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Commuting pairs of functions on a finite set

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Abstract. In this paper, we consider the problem of counting the number of ordered pairs (f, g) of commuting functions from $[n] = \{1, 2, \ldots, n\}$ to itself. We begin by considering the problem when the first function f is a permutation and derive an explicit formula in that case. As a consequence of our arguments, similar formulas may also be given when f belongs to one of several subsets of $\mathcal{S}_n$. An enumeration scheme and formula is developed for the problem of counting the ordered pairs of commuting functions with no restrictions. It relies on a classification of the functions from $[n]$ to $[n]$ based on how much a function differs from a permutation in some sense. Finally, it is shown that the number of ordered pairs of commuting functions from $[n]$ to itself is always divisible by n.

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1 Introduction

The problem of when the composition of two functions is commutative is a general one that has been addressed in various settings. For example, in analysis, one such question concerns the commutation of continuous functions from a closed interval to itself and determining various properties of these functions such as common fixed points (see [2]). In linear algebra, the problem concerns the commutativity of square matrices of the same size and one well-known result in this direction is that two real symmetric matrices commute if and only if they are simultaneously diagonalizable (see, e.g., [5, p. 356]). Here we consider the property of commutativity in a discrete setting and investigate various aspects concerning the enumeration of pairs of commuting functions defined on a finite set.

Our results are also related to recent ones concerning the commutativity of pairs of elements within finite semigroups (see [7, 1]). A theorem from algebra states that the number of ordered pairs of commuting elements within a finite group G is given by $|G|$ times the number of conjugacy classes of G (see, e.g., [6, p. 398]). When specialized to the symmetric group $\mathcal{S}_n$, this result states that the number of ordered pairs of commuting permutations is given by $n!$ times the number of partitions of

n. See entry A053529 in OEIS [8]. Various questions have been addressed in this direction; see, for example, the paper by Erdős and Straus [4] where they consider maximal commuting k-tuples within groups.

In this paper, we consider the problem of enumerating ordered pairs (f, g) of commuting functions from the set $[n] = \{1, 2, \ldots, n\}$ to itself. As a first step, we consider the problem of enumerating such pairs (f, g) where we require that f be a permutation. In the next section, we provide an explicit formula for the number of pairs in this case. Our arguments also yield formulas for the number of commuting pairs (f, g) where f belongs to one of several subsets of $\mathcal{S}_n$ and g is an arbitrary function. Using our formula, we can also show that the number of commuting pairs (f, g) on $[n]$ where f is a permutation is always divisible by n.

The problem of determining an explicit formula or generating function for the number of pairs of commuting functions from $[n]$ to itself with no restriction though appears to be more difficult. Note that the sequence counting such pairs occurs as entry A181162 in [8]. Here we provide a way of enumerating these pairs based on a certain partitioning of the self-maps of $[n]$. As a consequence, we obtain a seemingly new combinatorial identity relating n^n to the multinomial coefficients. Using the scheme we have developed, we are able to prove that the number of ordered pairs of commuting functions from $[n]$ to itself is always divisible by n. We can also show that the probability of a randomly chosen pair of elements commuting in the semigroup of self-maps on $[n]$ converges to zero as n increases without bound.

We will make use of the following notational conventions. If m and n are positive integers, then let $[m, n] = \{m, m+1, \ldots, n\}$ if $m \leqslant n$, with $[m, n] = \varnothing$ if $m > n$. Empty sums will assume the value zero and empty products the value one, with $0^0 = 1$. The multinomial coefficient, denoted $\binom{n}{n_1, \ldots, n_r}$, is given by $\frac{n!}{n_1! \cdots n_r!}$ if $n_1, \ldots, n_r$ are non-negative integers summing to n and is zero otherwise. Given a function f and a subset S of its codomain, the set of all elements x in the domain of f such that $f(x) \in S$ will be denoted by $f^{-1}(S)$.

2 Commuting functions and permutations

In this section, we consider the problem of counting the number of ordered pairs of commuting functions from $[n]$ to $[n]$ where the first function is a permutation. We begin with the following lemma, which we'll also need in the subsequent section.

