n+1}} = \sum_{\lambda\in\Lambda^{\leq n}} x^{\ell(\lambda)}u^{\mathrm{sum}(\lambda)}$. Fix $w \in X^n$, and define the partition $\mu(w)$ such that if $w_iw_{i+1} \in B$, then $\mu(w)$ has a part of size i. Clearly $\mathrm{sum}(\mu(w)) = \mathrm{maj}(w)$ and $\ell(\mu(w)) = \mathrm{des}(w)$. Then, for $\lambda \in \Lambda^{\leq n}$, form the partition $\lambda + \mu(w)$ by merging λ and $\mu(w)$; that is, for all i, $\lambda + \mu(w)$ will have one part of size i for each such part in either λ or $\mu(w)$. It is then clear that the left side of (3) is

$$\sum_{n\geq 0}\sum_{w\in X^n}\sum_{\lambda\in\Lambda^{\leq n}} x^{\ell(\lambda+\mu(w))}u^{\mathrm{sum}(\lambda+\mu(w))}w. \tag{4}$$

Next we want to compute the coefficient of x^k in (4). To this end, let $\nu = \lambda + \mu(w)$ for some $\lambda \in \Lambda^{\leq n}$ where $k = \ell(\nu)$. Then, we can use ν to factor w into subwords as $w = w^{(0)}w^{(1)} \cdots w^{(k)}$, where $w^{(i)} = w_{\nu_i+1} \cdots w_{\nu_{i+1}}$, using $\nu_0 = 0$ and $\nu_{k+1} = n$. For example, suppose that $X = N$, $A = \{w_1 w_2 : w_1 \leq w_2\}$, and $w = 021044203$ so that $\mu(w) = 2367$. Then in the case where λ is the empty partition, $\nu = \mu(w)$ so that $w^{(0)} = 02$, $w^{(1)} = 1$, $w^{(2)} = 004$, $w^{(3)} = 2$, and $w^{(4)} = 03$. This situation is best described pictorially by the diagram in Figure 2. Note that the descents of w have been broken into distinct subwords, so that each subword has no descents.

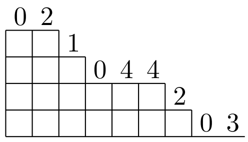


Figure 1: ν and $w^{(i)}$ for $w = 021044203$ and $\lambda = \epsilon$

Similarly if $\lambda = 001169$, then $\nu = 0011236679$ and $w^{(0)} = \epsilon$, $w^{(1)} = 0$, $w^{(2)} = \epsilon$, $w^{(3)} = 2$, $w^{(4)} = 1$, $w^{(5)} = 044$, $w^{(6)} = \epsilon$, $w^{(7)} = 2$, $w^{(8)} = 03$, and $w^{(9)}\epsilon$. This process is pictured in Figure 2.

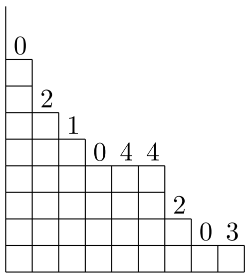


Figure 2: ν and $w^{(i)}$ for $w = 021024203$ and $\lambda = 001169$

Clearly $\mathrm{des}(w^{(j)}) = 0$ for each j, regardless of λ, since $\mu(w)$ (and thus ν) contains a part i if $w_i w_{i+1} \in B$.

Now, if we fix k and sum over all words w and corresponding ν of length k, we will show that we get every $k+1$-tuple of words with no descents exactly once. First, fix $w^{(0)}, \ldots, w^{(k)}$. From this, we can clearly obtain w by concatenation. We can also obtain ν, since $\nu_i = \ell(w^{(0)} \cdots w^{(i-1)})$. Moreover, ν must contain all of the parts of $\mu(w)$, since the subwords $w^{(i)}$ do not contain descents. Thus, we have a bijection.

Note that $\ell(\nu) = \sum_{i=0}^{k} \ell(w^{(i)})$. To obtain an expression for $\mathrm{sum}(\nu)$, we note that each letter of $w^{(i)}$ has $k-i$ squares beneath it in the corresponding diagram. Therefore, each letter of $w^{(i)}$ contributes 1 to the length and $k-i$ to the sum of ν. Thus the coefficient of x^k in (4) is

$$\sum_{\substack{w^{(0)}, w^{(1)}, \ldots, w^{(k)} \\ \mathrm{des}_A(w^{(i)}) = 0}} \prod_{i=0}^{k} (zu^{k-i})^{\ell(w^{(i)})} w^{(i)} = A(zu^k)A(zu^{k-1}) \cdots A(z).$$

Hence (4) is also equal to the right side of (3). □

In the case of $X = N$ and $A = \{w_1w_2 : w_1 \le w_2\}$, we see that A_n is the set of partitions of length n and $X^n = \sum_{\sigma\in S_n} \nabla_n(\sigma,\Lambda_n)$. Thus, if we substitute q^a for the letter $a \in N$ and use (2), then we see that

$$\begin{aligned}\sum_{w\in X^n} \frac{x^{\mathrm{des}(w)}u^{\mathrm{maj}(w)}z^n}{(x;u)_{n+1}} q^{\mathrm{sum}(w)} &= \sum_{\sigma\in S_n} \frac{x^{\mathrm{des}(\sigma)}u^{\mathrm{maj}(\sigma)}z^n}{(x;u)_{n+1}} \sum_{\lambda\in\Lambda_n} q^{\mathrm{sum}(\nabla_n(\sigma,\lambda))} \\ &= \sum_{\sigma\in S_n} \frac{x^{\mathrm{des}(\sigma)}u^{\mathrm{maj}(\sigma)}z^n}{(x;u)_{n+1}} \frac{q^{\mathrm{inv}(\sigma)}}{(q;q)_n}.\end{aligned}$$

Moreover, $A(z)$ becomes $e_q(z)$, so that Theorem 2.1 and (2) imply (1). If we vary the choice of A and instead substitute z_aq^a for the letter a and $z = 1$, Theorem 2.1 also implies the results of Fuller and Remmel.

