Some properties of Fröhlich's composition rings

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Abstract. We analyze the Fröhlich's composition rings 1968, and we identify the units of the related nearrings. By introduction of an infinite set of indeterminates, these structures are generalized, and particular substructure are singled out, which are local nearrings.

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1 Introduction

A (right) nearring is a structure $[M;+,\cdot]$ with two operations, addition and multiplication, defined onto M, such that

(i) $[M;+]$ is a group;

(ii) $[M;\cdot]$ is a semigroup;

(iii) $\forall x, y, z \in M$ $(x+y)\cdot z = x\cdot z + y\cdot z$.

A nearring M is called *zero-symmetric*, if $x \cdot 0 = 0$ for any $x \in M$. A zero-symmetric nearring M with (multiplicative) identity is said to be *local* ([5]) if the set $L(M)$ of elements of M without left inverses is a subgroup of $[M;+]$ such that $M \cdot L(M) \subseteq L(M)$. A composition ring ([2], page 329) is defined as a structure $[M;+,\cdot,\circ]$ such that $[M;+,\cdot]$ is a ring, $[M;+,\circ]$ is a nearring, and $\circ$ is right distributive with respect to $\cdot$. We consider the following structure, whose origin lies in [3]. Let G be the collection of formal power series with n indeterminates $X_1, X_2, \ldots, X_n$, over a commutative unitary ring R, having zero constant term, that is $G = R[[X_1, X_2, \ldots, X_n]]_0$. We put $N = G^n$. The structure $[N;+,\cdot,\circ]$ is a composition ring, where $+,\cdot$ are performed componentwise, while $\circ$ is given in the following manner. If $f,g \in N$, $f=(f_1,f_2,\ldots,f_n)$, $g=(g_1,g_2,\ldots,g_n)$, then $f\circ g$ is the element of N defined as follows (if $I_n=\{1,2,\ldots,n\}$)

$$\forall i \in I_n \quad (f\circ g)_i = f_i(g_1,g_2,\ldots,g_n).$$

In this article we determine the units of the nearring $[N;+,\circ]$. In Section 3, we introduce a generalization of the considered composition ring, by using an infinite set of indeterminates. For these last structures too, we characterize the units, and we identify a particular substructure also, which is a local nearring. As consequences of the above, some coupling maps ([7]), Dickson nearfields and other local nearrings are described in Section 4. For further results on local nearrings, besides the fundamental work [5], see the papers [1], [6], [8], [9] as well.

2 Units of the nearring $[N;+,\circ]$

The symbol u will be used to designate the identity of $[N;+,\circ]$, that is $u=(X_1,X_2,\ldots,X_n)$. If ξ is an arbitrary element of G, $m\in\mathbb{N}^\star$, we denote by $\xi^{[m]}$ the homogeneous component of degree m of ξ, and we put $\xi^{\{m\}}=\xi^{[1]}+\xi^{[2]}+\ldots+\xi^{[m]}$. For $f,g\in N$, in the sequel we suppose $f=(f_1,f_2,\ldots,f_n)$, $g=(g_1,g_2,\ldots,g_n)$; furthermore, if $i\in I_n$, $m\in\mathbb{N}$, let

$$f_i^{[m]} = \sum_{\substack{(k_1,k_2,\ldots,k_n)\in\mathbb{N}_0^n \\ k_1+k_2+\ldots+k_n=m}} a^{(i)}_{k_1k_2\ldots k_n} X_1^{k_1}X_2^{k_2}\ldots X_n^{k_n}$$

$$g_i^{[m]} = \sum_{\substack{(k_1,k_2,\ldots,k_n)\in\mathbb{N}_0^n \\ k_1+k_2+\ldots+k_n=m}} b^{(i)}_{k_1k_2\ldots k_n} X_1^{k_1}X_2^{k_2}\ldots X_n^{k_n}$$

Therefore:

$$\begin{aligned} f_1^{[1]} &= a^{(1)}_{10\ldots0}X_1 + a^{(1)}_{010\ldots0}X_2 + \ldots + a^{(1)}_{0\ldots01}X_n \\ f_2^{[1]} &= a^{(2)}_{10\ldots0}X_1 + a^{(2)}_{010\ldots0}X_2 + \ldots + a^{(2)}_{0\ldots01}X_n \\ &\cdots\cdots\cdots \\ f_n^{[1]} &= a^{(n)}_{10\ldots0}X_1 + a^{(n)}_{010\ldots0}X_2 + \ldots + a^{(n)}_{0\ldots01}X_n \end{aligned}$$