LEMMA 2.1 *Given any function $f : [n] \to [n]$, there exists a non-empty subset S of $[n]$ such that the restriction of f to S is a permutation of S.*

Proof. If $i \geqslant 0$, then let f^i denote the function f composed with itself i times, where f^0 is the identity function. Clearly, we have $range(f^{i+1}) \subseteq range(f^i)$ for all i. By finiteness, there exists some index j such that $range(f^j) = range(f^{j+1})$. Then the restriction of f to $range(f^j)$ is clearly onto $range(f^j)$ and thus a permutation of the set since it is finite, as desired. □

DEFINITION 2.2 We will refer to the restriction of f to $range(f^j)$ in the preceding proof as the permutation base of f and to the set $range(f^j)$ itself as the base set of f.

Remark. Note that the base set of f actually contains all subsets S such that the restriction of f to S is a permutation of S (such subsets being contained within $range(f^i)$ for all i).

DEFINITION 2.3 Given function $f : [n] \to [n]$, by a k-cycle of f, we will mean a sequence $(x_1, x_2, \ldots, x_k)$ of distinct entries in $[n]$ such that $x_r = f(x_{r-1})$ for $2 \leqslant r \leqslant k$, with $x_1 = f(x_k)$.

LEMMA 2.4 *Suppose f, g are functions from $[n]$ to $[n]$. If f and g commute, then for any k-cycle $(x_1, x_2, \ldots, x_k)$ of f, the sequence $(g(x_1), g(x_2), \ldots, g(x_\ell))$ forms an ℓ-cycle of f for some $\ell | k$.*

Proof. Suppose f and g commute. The fact that the images under g of a cycle of f form a cycle follows from $g(x_{i+1}) = g(f(x_i)) = f(g(x_i))$. If the length ℓ of the cycle doesn't divide k, then we have $g(x_k) \neq g(x_\ell)$ and thus $f(g(x_k)) \neq f(g(x_\ell))$ since $g(x_\ell)$ and $g(x_k)$ would be distinct elements belonging to the same cycle of f. But then $g(x_1) = f(g(x_\ell)) \neq f(g(x_k)) = g(f(x_k)) = g(x_1)$, a contradiction. □

Remark. The converse of Lemma 2.4 also holds if f is a permutation.

DEFINITION 2.5 Given a function $f : [n] \to [n]$, we will refer to the property possessed by a commuting function g of mapping any k-cycle $(x_1, x_2, \ldots, x_k)$ of f to an ℓ-cycle $(g(x_1), g(x_2), \ldots, g(x_\ell))$ of f for some $\ell | k$ as the cycle mapping property.

Lemma 2.4 shows that the cycle structure of a permutation base of a function limits which functions can commute with it. Given a permutation $\sigma \in \mathcal{S}_n$, we shall denote the cycle structure of σ by $(\lambda_1, \lambda_2, \ldots, \lambda_n)$, where λ_i is the number of cycles of σ of length i. Note that $\lambda_1 + 2\lambda_2 + \cdots + n\lambda_n = n$. We now revisit the problem of counting the number of commuting pairs of permutations of $[n]$ and provide a proof of the well-known formula using the prior lemma.

Example. Recall that the number of commuting pairs of elements of $\mathcal{S}_n$ is $n!$ times the number of partitions of n. To show this, note first that by symmetry the number of permutations of $[n]$ commuting with a given permutation is the same for all permutations possessing a given cycle structure. Let $L = L(\lambda_1, \ldots, \lambda_n)$ denote the number of members of $\mathcal{S}_n$ commuting with a permutation having a given cycle structure $(\lambda_1, \ldots, \lambda_n)$. Then the number of commuting pairs of permutations is given by

$$\sum_{\substack{1\lambda_1+\cdots+n\lambda_n=n \\ \lambda_i \geqslant 0}} \frac{n!}{\lambda_1! \cdots \lambda_n! 1^{\lambda_1} \cdots n^{\lambda_n}} L(\lambda_1, \ldots, \lambda_n),$$