3 Examples and Extensions

In this section, we will provide many examples utilizing Theorem 2.1. Some of our results will translate the notion of major index to other common combinatorial objects such as colored permutations or directed column-convex polyominoes. Other results will modify the notion of major index to sum the positions of features such as alternating descents or descents that occur at specified positions in the permutation.

3.1 Colored Permutations

A natural extension of the major index is to the set of colored permutations. That is, if you have c colors, a colored permutation is a permutation such that each letter has been assigned a color from the set $[c] = \{1, 2, \ldots, c\}$. Colored permutations can be thought of as elements of the set $[c]^n \times S_n$. When $c = 2$, the colored permutations are closely related to the signed permutations B_n, for which (1) has been treated separately (and distinctly) by Reiner [18] and by Mendes and Remmel [15]. Steingrímsson [21] also studied various statistics on $[c]^n \times S_n$. We will give yet another interpretation.

Let $\pi = (v, \sigma) \in [c]^n \times S_n$. We would like to compute the generating function

$$\sum_{n\ge 0} \sum_{\pi\in[c]^n\times S_n} \frac{x^{\mathrm{des}(\pi)}u^{\mathrm{maj}(\pi)}q^{\mathrm{inv}(\pi)}z^v}{(x;u)_{n+1}(q;q)_n}. \tag{5}$$

However, we first need to define the various colored permutation statistics. First, we will consider ordered pairs of words of the same length as words on an alphabet of ordered pairs. For example, we will consider $(v_1v_2, \sigma_1\sigma_2)$ and $(v_1, \sigma_1)(v_2, \sigma_2)$ to be the same word. Then, we can define statistics that depend on comparisons of these ordered pairs of letters.

We could define $\mathrm{des}(\pi)$, $\mathrm{maj}(\pi)$, and $\mathrm{inv}(\pi)$ to be $\mathrm{des}(\sigma)$, $\mathrm{maj}(\sigma)$, and $\mathrm{inv}(\sigma)$, respectively. However, in that case, (5) can be obtained from (1) merely by replacing z with $z_1 + z_2 + \cdots + z_c$, since the colors do not affect any of the statistics. Instead, we will use a lexicographic order on the letters of π, so that $(v_i, \sigma_i) > (v_j, \sigma_j)$ if $v_i > v_j$ or if $v_i = v_j$ and $\sigma_i > \sigma_j$. Thus, we define

- $\underline{\mathrm{des}}((v, \sigma)) = |\{i : (v_i, \sigma_i) > (v_{i+1}, \sigma_{i+1})\}|$,

- $\underline{\text{maj}}((v,\sigma)) = \sum_{i=1}^{n-1} i\chi((v_i,\sigma_i) > (v_{i+1},\sigma_{i+1}))$, and
- $\underline{\text{inv}}((v,\sigma)) = |\{(i,j) : i < j \ \& \ (v_i,\sigma_i) > (v_j,\sigma_j)\}|$.

We claim that

$$\sum_{n\geq 0}\sum_{\pi\in[c]^n\times S_n} \frac{x^{\underline{\text{des}}(\pi)}u^{\underline{\text{maj}}(\pi)}q^{\underline{\text{inv}}}z^v}{(x;u)_{n+1}(q;q)_n} \tag{6}$$

is still obtained from (1) by substituting $z_1 + \cdots + z_c$ for z. Consider the map f sending $\pi = (v,\sigma)$ to (v,τ), where $\tau \in S_n$ such that $\tau_i > \tau_j$ if and only if $\pi_i > \pi_j$. For example, if $(v,\sigma) = (31321, 53124)$, then $(3,5) > (3,1) > (2,2) > (1,4) > (1,3)$ so that $f((v,\sigma)) = (31321, 51432)$. Then, we have

$$\sum_{n\geq 0}\sum_{\pi\in[c]^n\times S_n} \frac{x^{\underline{\text{des}}(\pi)}u^{\underline{\text{maj}}(\pi)}q^{\underline{\text{inv}}(\pi)}z^v}{(x;u)_{n+1}(q;q)_n} = \sum_{n\geq 0}\sum_{\pi=f^{-1}((v,\tau))\in[c]^n\times S_n} \frac{x^{\text{des}(\tau)}u^{\text{maj}(\tau)}q^{\text{inv}(\tau)}z^v}{(x;u)_{n+1}(q;q)_n}. \tag{7}$$

Let V be the set of all $v \in [c]^n$ with a given value of z^v. Then, we will show that $|V \times \{\sigma\}| = |f^{-1}(V \times \{\sigma\})|$. Let $z^v = z_1^{c_1} \cdots z_n^{c_n}$. To construct an element of $V \times \{\sigma\}$, we must choose c_n positions for the color n, then c_{n-1} positions for the color $n-1$ from those that remain, and so on. To construct a preimage (v,ρ) of (v,σ), v must have n in the same positions as σ's largest c_n elements, $n-1$ in the same positions as σ's next largest c_{n-1} elements, and so on. Then, we must choose c_n values to have the color n in ρ and order them to agree with σ, then c_{n-1} remaining values to have color $n-1$, and so on. Thus, we see the claim is true. But then, for a given z^v, any permutation occurs as σ the same number of times as it occurs as τ on the right side of (7), so

$$\sum_{n\geq 0}\sum_{\pi=f^{-1}((v,\tau))\in[c]^n\times S_n} \frac{x^{\text{des}(\pi)}u^{\text{maj}(\pi)}q^{\text{inv}(\pi)}z^v}{(x;u)_{n+1}(q;q)_n} = \sum_{n\geq 0}\sum_{\pi=(v,\sigma)\in[c]^n\times S_n} \frac{x^{\text{des}(\sigma)}u^{\text{maj}(\sigma)}q^{\text{inv}(\sigma)}z^v}{(x;u)_{n+1}(q;q)_n}.$$

Thus, we see the generating function is unchanged from the previous case.