We define

1) $V(f)=(f_1^{[1]},f_2^{[1]},\ldots,f_n^{[1]})$, $V(g)=(g_1^{[1]},g_2^{[1]},\ldots,g_n^{[1]})$;

2) $M(f)$ as the following $n\times n$ matrix with entries in R

$$\begin{pmatrix} a^{(1)}_{10\ldots0} & a^{(1)}_{010\ldots0} & \cdots & a^{(1)}_{00\ldots01} \\ a^{(2)}_{10\ldots0} & a^{(2)}_{010\ldots0} & \cdots & a^{(2)}_{00\ldots01} \\ \cdots & \cdots & \cdots & \cdots \\ a^{(n)}_{10\ldots0} & a^{(n)}_{010\ldots0} & \cdots & a^{(n)}_{00\ldots01} \end{pmatrix}$$

*For us, $\mathbb{N}_0$ is the set of all natural numbers, while $\mathbb{N}$ is the set of natural numbers, 0 excepted.

THEOREM 2.1 *For $f \in N$, with respect to $\circ$ the following assertions are equivalent:*

1) f is right-invertible;

2) f is left-invertible;

3) f is unit;

4) $\det(M(f))$ is unit in R.

Proof. Let f be right-invertible. Then, there exists some $g \in N$ such that $f \circ g = u$. Hence, $V(f) \circ V(g) = u$, and consequently $M(f) \cdot M(g) = J_n$, where J_n is the identity $n \times n$ matrix with entries in R. By explanations contained in [4] (see proof of Lemma 2.1, page 104) the last equation is equivalent to the condition $\det(M(f))$ unit in R. Similarly, f left-invertible implies $\det(M(f))$ unit in R.

Conversely, suppose that condition 4) of the statement holds. We intend to determine some $g \in N$ such that

$$f \circ g = u \tag{1}$$

Here, the coefficients of g are the unknowns of the problem. From (1), we must have $V(f) \circ V(g) = u$, hence $M(f) \cdot M(g) = J_n$; by 4), the coefficients of $V(g)$ are uniquely identified, if we bear in mind that we now have $M(g) = M(f)^{-1}$. We proceed by induction. Let $m \in \mathbb{N}$, $m \geq 2$. We suppose that $\forall i \in I_n$ $g_i^{\{m-1\}}$ is uniquely determined. For every $i \in I_n$, we consider $g_i^{[m]}$. The following equality holds

$$(f \circ g)_i^{[m]} = f_i^{[1]}(g_1^{[m]}, g_2^{[m]}, \ldots, g_n^{[m]}) + H_i \tag{2}$$

where H_i is a sum of terms each of which appears in the development of

$$f_i^{\{m-1\}}(g_1^{\{m-1\}}, g_2^{\{m-1\}}, \ldots, g_n^{\{m-1\}}).$$

By inductive hypothesis, the set $D_i \subseteq R$ of the coefficients in H_i is known. Let $(t_1, t_2, \ldots, t_n)$ be an element of $\mathbb{N}_0^n$ such that $t_1 + t_2 + \ldots + t_n = m$. We try to determine $b^{(j)}_{t_1 t_2 \ldots t_n}$, for any $1 \leq j \leq n$. To this purpose, we consider the term in $X_1^{t_1} X_2^{t_2} \ldots X_n^{t_n}$ in (2). Since, because of (1), the (2) must be null, we obtain to the following linear system in the unknowns $b^{(j)}_{t_1 t_2 \ldots t_n}$, $j \in I_n$

$$\begin{aligned}
a^{(1)}_{10\ldots0} b^{(1)}_{t_1 t_2 \ldots t_n} + a^{(1)}_{010\ldots0} b^{(2)}_{t_1 t_2 \ldots t_n} + \ldots + a^{(1)}_{00\ldots01} b^{(n)}_{t_1 t_2 \ldots t_n} + h_1 = 0 \\
a^{(2)}_{10\ldots0} b^{(1)}_{t_1 t_2 \ldots t_n} + a^{(2)}_{010\ldots0} b^{(2)}_{t_1 t_2 \ldots t_n} + \ldots + a^{(2)}_{00\ldots01} b^{(n)}_{t_1 t_2 \ldots t_n} + h_2 = 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
a^{(n)}_{10\ldots0} b^{(1)}_{t_1 t_2 \ldots t_n} + a^{(n)}_{010\ldots0} b^{(2)}_{t_1 t_2 \ldots t_n} + \ldots + a^{(n)}_{00\ldots01} b^{(n)}_{t_1 t_2 \ldots t_n} + h_n = 0
\end{aligned}$$