by the well-known fact (see, e.g., [9, Prop. 1.3.2]) that there are $\frac{n!}{\lambda_1! \cdots \lambda_n! 1^{\lambda_1} \cdots n^{\lambda_n}}$ permutations of length n having cycle structure $(\lambda_1, \ldots, \lambda_n)$. To complete the proof, it suffices to show that $L = \lambda_1! \cdots \lambda_n! 1^{\lambda_1} \cdots n^{\lambda_n}$. Suppose $\sigma \in \mathcal{S}_n$ has cycle structure $(\lambda_1, \ldots, \lambda_n)$ and that ρ commutes with σ. By Lemma 2.4 and injectivity, it is seen for each k that ρ permutes the k-cycles of σ. For each k-cycle of σ, there are k ways to map it under ρ to any other k-cycle, whence there are $k^{\lambda_k} \lambda_k!$ ways to map k-cycles. Considering cycles of all lengths implies $L = 1^{\lambda_1} \lambda_1! 2^{\lambda_2} \lambda_2! \cdots n^{\lambda_n} \lambda_n!$, as desired. □

Using a similar argument, we can obtain the following result.

PROPOSITION 2.6 *The number of commuting pairs of functions from $[n]$ to $[n]$ where the first function is a permutation is given by*

$$\sum_{\substack{1\lambda_1+\cdots+n\lambda_n=n \\ \lambda_i \geqslant 0}} \frac{n!}{\lambda_1! \cdots \lambda_n! 1^{\lambda_1} \cdots n^{\lambda_n}} \prod_{k=1}^{n} \left(\sum_{j|k} j\lambda_j \right)^{\lambda_k}.$$

Proof. For a permutation σ of $[n]$ with cycle structure $(\lambda_1, \lambda_2, \dots, \lambda_n)$, the number of functions $g : [n] \to [n]$ commuting with σ is

$$\prod_{k=1}^{n} \left(\sum_{j|k} j\lambda_j \right)^{\lambda_k}.$$

Indeed, the image under g of each k-cycle of σ is determined by choosing the image of a single element in that cycle. By Lemma 2.4, this element may be mapped to any element of any j-cycle where $j|k$. Summing over all possible cycle structures yields the result. □

A *partition* of a finite set is a collection of non-empty pairwise disjoint subsets, called *blocks*, whose union is the set. A partition of $[n]$ is said to have block structure $(\lambda_1, \dots, \lambda_n)$ if it has λ_i blocks of size i for each i. Recall that there are $\frac{n!}{\lambda_1! \cdots \lambda_n! 1!^{\lambda_1} \cdots n!^{\lambda_n}}$ partitions of $[n]$ having block structure $(\lambda_1, \dots, \lambda_n)$. Note that partitions may be regarded as permutations all of whose cycles are in ascending order. By restricting permutations to those corresponding to set partitions in the proof of Proposition 2.6 above, we obtain the following result.

Corollary 2.7 *The number of commuting pairs of functions from $[n]$ to $[n]$ where the first function is a partition of $[n]$ is given by*

$$\sum_{\substack{1\lambda_1 + \cdots + n\lambda_n = n \\ \lambda_i \geqslant 0}} \frac{n!}{\lambda_1! \cdots \lambda_n! 1!^{\lambda_1} \cdots n!^{\lambda_n}} \prod_{k=1}^{n} \left(\sum_{j|k} j\lambda_j \right)^{\lambda_k}.$$

Remark. Note that the formula in Corollary 2.7 is that of the expression for the n-th complete Bell polynomial $B_n(t_1, t_2, \dots, t_n)$ (see, e.g., [3]), but with the indeterminates t_k replaced by the sums $\sum_{j|k} j\lambda_j$.

Recall that an involution of $[n]$ is a permutation all of whose cycles have length one or two. By restricting the outer sum in the formula given in Proposition 2.6 above so that only λ_1 or λ_2 may be non-zero, we obtain the following result.