We will instead seek the generating function

$$\sum_{n\geq 0}\sum_{\pi\in[c]^n\times S_n} \frac{x^{\underline{\text{des}}(\pi)}u^{\underline{\text{maj}}(\pi)}q^{\text{inv}(\sigma)}z^v}{(x;u)_{n+1}(q;q)_n}, \tag{8}$$

where we track the inversions of the underlying permutation σ. First, we will prove a necessary extension to Theorem 2.1.

THEOREM 3.1 *Let X be an alphabet with complementary pair (A,B). Let $Y = [c] \times X$ and define $C = \{(v_1v_2, w_1w_2) \in Y^2 : v_1 < v_2$ or $v_1 = v_2$ and $w_1w_2 \in A\}$. If $C^{(i)} = \{(ii, w_1w_2) : w_1w_2 \in A\}$, then*

$$C(z) = C^{(1)}(z)C^{(2)}(z)\cdots C^{(c)}(z)$$

is the generating function for all pairs (v,σ) such that $\text{des}_c((v,\sigma)) = 0$. *Thus*

$$\sum_{n\geq 0}\sum_{w\in Y^n} \frac{x^{\text{des}_C(w)}u^{\text{maj}_C(w)}z^n}{(x;u)_{n+1}} w = \sum_{k\geq 0} x^k C(zu^k)C(zu^{k-1})\cdots C(z). \tag{9}$$

Proof. By definition, any word (v, w) with no C-descents must have the letters of v weakly increasing. Further, for letters with the same element of $[c]$, the corresponding subword of w must have no A-descents. Thus, the first part of the theorem is clear. It is then easy to see that (9) is just a special case of Theorem 2.1. □

COROLLARY 3.2

$$\sum_{n\geq 0}\sum_{\pi=(v,\sigma)\in[c]^n\times S_n}\frac{x^{\underline{\mathrm{des}}(\pi)}u^{\underline{\mathrm{maj}}(\pi)}q^{\mathrm{inv}(\sigma)}z^v}{(x;u)_{n+1}(q;q)_n}=\sum_{k\geq 0}x^k\prod_{i=1}^{c}e_q(z_iu^k)e_q(z_iu^{k-1})\cdots e_q(z_i).$$

Proof. As usual, we will take $X = N$ and $A = \{w_1w_2 : w_1 \leq w_2\}$. Then, since $A(z) = e_q(z)$, we have $C(z) = e_q(z_1)\cdots e_q(z_c)$. If we set $z = 1$ and replace the letters (v_i, w_i) by $q^{w_i}z_{v_i}$ in w in Theorem 3.1, we have

$$\sum_{n\geq 0}\sum_{(v,w)\in([c]\times N)^n}\frac{x^{\underline{\mathrm{des}}((v,w))}u^{\underline{\mathrm{maj}}((v,w))}q^{\mathrm{sum}(w)}z^v}{(x;u)_{n+1}}=\sum_{k\geq 0}x^k\prod_{i=1}^{c}e_q(z_iu^k)e_q(z_iu^{k-1})\cdots e_q(z_i).$$

In order to obtain a statement about colored permutations, we must modify Fédou's bijection. For $(v, \sigma) \in [c]^n \times S_n$ and $\lambda \in \Lambda_n$, we will associate the colored word $(v, \nabla_n(\sigma, \lambda)) \in ([c] \times N)^n$. Then, since ∇_n preserves descents, it is clear that $\underline{\mathrm{des}}((v, w)) = \underline{\mathrm{des}}((v, \sigma))$ and $\underline{\mathrm{maj}}((v, w)) = \underline{\mathrm{maj}}((v, \sigma))$. Thus, if we multiply the left of (2) by $x^{\underline{\mathrm{des}}((v,\sigma))}u^{\underline{\mathrm{maj}}((v,\sigma))}$ and the right by $x^{\underline{\mathrm{des}}((v,w))}u^{\underline{\mathrm{maj}}((v,w))}$, then sum over all (v, σ), we get

$$\sum_{(v,w)\in([c]\times N)^n}x^{\underline{\mathrm{des}}((v,w))}u^{\underline{\mathrm{maj}}((v,w))}q^{\mathrm{sum}(w)}=\sum_{(v,\sigma)\in[c]^n\times S_n}\frac{x^{\underline{\mathrm{des}}((v,\sigma))}u^{\underline{\mathrm{maj}}((v,\sigma))}q^{\mathrm{inv}(\sigma)}}{(q;q)_n}.$$

Substituting this expression above completes the proof. □

3.2 Pairs of Permutations

Another natural extension of the number of descents and the major index is to the number of common descents and the common major index, respectively. That is, for $\sigma, \tau \in S_n$, let $\mathrm{comdes}(\sigma, \tau)$ be the number of positions i such that $\sigma_i > \sigma_{i+1}$ and $\tau_i > \tau_{i+1}$, and let $\mathrm{commaj}(\sigma, \tau)$ be the sum of those i. This type of statistic has been studied by Carlitz and Scoville [1], together and with Vaughan [2], in the 1970s and revisited by Fédou and Rawlings [6, 7] in the 1990s. Mendes and Remmel [15] derived the generating function analogous to (1), which we will re-derive here.