where $h_i \in D_i$ for every i. Since $\det(M(f))$ is unit in R, this system has one and only one solution. Therefore, $g_i^{[m]}$ is uniquely determined, for arbitrary $i \in I_n$. From the induction principle, the element $g \in N$ such that $f \circ g = u$, is then uniquely determined. Since $\det(M(g))$ is unit in R, from what precedes there exists one and only one $f_1 \in N$ such that $g \circ f_1 = u$; therefore $f \circ g \circ f_1 = u \circ f_1 = f_1$, which implies $f \circ u = f_1$, that is $f = f_1$, so $f \circ g = g \circ f = u$, and f is unit in $[N; +, \circ]$. If an element $f \in N$ is left-invertible, since f must then satisfy condition 4), f is also right-invertible, namely f is unit in the nearring $[N; +, \circ]$. The corresponding result holds if f is right invertible. $\square$

3 Some local nearrings

We consider the ring $F = R[[S]]_0$ of formal power series, with constant term null, in the infinite countable set of indeterminates $S = \{X_1, X_2, \ldots, X_n, \ldots\}$. Here, if $\xi \in F$, then $\xi = \xi^{[1]} + \xi^{[2]} + \ldots$, where, in any homogeneous component $\xi^{[j]}$, only a finite number of elements of S effectively appears. Let $(n_1, n_2, \ldots, n_s, \ldots)$ be a strictly increasing infinite sequence of elements of $\mathbb{N}$. For all $i \in \mathbb{N}$, define $G_i = R[[X_1, X_2, \ldots, X_{n_i}]]_0$.
Put $N' = \{f = (f_1, f_2, \ldots, f_n, \ldots) \in F^{\mathbb{N}} \mid \forall i \in \mathbb{N}\, (f_1, f_2, \ldots, f_{n_i}) \in G_i^{n_i}\}$. We introduce two operations $+$ and $\cdot$, addition and multiplication, onto N', componentwise. Moreover, let f, g be two generic elements of N', and we suppose, here and in what follows, $f = (f_1, f_2, \ldots, f_n, \ldots)$, $g = (g_1, g_2, \ldots, g_n, \ldots)$. Then, let $f \circ g$ be the element of $F^{\mathbb{N}}$ such that $\forall i \in \mathbb{N}$ $(f \circ g)_i = f_i(g_1, g_2, \ldots, g_n, \ldots)$.

Lemma 3.1 *For $f, g \in N'$, we have $f \circ g \in N'$.*

Proof. Let j be an arbitrary element of $\mathbb{N}$. Then $(f_1, f_2, \ldots, f_{n_j}) \in G_j^{n_j}$. Hence, if $i \leq n_j$, no X_k appears in f_i for $k \geq n_j + 1$, which implies $(f \circ g)_i = f_i(g_1, g_2, \ldots, g_{n_j})$; consequently, $((f \circ g)_1, (f \circ g)_2, \ldots, (f \circ g)_{n_j}) \in G_j^{n_j}$, so $f \circ g \in N'$. □

From Lemma 3.1, we immediately have

Theorem 3.2 *The structure $[N'; +, \circ]$ is a composition ring.*

We remark that N' (with respect to $\circ$) has identity $u' = (X_1, X_2, \ldots, X_n, \ldots)$. If we recall Theorem 2.1, we observe that the following theorem also holds.

Theorem 3.3 *For $f \in N'$ (with respect to $\circ$) the following assertions are equivalent:*

1) f is right-invertible;

2) f is left-invertible;

3) f is unit;

4) for all $i \in \mathbb{N}$, $\det(M((f_1, f_2, \ldots, f_{n_i})))$ is unit in R.

Let now N'' be the set of $f \in N'$ such that, for every $i \in \mathbb{N}$, $f_i^{[1]} = aX_i$, where $a \in R$.