Corollary 2.8 *The number of commuting pairs of functions from $[n]$ to $[n]$ where the first function is an involution is given by*

$$n! \sum_{\substack{k=0 \\ k \equiv n (mod\ 2)}}^{n} \frac{k^k}{k! \left(\frac{n-k}{2}\right)!} \left(\frac{n}{2}\right)^{\frac{n-k}{2}}.$$

Remark. By disallowing $\lambda_1 > 0$ in the formula in Proposition 2.6, or by taking $\lambda_2 = \frac{n}{2}$ when n is even, one obtains expressions for the number of commuting pairs of functions where the first function is either a *derangement* (i.e., a permutation having no fixed points) or a *perfect matching* (i.e., a permutation all of whose cycles have length two). Furthermore, taking $\lambda_n = 1$, it is seen that there are $n!$ pairs of commuting functions, where the first function is a permutation containing a single cycle.

We have the following divisibility result concerning the number of commuting permutation function pairs.

Theorem 2.9 *The number of commuting pairs of functions from $[n]$ to $[n]$ where the first function is a permutation is divisible by n for all $n \geqslant 1$.*

Proof. We make use of the formula given in Proposition 2.6. Let us group together all terms for which ℓ is the smallest index such that $\lambda_\ell > 0$, where $1 \leqslant \ell \leqslant n$ is fixed. Let us consider sums of this type. Note that the factors corresponding to $1 \leqslant k \leqslant \ell - 1$ in the product are all one, by $0^0 = 1$, and that the $k = \ell$ factor is given by $(\ell\lambda_\ell)^{\lambda_\ell}$ since $\lambda_j = 0$ if $j < \ell$. Thus, the formula in this case may be written as

$$\sum_{\substack{\ell\lambda_\ell+\cdots+n\lambda_n=n\\ \lambda_\ell>0}} \frac{n^{\underline{\ell}}(n-\ell)!}{\ell\lambda_\ell(\lambda_\ell-1)!\lambda_{\ell+1}!\cdots\lambda_n!\ell^{\lambda_\ell-1}(\ell+1)^{\lambda_{\ell+1}}\cdots n^{\lambda_n}}(\ell\lambda_\ell)^{\lambda_\ell}\prod_{k=\ell+1}^{n}\left(\sum_{j|k,j\geqslant\ell} j\lambda_j\right)^{\lambda_k}.$$

Since $\lambda_\ell > 0$, by assumption, note that the $\ell\lambda_\ell$ factor written at the beginning of the denominator of the fraction is canceled out by $(\ell\lambda_\ell)^{\lambda_\ell}$.

If $\ell \leqslant \frac{n}{2}$, then $\lambda_\ell > 0$ implies $\lambda_n = \lambda_{n-1} = \cdots = \lambda_{n-\ell+1} = 0$. In this case we may then write

$$\begin{aligned}&\frac{(n-\ell)!}{(\lambda_\ell-1)!\lambda_{\ell+1}!\cdots\lambda_n!\ell^{\lambda_\ell-1}(\ell+1)^{\lambda_{\ell+1}}\cdots n^{\lambda_n}}\\ &\qquad= \frac{(n-\ell)!}{(\lambda_\ell-1)!\lambda_{\ell+1}!\cdots\lambda_{n-\ell}!\ell^{\lambda_\ell-1}(\ell+1)^{\lambda_{\ell+1}}\cdots(n-\ell)^{\lambda_{n-\ell}}},\end{aligned}$$

with the second expression seen to be an integer since it counts the number of permutations of $[n-\ell]$ having cycle structure $(0, \ldots, 0, \lambda_\ell - 1, \lambda_{\ell+1}, \ldots, \lambda_{n-\ell})$, where it is understood that there are 0's in the first $\ell - 1$ components of the vector. Since the product from $k = \ell + 1$ to $k = n$ in the sum above is clearly integral, it follows that the product of all the factors is an integral multiple of $n^{\underline{\ell}}$, and hence of n, since $\ell > 0$.

If $\ell > \frac{n}{2}$, then the minimality of ℓ implies $\ell = n$ and $\lambda_\ell = 1$, in which case we get a single term of $n!$.