THEOREM 3.3 *Let X be an alphabet with complementary pair (A, B), and let Y be another alphabet with complementary pair (C, D). Let $Z = X\times Y$ with complementary pair (E, F), where $F = \{(x_1x_2, y_1y_2) : x_1x_2 \in B$ and $y_1y_2 \in D\}$. Then*

$$\sum_{n\geq 0}\sum_{w\in Z^n}\frac{x^{\mathrm{des}_E(w)}u^{\mathrm{maj}_E(w)}z^n}{(x;u)_{n+1}}w=\sum_{k\geq 0}x^kE(zu^k)E(zu^{k-1})\cdots E(z),$$

where $E(z) = (F(-z))^{-1}$ and $F_n = B_n \times D_n$.

Proof. Rawlings and Tiefenbruck [17] proved the relationship $A(z) = (B(-z))^{-1}$ for all complementary pairs (A, B), so $E(z), F(z)$ satisfy the same relationship. It should also be noted that the same result implies $B(z) = (A(-z))^{-1}$ and $D(z) = (C(-z))^{-1}$, from which B_n and D_n can be obtained. The claim $F_n = B_n \times D_n$ is obvious, since a pair of letters $(x_1x_2, y_1y_2) \in F$ must have $x_1x_2 \in B$ and $y_1y_2 \in D$. Apply Theorem 2.1 to complete the proof. □

It should be noted that, if X and Y are partially ordered sets (posets) such that $B = \{w_1w_2 \in X : w_1 \leq w_2\}$ and $D = \{w_1w_2 \in Y : w_1 \leq w_2\}$, then F is the product order on $X \times Y$. Since Theorem 2.1 may be applied equally well to posets, Theorem 3.3 thus gives us an easy way to compute results for product posets.

Corollary 3.4

$$\sum_{n\geq 0}\sum_{\sigma,\tau\in S_n} \frac{x^{\mathrm{comdes}(\sigma,\tau)}u^{\mathrm{commaj}(\sigma,\tau)}q_1^{\mathrm{inv}(\sigma)}q_2^{\mathrm{inv}(\tau)}z^n}{(x;u)_{n+1}(q_1;q_1)_n(q_2;q_2)_n} = \sum_{k\geq 0} x^k E(zu^k)E(zu^{k-1})\cdots E(z),$$

where

$$E(z) = \left(\sum_{n\geq 0} \frac{q_1^{\binom{n}{2}}q_2^{\binom{n}{2}}(-z)^n}{(q_1;q_1)_n(q_2;q_2)_n}\right)^{-1}.$$

Proof. First, apply (2) to both σ and τ. Then, we want to enumerate pairs of compositions, keeping track of where they have common descents. Then, to apply Theorem 3.3, we must compute B_n in the standard case $X = N$, $A = \{w_1w_2 : w_1 \leq w_2\}$. B_n is the set of words w of length n such that every position is a descent. This is the reverse of a partition with distinct parts. Subtracting $n - i$ from w_i, we obtain the reverse of an arbitrary partition. Replacing the letter a with q^a, we then see that b_n becomes $\frac{q^{\binom{n}{2}}}{(q;q)_n}$. Now, if we use Theorem 3.3 with $X = Y$ and $A = C$, and then replace the letters of X with q_1^a and the letters of Y with q_2^a, we obtain the given expression. □

It is straightforward to see how Theorem 3.3 and Corollary 3.4 extend to more than two sets of words. We also note that Theorem 3.3 can be applied to the colored permutations and words in the previous subsection to obtain a different result.

3.3 Descents at Positions Congruent to i mod j

Another interesting extension of the major index is to count only those descents that occur at specific positions. In this subsection, we will consider the particular case where we count descents that occur at positions congruent to i mod j. That is, for $1 \leq i \leq j$, let $\mathrm{des}_{i\Gamma j}(\sigma)$ be the number of k such that $\sigma_{i+jk} > \sigma_{i+jk+1}$, and let $\mathrm{maj}_{i\Gamma j}(\sigma)$ be the sum of those $i + jk$. Then, we would naturally be interested in computing

$$\sum_{n\geq 0}\sum_{\sigma\in S_n} \frac{x^{\mathrm{des}_{i\Gamma j}(\sigma)}u^{\mathrm{maj}_{i\Gamma j}(\sigma)}q^{\mathrm{inv}(\sigma)}z^n}{(x;u)_{n+1}(q;q)_n}. \tag{10}$$

However, our analysis will lead us to a generating function of a different form.

Theorem 3.5 *Let X be an alphabet with complementary pair (A, B). Then,*

$$\sum_{n\geq 0}\sum_{w\in X^n} \frac{x^{\mathrm{des}_{i\Gamma j}(w)} u^{\mathrm{maj}_{i\Gamma j}(w)} z^n}{(xu^i; u^j)_{\lfloor \frac{n-i}{j}\rfloor+1}} w = \sum_{k\geq 0} x^k A^{0,k}(zu^k) A^{1,k}(zu^{k-1}) \cdots A^{k,k}(z), \tag{11}$$

where

$$\begin{aligned}
A^{0,0}(z) &= (1-(a_1z)^{i+1})(1-a_1z)^{-1} + a_1^{i-1}a_2z^{i+1}(1-a_1^{j-2}a_2z^j)^{-1}(1-(a_1z)^j)(1-a_1z)^{-1},\\
A^{0,k}(z) &= a_1^iz^i + a_1^{i-1}a_2z^{i+1}(1-a_1^{j-2}a_2z^j)^{-1}a_1^{j-1}z^{j-1} \text{ if } k>0,\\
A^{m,k}(z) &= 1 + a_1^jz^j + a_1^{j-1}a_2z^{j+1}(1-a_1^{j-2}a_2z^j)^{-1}a_1^{j-1}z^{j-1}, \text{ if } 0<m<k,\\
A^{k,k}(z) &= 1 + a_1z(1-a_1^{j-2}a_2z^j)^{-1}(1-(a_1z)^j)(1-a_1z)^{-1}
\end{aligned}$$

where $a_n = \sum_{w\in A_n} w$.