Theorem 3.4 *The structure $[N''; +, \cdot, \circ]$ is a sub-composition ring of $[N'; +, \cdot, \circ]$. If R is a local ring (in particular, if R is a field), then $[N''; +, \circ]$ is a local nearring.*

Proof. The proof of the first assertion is trivial.
Let R be a local ring, and let $L(R)$ be the ideal of R of non-units in R. We prove that $[N''; +, \circ]$ is a local nearring. To this purpose, we first show that the subset $L(N'')$ of N'' of elements without left inverses (with respect to $\circ$) coincides with the set $\{f \in N'' | \forall i \in \mathbb{N} \quad f_i^{[1]}$ has coefficients in $L(R)\}$. Indeed, if we call T this last set, then we have

a) $N'' \circ T \subseteq T$;

b) T is subgroup of $[N''; +]$;

c) T consists of elements without left inverses.

Part c) is valid since, if $f \in T$, and $\forall i \in \mathbb{N}$ $f_i^{[1]} = cX_i$ (for c fixed in $L(R)$), then for every $j \in \mathbb{N}$ $\det(M((f_1, f_2, \dots, f_{n_j})))$ is c^{n_j}, which does not have multiplicative inverse in R, so by Theorem 3.3 f is not left-invertible in the nearring $[N''; +, \circ]$. Let now g be an element belonging to $N'' \setminus T$. Then there exists some $a \in R \setminus L(R)$ such that for all $i \in \mathbb{N}$ $g_i^{[1]} = aX_i$. For an arbitrary $j \in \mathbb{N}$, $\det(M((g_1, g_2, \dots, g_{n_j})))$ equals a^{n_j}, which is unit in R. By Theorem 3.3, g is left-invertible, which completes the proof. □

4 Some consequences

Throughout this section, the symbol B is used to denote the ring $R[[X_1, X_2, \dots, X_n]]$. Let f be an (arbitrarily) fixed element of N, $f = (f_1, f_2, \dots, f_n)$. If $\xi \in B$, $\xi \neq \underline{0}$, we will write $\sigma(\xi)$ for the minimum of the degrees of non null homogeneous components in ξ, and we put $\sigma(\underline{0}) = +\infty$.

LEMMA 4.1 *For any $\xi \in B$, we have $\xi(f_1, f_2, \dots, f_n) \in B$.*

Proof. We can uniquely write ξ as $\xi = \xi_0 + \xi_1$ where $\xi_0 \in R$, $\xi_1 \in G$, while $\xi(f_1, f_2, \dots, f_n) = \xi_0 + \xi_1(f_1, f_2, \dots, f_n)$, in which $\xi_1(f_1, f_2, \dots, f_n) \in G$. □

From Lemma 4.1, the function

$$\begin{array}{llll} T: & B & \to & B \\ T: & \xi & \to & \xi(f_1, f_2, \dots, f_n) \end{array}$$

is an endomorphism of the ring $[B; +, \cdot]$. Clearly, the following obvious lemma also holds.

LEMMA 4.2 *For any $\xi \in B$, we have $\sigma(\xi) \leq \sigma(T(\xi))$.*

From now on, we suppose $\det(M(f))$ to be unit in R.

THEOREM 4.3 *Under the previous hypothesis, T is an automorphism of the ring $[B; +, \cdot]$. Moreover,*

$$\forall \xi \in B \quad \sigma(\xi) = \sigma(T(\xi))$$

Proof. By Theorem 2.1, f is unit in $[N; +, \circ]$. Let $\overline{f} = (\overline{f}_1, \overline{f}_2, \dots, \overline{f}_n) \in N$ be the inverse of f with respect to $\circ$. Then (with notations as above) we can say that

$$\begin{aligned} \forall \xi = \xi_0 + \xi_1 \in B \quad T(\xi(\overline{f}_1, \overline{f}_2, \dots, \overline{f}_n)) &= (\xi(\overline{f}_1, \overline{f}_2, \dots, \overline{f}_n))(f_1, f_2, \dots, f_n) = \\ &(\xi_0 + \xi_1(\overline{f}_1, \overline{f}_2, \dots, \overline{f}_n))(f_1, f_2, \dots, f_n) = \\ &\xi_0 + \xi_1((\overline{f}_1, \overline{f}_2, \dots, \overline{f}_n) \circ (f_1, f_2, \dots, f_n)) = \\ &\xi_0 + \xi_1(X_1, X_2, \dots, X_n) = \xi_0 + \xi_1 = \xi \end{aligned}$$

so T is surjective. Furthermore, if $\xi \in \operatorname{Ker} T$, then $\underline{0} = T(\xi) = \xi(f_1, f_2, \dots, f_n)$. As consequence, by similar steps to the former, we determine

$$\underline{0} = (\xi(f_1, f_2, \dots, f_n))(\overline{f}_1, \overline{f}_2, \dots, \overline{f}_n) = \xi,$$

which gives $\operatorname{Ker} T = \{\underline{0}\}$, and T is an automorphism.