Thus the sum corresponding to each possible ℓ value is divisible by n. Considering all possible ℓ, it follows that the total of all these sums is divisible by n, which implies the result. □

Remark. Using Corollary 2.8, one can show that the number of commuting pairs of functions from $[n]$ to $[n]$ where the first function is an involution is divisible by n^2 for all $n \geqslant 3$, with n^3 dividing the number of such pairs if and only if $n = 2^i$ and $i \geqslant 3$.

3 Commuting pairs of functions

In this section, we address the problem of counting ordered pairs of commuting functions defined on the same finite set. Given a function $f : [n] \to [n]$, we consider the following sequence of subsets of $[n]$.

Definition 3.1 Given $f : [n] \to [n]$, define the sequence of sets $(B_i)_{i\geqslant 1}$ by letting B_1 be the base set of f and $B_{i+1} = f^{-1}(B_i)$ for $i \geqslant 1$.

Lemma 3.2 *If $i \geqslant 1$ and $B_i \neq [n]$, then $B_i \subset B_{i+1}$.*

Proof. We first show by induction that $B_i \subseteq B_{i+1}$ for $i \geqslant 1$. Note that this holds if $i = 1$ since B_1 the base set of f implies $f(B_1) = B_1$ and thus $B_1 \subseteq f^{-1}(B_1) = B_2$. If $i \geqslant 2$, then $f(B_i) \subseteq B_{i-1} \subseteq B_i$ implies $B_i \subseteq f^{-1}(B_i) = B_{i+1}$, which completes the induction. Now suppose to the contrary that $B_i = B_{i+1}$ for some i with $B_i \neq [n]$. Then the restriction of f to $[n] - B_i$ has range contained within $[n] - B_i$. By Lemma 2.1, there exists some non-empty subset S of $[n] - B_i$ such that f restricted to S is a permutation of the set. But this contradicts the earlier observation that B_1 contains all such subsets S, which completes the proof. □

We now consider the following way of classifying functions from $[n]$ to $[n]$.

Definition 3.3 We will say that $f : [n] \to [n]$ has rank r if $B_r = [n]$, where r is minimal. Given a sequence $(n_1, n_2, \ldots, n_r)$ of positive integers summing to n, we will say that f is of type $(n_1, n_2, \ldots, n_r)$ if $n_i = |B_i| - |B_{i-1}|$ for $i \geqslant 1$ (where $B_0 = \varnothing$).

Example. Let $f : [10] \to [10]$ be given as follows:

$$f(1) = 1, \quad f(2) = 1, \quad f(3) = 1, \quad f(4) = 3, \quad f(5) = 2,$$
$$f(6) = 4, \quad f(7) = 1, \quad f(8) = 6, \quad f(9) = 5, \quad f(10) = 9.$$

Then $B_1 = \{1\}$, $B_2 = \{1, 2, 3, 7\}$, $B_3 = \{1, 2, 3, 4, 5, 7\}$, $B_4 = [10] - \{8, 10\}$, and $B_5 = [10]$. The function f would have rank 5 and be of type $(1, 3, 2, 2, 2)$. Note that the rank 1 functions correspond to members of $\mathcal{S}_n$ and are all of type (n).

Classifying the functions from $[n]$ to $[n]$ according to the rank and type yields the following combinatorial identity which we were unable to find in the literature.

Corollary 3.4 *If $n \geqslant 1$, then*

$$n^n = \sum_{r=1}^{n} \sum_{\substack{\sum_{i=1}^{r} n_i = n \\ n_i > 0}} \binom{n}{n_1, n_2, \ldots, n_r} n_1! n_1^{n_2} n_2^{n_3} \cdots n_{r-1}^{n_r}. \tag{1}$$

As we have seen, the permutation base of a function and its cycle structure limit which functions can commute with it. On the other hand, any appropriate mapping of the cycles of the permutation base of a function f can be extended to the elements outside of the base of f to produce a function which commutes with f. We start with a simple case. Suppose f has rank 2. Let us fix a base of f and assume that it is a permutation of $[k]$ having cycle structure $(\lambda_1, \ldots, \lambda_k)$. We enumerate the ordered pairs (f, g), where f is of rank 2 and having a fixed permutation base as described. Note that each member of $[k+1, n]$ maps to an element of $[k]$ under f (k^{n-k} possibilities) since f is of rank 2 whose base set is $[k]$. To determine the functions commuting with f as described, we first choose a map $\alpha : [k] \to [k]$ which possesses the cycle mapping property (so that there are $\prod_{i=1}^{k} \left(\sum_{j|i} j\lambda_j \right)^{\lambda_i}$ possibilities for α).