Proof. Consider the proof of Theorem 2.1. For the statistic $\mathrm{des}_{i\Gamma j}(w)$, we define $\mu_{i,j}(w)$ to be the partition which has a part of size $i+kj$ if and only if $w_{i+kj}w_{i+kj+1} \in B$. This is the analogue of $\mu(w)$ in the proof of theorem 2.1. Let $\Lambda_{i,j}^{\leq n}$ be the set of partitions whose parts of size $\leq n$ and whose parts all equivalent to $i \mod j$. Then it is easy to see

$$(xu^i; u^j)_{\lfloor \frac{n-i}{j}\rfloor+1} = \sum_{\lambda\in\Lambda_{i,j}^{\leq n}} x^{\ell(\lambda)} u^{\mathrm{sum}(\lambda)}.$$

It is then clear that the left side of (11) is

$$\sum_{n\geq 0}\sum_{w\in X^n}\sum_{\lambda\in\Lambda_{i,j}^{\leq n}} x^{\ell(\lambda+\mu_{i,j}(w))} u^{\mathrm{sum}(\lambda+\mu_{i,j}(w))} \tag{12}$$

Next we want to take the coefficient of x^k in (12). To this end, let $\nu = \lambda + \mu_{i,j}(w)$ where $\lambda \in \Lambda_{i,j}^{\leq n}$ and $\ell(\nu) = k$. Then as in the proof of Theorem 2.1, we can use μ to factor w into subwords $w^{(0)}w^{(0)}\cdots w^{(k)}$, each of which have no A-descents at positions of the form $i+kj$. If $k=0$, we see that w is a word with no descents at positions congruent to i mod j, which are counted by $A^{0,0}(z)$. That is, such a word $w^{(0)}$ either has length $\leq i$, which gives rise to the term $(1-(a_1z)^{i+1})(1-a_1z)^{-1}$, or can be factored into words $u^{(1)}u^{(2)}\cdots u^{(s)}u^{(s+1)}$ such that $u^{(1)} = u_1^{(1)}\cdots u_{i+1}^{(1)}$ where $u_i^{(1)} \leq_A u_{i+1}^{(1)}$, each $u^{(r)} = u_1^{(r)}\cdots u_j^{(r)}$ where $u_{j-1}^{(1)} \leq_A u_j^{(1)}$ if $2\leq i\leq s$, and $u^{(s+1)}$ has length $\leq j-1$ which gives rise to the factor $a_1^{i-1}a_2z^{i+1}(1-a_1^{j-2}a_2z^j)^{-1}(1-(a_1z)^j)(1-a_1z)^{-1}$.

If $k>0$, then:

- $w^{(0)}$ has length congruent to i mod j and no descents at positions congruent to i mod j.
- For $0<m<k$, $w^{(m)}$ has length congruent to 0 mod j and no descents at positions congruent to 0 mod j.
- $w^{(k)}$ has arbitrary length and no descents at positions congruent to 0 mod j.

The same type of reasoning used to compute $A^{0,0}(z)$ will show that these three types of words are counted respectively by $A^{0,k}(z)$, $A^{m,k}(z)$, and $A^{k,k}(z)$. □

It is clear how to apply Theorem 3.5 to permutations. Apply Theorem 3.5 to compositions, letting the letter a be replaced by q^a, so that $a_1 = \frac{1}{1-q}$ and $a_2 = \frac{1}{(q;q)_2}$. Fédou's bijection completes the result.

Now, let $S_n^{i\upharpoonright j}$ be the set of permutations whose only descents occur at positions congruent to i mod j. Mendes, Remmel, and Riehl [16] derived the generating function for permutations in $S_n^{i\upharpoonright j}$ by descents and inversions. In these permutations, $\mathrm{des}_{i\upharpoonright j}(\sigma) = \mathrm{des}(\sigma)$ and $\mathrm{maj}_{i\upharpoonright j}(\sigma) = \mathrm{maj}(\sigma)$, so restricting (11) to $S_n^{i\upharpoonright j}$, we obtain the natural extension of their result to include the major index.

Theorem 3.6 *Let X be an alphabet with complementary pair (A, B), and let $X^n_{i\upharpoonright j} = \{w \in X^n : w_k w_{k+1} \in B$ implies $k = i$ mod $j\}$. Then,*

$$\sum_{n\geq 0}\sum_{w\in X^n_{i\upharpoonright j}} \frac{x^{\mathrm{des}_{i\upharpoonright j}(w)}u^{\mathrm{maj}_{i\upharpoonright j}(w)}z^n}{(xu^i;u^j)_{\lfloor\frac{n-i}{j}\rfloor+1}} w = \sum_{k\geq 0} x^k A^{0,k}(zu^k)A^{1,k}(zu^{k-1})\cdots A^{k,k}(z),$$

where

$$\begin{aligned} A^{k,k}(z) &= A(z), \\ A^{0,k}(z) &= a_i z^i + a_{i+j}z^{i+j} + a_{i+2j}z^{i+2j} + \cdots \quad \text{if } k > 0, \\ A^{m,k}(z) &= 1 + a_j z^j + a_{2j}z^{2j} + \cdots \quad \text{if } 0 < m < k. \end{aligned}$$

Proof. The proof is identical to that of Theorem 3.5, except that now the subwords $w^{(m)}$ may not have any descents. It is clear then that $A^{k,k}(z)$ reduces to $A(z)$ and that $A^{0,k}(z)$ and $A^{m,k}(z)$ are as given. □

It should now be clear how to apply Theorem 3.6 to permutations. We see that a_n becomes $\frac{1}{(q;q)_n}$, so that $A^{0,k}(z)$ and $A^{m,k}(z)$ can actually be written as linear combinations of complex q-exponential functions. Recall that the $\frac{q^{\mathrm{inv}(\sigma)}}{(q;q)_n}$ term comes from the term $q^{\mathrm{sum}(w)}$ with compositions. Thus, the result above really does extend the result of Mendes, Remmel, and Riehl [16].