Given $\xi \in B \setminus \{\underline{0}\}$, from Lemma 4.2 $\sigma(\xi) \leq \sigma(T(\xi))$. We assert that $\sigma(\xi) = \sigma(T(\xi))$. Assume, on the contrary, $\sigma(\xi) < \sigma(T(\xi))$. Then, if we apply once again Lemma 4.2, we can write $\sigma(\xi) < \sigma(T(\xi)) = \sigma(\xi(f_1, f_2, \dots, f_n)) \leq \sigma((\xi(f_1, f_2, \dots, f_n))(\overline{f}_1, \overline{f}_2, \dots, \overline{f}_n)) = \sigma(\xi)$, which is impossible. □

COROLLARY 4.4 *Let R be an unitary integral domain. Then the following function $\varphi : B \to \mathrm{End}[B; +, \cdot]$ $(\xi \to \varphi_\xi)$ is a coupling map for the domain $[B; +, \cdot]$, where φ is defined by:*
$\varphi_{\underline{0}}$ equal to the zero-map 0_B, while $\forall \xi \neq \underline{0}$ $\varphi_\xi = T^{\sigma(\xi)}$.

Proof. It is necessary to demonstrate that*, for each $\xi, \eta \in B$

$$\varphi_\xi \circ \varphi_\eta = \varphi_{\varphi_\eta(\xi) \cdot \eta} \tag{3}$$

If $\xi = \underline{0}$ or $\eta = \underline{0}$, the (3) is obvious.
For $\xi \neq \underline{0} \neq \eta$ we have

$$\varphi_\xi \circ \varphi_\eta = T^{\sigma(\xi)} \circ T^{\sigma(\eta)} = T^{\sigma(\xi)+\sigma(\eta)}$$

while, by Theorem 4.3, $\sigma(T^{\sigma(\eta)}(\xi)) = \sigma(\xi)$; hence

$$\varphi_{\varphi_\eta(\xi)\cdot\eta} = \varphi_{T^{\sigma(\eta)}(\xi)\cdot\eta} = T^{\sigma(T^{\sigma(\eta)}(\xi)\cdot\eta)} = T^{\sigma(T^{\sigma(\eta)}(\xi))+\sigma(\eta)} = T^{\sigma(\xi)+\sigma(\eta)}$$

thus condition (3) is proved in all cases. □

If R is a local ring, it is well known that $[B; +, \cdot]$ is a local ring, in which the ideal of non units is $L(R) + G$. From this consideration and Theorem 2.3 of [5], we obtain

COROLLARY 4.5 *If R is a local integral domain, then the nearring $[B; +, \circ]$, coupled to the ring $[B; +, \cdot]$ by the φ of the Corollary 4.4, is a local nearring (where $L(R) + G$ is the set of non units).*

Proof. We recall that, in the nearring $[B; +, \circ]$, we have $\xi \circ \eta = \varphi_\eta(\xi) \cdot \eta$ ([7], page 154). Let $\eta \in B$, $\eta = \eta^{[0]} + \eta^{[1]} + \ldots + \eta^{[m]} + \ldots$, where, as usual, $\eta^{[i]}$ is the homogeneous component of degree i of η. We take $\eta \notin L(R) + G$; in this case, $\eta^{[0]} \in R \setminus L(R)$ and φ_η is the identity map on B. We look for a $\xi = \xi^{[0]} + \xi^{[1]} + \ldots$ in B, such that

$$\xi \circ \eta = 1 \tag{4}$$

Here, we have $\xi \circ \eta = \xi \cdot \eta$, and $\xi \circ \eta = \xi^{[0]} \cdot \eta^{[0]} + (\xi^{[1]} \cdot \eta^{[0]} + \xi^{[0]} \cdot \eta^{[1]}) + \ldots$
Condition (4) requires that $\xi^{[0]} = \eta^{[0]^{-1}}$ is the unique inverse of $\eta^{[0]}$ in R. We proceed by induction, and we suppose that $\xi^{[0]}, \xi^{[1]}, \ldots, \xi^{[m-1]}$ are uniquely determined, for an $m \geq 0$. Condition (4) implies that

$$\xi^{[0]} \cdot \eta^{[m]} + \xi^{[1]} \cdot \eta^{[m-1]} + \ldots + \xi^{[m]} \cdot \eta^{[0]} = 0 \tag{5}$$