Now we need to determine how to map the elements of $[k+1, n]$. Let us denote the restriction of f to $[k+1, n]$ by β. If $i \in [k+1, n]$ and $\beta(i) = j$, then a function g commuting with f and extending α may map i to any element which maps to $\alpha(j)$ under f. Thus, there are $|\beta^{-1}(\{\alpha(\beta(i))\})| + 1$

possibilities for the image of i. This implies that there are

$$A(k;\lambda_1,\dots,\lambda_k) := \sum_{\substack{\beta:[k+1,n]\to[k]\\ \alpha:[k]\to[k]\\ \alpha \text{ possesses c.m. property}}} \prod_{k+1\leqslant i\leqslant n} (|\beta^{-1}(\{\alpha(\beta(i))\})|+1)$$

commuting pairs f and g, where f is of rank 2 and having a fixed permutation base on $[k]$ with cycle structure $(\lambda_1,\dots,\lambda_k)$. Upon choosing the elements comprising the base set of f, along with a permutation of those elements possessing a given cycle structure, it follows that there are

$$\sum_{k=1}^{n} \binom{n}{k} \sum_{\substack{1\lambda_1+\cdots+k\lambda_k=k\\ \lambda_i\geqslant 0}} \frac{k!}{\lambda_1!\cdots\lambda_k! 1^{\lambda_1}\cdots k^{\lambda_k}} A(k;\lambda_1,\dots,\lambda_k)$$

commuting pairs f and g, where f has rank 2.

In general, suppose f is a function of rank r and type $(n_1,\dots,n_r)$. Assume that $B_i=[n_1+\cdots+n_i]$ for $i\geqslant 1$, where the B_i are as defined above with $B_0=\varnothing$. Then there is a collection of functions $f_1 : B_1\to B_1$ and $f_i : B_i - B_{i-1}\to B_{i-1}-B_{i-2}$ for $2\leqslant i\leqslant r$, where f_1 is a permutation, whose composite is f.

To form a function g commuting with the function f we have created, we need another set of maps g_i giving the images of each element under the commuting map. The following lemma provides some structure for these maps.

Lemma 3.5 *Let $f:[n]\to[n]$ have rank r and type $(n_1,\dots,n_r)$. Suppose f is decomposed into the functions f_i as described. If g commutes with f, then g may be decomposed into functions $g_1 : B_1\to B_1$ and $g_i : B_i-B_{i-1}\to B_i$ for $2\leqslant i\leqslant r$.*

Proof. Suppose $g(B_1)\subseteq B_{i+1}$ for some $1\leqslant i\leqslant r-1$. Then $g(f(x))=f(g(x))$ for all $x\in B_1$ implies $g(B_1)\subseteq B_i$ since f permutes the elements of B_1 and $f(B_{i+1})\subseteq B_i$. That $g(B_1)\subseteq B_1$ now follows by induction since $g(B_1)\subseteq B_r=[n]$. To complete the proof, suppose to the contrary that $g(B_i-B_{i-1})$ is not contained within B_i for some i. Let i' denote the smallest such index i. By the preceding, we have $i'>1$. Then there exists some $b\in B_{i'}-B_{i'-1}$ such that $g(b)\in B_j-B_{j-1}$ where $j>i'$. But then $x=g(f(b))=f(g(b))$ implies $x\in B_{i'-1}\cap(B_{j-1}-B_{j-2})=\varnothing$, a contradiction. □

Note that the function g_1 in Lemma 3.5 possesses the cycle mapping property with respect to the function f_1, with $f_\ell(g_i(x))=g_{i-1}(f_i(x))$ for all $x\in B_i-B_{i-1}$ and $i\geqslant 2$, where the index ℓ, $1\leqslant\ell\leqslant i$, is determined by the set containing $g_i(x)$. The following definition summarizes the properties discussed above that are possessed by commuting pairs of functions.