It should be noted that both Theorem 3.5 and Theorem 3.6 can be thought of as special cases of a more general result. That is, in the context of compositions, we can rewrite w in the proofs as a word w' on the alphabets N^j or Λ_j, respectively, except with modifications to the first and last letters of w'.

3.4 Alternating Descents

Chebikin [3] defined the alternating descent set of a permutation σ by $\mathrm{AltDes}(\sigma) = \{2i-1 : \sigma_{2i-1} < \sigma_{2i}\} \cup \{2i : \sigma_{2i} > \sigma_{2i+1}\}$. In words, it is the set of odd positions that are ascents and the even positions that are descents. One could also think of the set of common ascents and descents with an up-down permutation. We then define $\mathrm{altdes}(\sigma) = |\mathrm{AltDes}(\sigma)|$ and $\mathrm{altmaj}(\sigma) = \sum_{i\in\mathrm{AltDes}(\sigma)} i$. We wish to compute

$$\sum_{n\geq 0}\sum_{\sigma\in S_n} \frac{x^{\mathrm{altdes}(\sigma)}u^{\mathrm{altmaj}(\sigma)}q^{\mathrm{inv}(\sigma)}z^n}{(x;u)_{n+1}(q;q)_n}. \tag{13}$$

Remmel [19] computed the version of (13) with inversions omitted.

We will define altdes and altmaj similarly for words. In particular, let X be an alphabet with complementary pair (A, B). If $w = w_1 \cdots w_n \in X^n$, then we define $\mathrm{AltDes}_A(w) = \{2i-1 : w_{2i-1}w_{2i} \in A\} \cup \{2i : w_{2i}w_{2i+1} \in B\}$ and set $\mathrm{altdes}_A(w) = |\mathrm{AltDes}_A(w)|$ and $\mathrm{altmaj}_A(w) = \sum_{i\in \mathrm{AltDes}_A(w)} i$.

THEOREM 3.7 *Let X be an alphabet with complementary pair (A, B). Define*

$$A^{\cos}(z) = \sum_{n\geq 0}(-1)^n a_{2n} z^{2n},$$
$$A^{\sin}(z) = \sum_{n\geq 0}(-1)^n a_{2n+1} z^{2n+1},$$

and let $A^M(z)$ be the matrix

$$A^M(z) = \begin{bmatrix} (A^{\cos}(z))^{-1} & (A^{\cos}(z))^{-1}A^{\sin}(z) \\ A^{\sin}(z)(A^{\cos}(z))^{-1} & A^{\cos}(z) + A^{\sin}(z)(A^{\cos}(z))^{-1}A^{\sin}(z) \end{bmatrix}.$$

Then,

$$\sum_{n\geq 0}\sum_{w\in X^n} \frac{x^{\mathrm{altdes}_A(w)} u^{\mathrm{altmaj}_A(w)} z^n}{(x;u)_{n+1}} w = \sum_{k\geq 0} x^k [1 \ \ 0] A^M(zu^k)A^M(zu^{k-1})\cdots A^M(z)\begin{bmatrix}1\\1\end{bmatrix}. \tag{14}$$

Proof. Consider the proof of Theorem 2.1. In this case, the set of words that may be used for $w^{(i)}$ depends on whether $w^{(i)}$ starts at an even or odd position in w. Thus, we will generate our word w using a two-state Markov process. One state will generate subwords starting at an odd position, and the other will generate subwords starting at an even position. Then, the transitions between these states will consist of words $w^{(i)}$ with the appropriate starting position and length. Therefore, we need to compute the generating functions for words with no alternating descents for each combination of even and odd length and starting at an even or odd position.

One can use Goulden and Jackson's Pattern Algebra method, as described in [13] or [17], to find these generating functions. We have computed these functions and placed them in the matrix $A^M(z)$, which is a transition matrix for our Markov process. The top-left entry is the generating function for subwords of even length starting at an odd position. The top-right entry is the generating function for subwords of odd length starting at an odd position. The bottom-left entry is the generating function for subwords of odd length starting at an even position, and the bottom-right entry is the generating function for subwords of even length starting at an even position. Since $A^M(z)$ is a transition matrix, the matrix multiplication in (14) results in subwords being lined up properly. The leading $[1 \ \ 0]$ restricts the sum to words that start at an odd position, *i.e.* 1, whereas the trailing vector combines words of all lengths. The remainder of the proof remains unchanged. □

COROLLARY 3.8 *Define*

$$\cos_q(z) = \sum_{n\geq 0} \frac{(-1)^n z^{2n}}{(q;q)_{2n}},$$
$$\sin_q(z) = \sum_{n\geq 0} \frac{(-1)^n z^{2n+1}}{(q;q)_{2n+1}}.$$

Then,

$$\sum_{n\geq 0}\sum_{\sigma\in S_n}\frac{x^{\mathrm{altdes}(\sigma)}u^{\mathrm{altmaj}(\sigma)}q^{\mathrm{inv}(\sigma)}z^n}{(x;u)_{n+1}(q;q)_n}=\sum_{k\geq 0}x^k\begin{bmatrix}1 & 0\end{bmatrix}A^M(zu^k)A^M(zu^{k-1})\cdots A^M(z)\begin{bmatrix}1\\1\end{bmatrix},$$

where

$$A^M(z)=\begin{bmatrix}\frac{1}{\cos_q(z)} & \frac{\sin_q(z)}{\cos_q(z)}\\ \frac{\sin_q(z)}{\cos_q(z)} & \frac{(\cos_q(z))^2+(\sin_q(z))^2}{\cos_q(z)}\end{bmatrix}.$$

Proof. Apply Theorem 3.7 to the case $X=N$, $A=\{w_1w_2 : w_1\leq w_2\}$, and then use (2). □

We note that a similar method could have been used to compute (10).