Since $\eta^{[0]}$ is invertible in R, $\xi^{[m]}$ is uniquely determined by condition (5). Therefore, η has a left inverse with respect to $\circ$.

On the other hand, for $\eta \in L(R) + G$, suppose that there exists a $\xi = \xi^{[0]} + \xi^{[1]} + \ldots$ in B such that $\xi \circ \eta = 1$. Then $\xi^{[0]} \cdot \eta^{[0]} = 1$, from which $\eta^{[0]} \notin L(R)$ which is a contradiction. At this point, it is enough to use Theorem 2.3 of [5] and our assertion is proved. □

*See [7], page 155.

Corollary 4.6 *Let R be an unitary integral domain, and let $Q(B)$ be the field of fractions of the integral domain B. Then, the function*

$$\begin{array}{llcl} U: & Q(B) & \rightarrow & Q(B) \\ U: & \frac{\xi}{\eta} & \rightarrow & \frac{T(\xi)}{T(\eta)} \end{array}$$

is an automorphism of $Q(B)$.
Furthermore, the following function $\psi : Q(B) \rightarrow \mathrm{End}[Q(B); +, \cdot]$ $(\alpha \rightarrow \psi_\alpha)$ is a coupling map for the field $Q(B)$ (with coupled nearring $[Q(B); +, \circ]$ nearfield), where ψ is defined by:
$\psi_{\underline{0}}$ equal to the zero-map $0_{Q(B)}$, while $\psi_\alpha = U^{\sigma(\xi)-\sigma(\eta)}$ if $\alpha \neq \underline{0}$, $\alpha = \frac{\xi}{\eta}$ for $\xi, \eta \in B \setminus \{\underline{0}\}$.

Proof. We have that U is an automorphism by Theorem 4.3. Let $\alpha = \frac{\xi}{\eta}$, $\beta = \frac{\omega}{\lambda}$ be non null fractions belonging to $Q(B)$. The definition of ψ gives

$$\psi_\alpha \circ \psi_\beta = U^{\sigma(\xi)-\sigma(\eta)} \circ U^{\sigma(\omega)-\sigma(\lambda)} = U^{\sigma(\xi)+\sigma(\omega)-\sigma(\eta)-\sigma(\lambda)}$$

Suppose $U^{\sigma(\omega)-\sigma(\lambda)}\left(\frac{\xi}{\eta}\right) = \frac{\xi'}{\eta'}$ (with $T^{\sigma(\omega)-\sigma(\lambda)}(\xi) = \xi'$, $T^{\sigma(\omega)-\sigma(\lambda)}(\eta) = \eta'$).
By Theorem 4.3 $\sigma(\xi) = \sigma(\xi')$ and $\sigma(\eta) = \sigma(\eta')$, so we can write

$$\psi_{\psi_\beta(\alpha)\cdot\beta} = \psi_{U^{\sigma(\omega)-\sigma(\lambda)}\left(\frac{\xi}{\eta}\right)\cdot\frac{\omega}{\lambda}} = \psi_{\frac{\xi'}{\eta'}\cdot\frac{\omega}{\lambda}} = \psi_{\frac{\xi'\omega}{\eta'\lambda}} = U^{\sigma(\xi'\omega)-\sigma(\eta'\lambda)} =$$

$$= U^{\sigma(\xi')+\sigma(\omega)-\sigma(\eta')-\sigma(\lambda)} = U^{\sigma(\xi)+\sigma(\omega)-\sigma(\eta)-\sigma(\lambda)}$$

Therefore $\psi_\alpha \circ \psi_\beta = \psi_{\psi_\beta(\alpha)\cdot\beta}$, and so we can say that ψ is a coupling map for the field $Q(B)$. □

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