3.5 Compositions, with Number of Even-to-Odd and Odd-to-Even Transitions

Theorem 2.1 has enough generality to keep track of extra statistics that sum over subword boundaries, such as the number of occurrences of a particular letter. However, we may use the approach of the previous section to track some more interesting statistics, such as the number of transitions between subsets of the alphabet X. While we could state a very general result, we will content ourselves with a simple example.

THEOREM 3.9 *For $w\in N^n$, let* $\mathrm{oe}(w)$ *be the number of i such that w_i is odd and w_{i+1} is even, and let* $\mathrm{eo}(w)$ *be the number of i such that w_i is even and w_{i+1} is odd. Let $\left[{n \atop k}\right]_q=\frac{(q;q)_n}{(q;q)_{n-k}(q;q)_k}$, and define*

$$A^{ee}(z)=\sum_{i\geq 0}r^is^i\sum_{n\geq 1}\frac{q^{\binom{2i+1}{2}}\left[{n-1 \atop 2i}\right]_q z^n}{(q^2;q^2)_n},$$
$$A^{eo}(z)=\sum_{i\geq 0}r^{i+1}s^i\sum_{n\geq 1}\frac{q^{\binom{2i+2}{2}}\left[{n-1 \atop 2i+1}\right]_q z^n}{(q^2;q^2)_n},$$
$$A^{oe}(z)=\frac{s}{r}A^{eo}(zq),\ \ and$$
$$A^{oo}(z)=A^{ee}(zq),$$

and define the matrices

$$A^M(z)=\begin{bmatrix}1+A^{ee}(z)+rA^{oe}(z) & A^{eo}(z)+rA^{oo}(z)\\ sA^{ee}(z)+A^{oe}(z) & 1+sA^{eo}(z)+A^{oo}(z)\end{bmatrix},$$
$$A^{M0}(z)=\begin{bmatrix}A^{ee}(z) & A^{eo}(z)\\ A^{oe}(z) & A^{oo}(z)\end{bmatrix}.$$

For $w\in N^n$, define the weight *of w,* $\mathrm{wt}(w)$, *to be $r^{\mathrm{eo}(w)}s^{\mathrm{oe}(w)}q^{\mathrm{sum}(w)}z^n$. Then,*

$$\sum_{w\in N^*}x^{\mathrm{des}(w)}u^{\mathrm{maj}(w)}\,\mathrm{wt}(w)=1+\sum_{k\geq 0}\frac{x^k}{1-x}[1\ \ 1]A^{M0}(zu^k)A^M(zu^{k-1})\cdots A^M(z)\begin{bmatrix}1\\1\end{bmatrix}. \tag{15}$$

Proof. Similar to the proof of Theorem 3.7, we will generate w with a Markov process with two states. One state assumes the previous subword ended with an even letter, and the other state assumes the previous subword ended with an odd letter. For our transition matrix, we will need generating functions for the subwords that begin and end with all combinations of even and odd numbers. Moreover, these generating functions will need to track $eo(w^{(i)})$ and $oe(w^{(i)})$. Since subwords have no descents, we will enumerate partitions that switch between even and odd parts a fixed number of times. Since every partition whose first part is odd can be obtained by adding 1 to every letter of a partition whose first part is even, we can consider only those partitions starting with an even part.

Consider $\mu \in \Lambda_n$ that starts with at least one 0, followed by at least one 1, etc., up to at least one i. Let λ be a partition of length n with all even parts. By varying μ and λ and letting $\nu_j = \mu_j + \lambda_j$, we can form every partition $\nu \in \Lambda_n$ starting with an even part and switching between even and odd parts i times. Thus, we will count the number of such μ and λ. Removing one 0, one 1, etc. from μ (thus removing a total of $\binom{i+1}{2}$), we obtain an arbitrary partition with $n-i-1$ parts of size up to i. It is well-known that the generating function for these by sum is $\begin{bmatrix} n-1 \\ i \end{bmatrix}_q$. Partitions with even parts can be obtained by doubling each part in an ordinary partition, so the generating function for these is $\frac{1}{(q^2;q^2)_n}$. Putting this analysis together and summing over positive n, we therefore obtain the functions $A^{ee}(z), A^{eo}(z), A^{oe}(z), A^{oo}(z)$ given above.

The matrix $A^M(z)$ is the transition matrix for our Markov process. The top-left entry is the generating function for subwords ending with an even letter, assuming the previous letter was even. The top-right entry is the generating function for subwords ending with an odd letter, assuming the previous letter was even. The bottom-left entry is the generating function for subwords ending with an even letter, assuming the previous letter was odd, and the bottom-right entry is the generating function for subwords ending with an odd letter, assuming the previous letter was odd. Finally, the matrix $A^{M0}(z)$ ensures that the subword $w^{(0)}$ is non-empty, so we can enter the correct state in our Markov process. We adjust for this by adding the factor of $\frac{1}{1-x}$, which effectively adds an arbitrary number of empty subwords at the beginning. Since the matrices $A^M(z)$ and $A^{M0}(z)$ are transition matrices, the matrix multiplication in (15) results in subwords being lined up properly with the correct count of even-to-odd and odd-to-even transitions. □

3.6 Directed Column-Convex Polyominoes

A column-convex polyomino (CCP) is constructed by successively gluing a finite sequence of columns, each consisting of a finite number of unit square cells, together in the xy-plane so that (i) the lower left vertex of the leftmost column has coordinates (0,0), and (ii) each pair of adjacent columns share an edge of positive integer length. Figure 3 gives a diagram of a CCP. The enumeration of CCPs and of subclasses of CCPs by various statistics has been widely studied. Polyomino enumeration has been surveyed by Delest [4], Guttmann [12], Rensburg [20], and Viennot [22].

Let CCP be the set of column-convex polyominoes, which we will regard as a set of words whose letters are columns. In particular, Q_k will represent the kth column of Q. The area of $Q \in CCP$ is denoted by area(Q). We also say that an upper ascent (upper level, upper descent) occurs at index k if the top cell in Q_k is lower than (respectively level with, higher than) the top cell in Q_{k+1}. Lower ascents, lower levels, and lower descents are similarly defined along the lower ridge. For example, the CCP in Figure 3 has lower descents at positions $1, 2, 5$, and 7. A directed column-convex polyomino

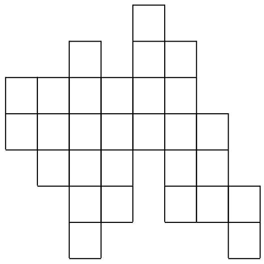
Q =


Figure 3: A column-convex polyomino

(DCCP) is a CCP with no lower descents, and we will denote the set of DCCPs by $DCCP$.

Consider the alphabet $X = \{(j, m) : j, m \text{ are positive integers}\}$, and let

$$\mathcal{Y} = \bigcup_{n\geq 0} \Big\{ (j_1 j_2 \cdots j_n, m_1 m_2 \cdots m_n) \in X^n : m_n = 1 \text{ and } j_k + j_{k+1} > m_k \text{ for } 1 \leq k < n \Big\}.$$

Rawlings and Tiefenbruck [17] constructed a bijection $\delta : CCP \to \mathcal{Y}$ and used it to study consecutive patterns in CCPs and DCCPs. For a column-convex polyomino Q with n columns, they define

$$\delta(Q) = (j_1 j_2 \cdots j_n, m_1 m_2 \cdots m_n), \tag{16}$$

where j_k is the number of cells in Q_k, $m_n = 1$, and for $1 \leq k < n$, m_k is the change in the y-ordinate from the bottom edge of Q_{k+1} to the top edge of Q_k. For Q in Figure 3, $\delta(Q) = (23644532, 35526541)$. Using δ, we can derive a version of (1) for DCCPs.

THEOREM 3.10 *For* $Q \in DCCP$*, let* $\operatorname{des}(Q)$ *be the number of upper descents in* Q *and* $\operatorname{maj}(Q)$ *be the sum of their column numbers. Then,*

$$\sum_{Q\in DCCP} x^{\operatorname{des}(Q)} u^{\operatorname{maj}(Q)} q^{\operatorname{area}(Q)} z^{\ell(Q)} = 1 + \sum_{k\geq 0} \sum_{j=0}^{k} x^k A(zu^k) A(zu^{k-1}) \cdots A(zu^{j+1}) A_n(zu^j),$$

where

$$A(z) = \left(\sum_{k\geq 0} \frac{(-z)^k q^{\binom{k+1}{2}}}{(q;q)_k^2} \right)^{-1},$$

$$A_n(z) = z \left(\sum_{k\geq 0} \frac{(-z)^k q^{\binom{k+2}{2}}}{(q;q)_{k+1}(q;q)_k} \right) \left(\sum_{k\geq 0} \frac{(-z)^k q^{\binom{k+1}{2}}}{(q;q)_k^2} \right)^{-1}.$$

Proof. Using δ, it is clear that DCCPs correspond to words on the alphabet $Y = \{(j, m) \in X : j \geq m\}$ such that $m_n = 1$. Therefore, following the proof of Theorem 2.1, we need generating functions for words on Y with and without the restriction that $m_n = 1$. Rawlings and Tiefenbruck computed them, obtaining $A(z)$ and $A_n(z)$ above. The inclusion of the sum over j handles the condition that the final non-empty subword must have $m_n = 1$. □

We will note two other methods by which we could have dealt with the restriction on the final letter. First, we could have written the expression using a product of matrices. Second, we could have modified the proof of Theorem 2.1 as we did in Theorem 3.5. That is, when $\mu(w)$ is used to factor w into subwords, w_n always appears in the final subword. Therefore, we could have restricted λ to $\Lambda^{\leq n-1}$. In that case, $w^{(k)}$ would always be non-empty, allowing us to modify only the final term in the product.

4 Open Problems

In the course of our study, there are a few problems of interest that we haven't been able to solve. We state them here as encouragement to other researchers.

Problem 1. A peak in a permutation σ is a position i such that $\sigma_i < \sigma_{i+1}$ and $\sigma_{i+1} > \sigma_{i+2}$. Let $\text{peak}(\sigma)$ be the number of peaks in σ and $\text{maj}_{\text{peak}}(\sigma)$ be the sum of their positions. Find a generating function for permutations by peak and maj_{peak}.

Problem 2. A 231-pattern in a permutation σ is a sequence of indices $i < j < k$ such that $\sigma_k < \sigma_i < \sigma_j$. Find a generating function for des and maj in the subset of permutations with no 231-patterns.

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