Definition 3.6 Let $B_i=[n_1+n_2+\cdots+n_i]$ for $1\leqslant i\leqslant r$, with $B_0=\varnothing$. Suppose $\{f_i\}_{i=1}^r$ is a collection of functions such that $f_1 : B_1\to B_1$ is a permutation of type $(\lambda_1,\lambda_2,\dots,\lambda_{n_1})$ and $f_i : B_i-B_{i-1}\to B_{i-1}-B_{i-2}$ for $2\leqslant i\leqslant r$. Let $\{g_i\}_{i=1}^r$ be a collection of functions such that $g_1 : B_1\to B_1$ commutes with f_1 and $f_\ell(g_i(x))=g_{i-1}(f_i(x))$ for all $x\in B_i-B_{i-1}$ and $i\geqslant 2$ (the index ℓ, $1\leqslant\ell\leqslant i$, being determined by the set containing $g_i(x)$). Then we will denote the collection of all permissible sets of functions $\{f_i\}_{i=1}^r$ and $\{g_i\}_{i=1}^r$ by $S(n_1,\dots,n_r;\lambda_1,\dots,\lambda_{n_1})$.

Allowing the elements in the B_i to vary as well as the cycle structure of the permutation base of f implies the following result.

Proposition 3.7 *The number of commuting pairs of functions from $[n]$ to $[n]$ is given by*

$$\sum_{r=1}^{n} \sum_{\substack{n_1+\cdots+n_r=n \\ n_i>0}} \binom{n}{n_1,\ldots,n_r} \sum_{\substack{1\lambda_1+\cdots+n_1\lambda_{n_1}=n_1 \\ \lambda_i\geqslant 0}} |S(n_1,\ldots,n_r;\lambda_1,\ldots,\lambda_{n_1})|.$$

The following result concerns the probability that two elements in the semigroup of self-maps on $[n]$ commute (for related results, see [7]).

Theorem 3.8 *The probability that a randomly chosen pair of functions from $[n]$ to $[n]$ commutes converges to zero as $n \to \infty$.*

Proof. Considering the size k of the base set of f in a commuting pair (f, g) of functions on $[n]$, it follows from Lemma 3.5 that the number of such pairs is bounded above by $\sum_{k=1}^{n} \binom{n}{k} k! k^k n^{2n-2k}$. To complete the proof, it is enough to show

$$\sum_{k=1}^{n} \frac{n^{\underline{k}} k^k}{n^{2k}} \to 0 \qquad \text{as} \quad n \to \infty.$$

Given $n \geqslant 1$, let $a_k = \frac{n^{\underline{k}} k^k}{n^{2k}}$, $1 \leqslant k \leqslant n$. Using the fact that $\left(\frac{1+k}{k}\right)^k$ is an increasing function of k having limit $e < 3$, along with the obvious inequality $x(1-x) \leqslant \frac{1}{4}$ for all x, we have

$$\frac{a_{k+1}}{a_k} = \frac{(n-k)(k+1)\left(\frac{1+k}{k}\right)^k}{n^2} < \frac{3}{4} + \frac{3(n-k)}{n^2} < c$$

for all $k \in [n-1]$ if n is sufficiently large where $\frac{3}{4} < c < 1$ is any fixed constant. Thus,

$$\sum_{k=1}^{n} \frac{n^{\underline{k}} k^k}{n^{2k}} < \frac{1}{n} \sum_{k=1}^{n} c^{k-1} < \frac{1}{n(1-c)} \to 0 \qquad \text{as} \quad n \to \infty,$$

as desired. □

We conclude this section with the following divisibility result concerning the number of ordered pairs of commuting functions.

Theorem 3.9 *The number of commuting pairs of functions from $[n]$ to $[n]$ is divisible by n for all $n \geqslant 1$.*

Proof. For $k = 1, \ldots, n$ and λ_i such that $1\lambda_1 + \cdots + k\lambda_k = k$, let $C(\lambda_1, \ldots, \lambda_k)$ be the number of commuting pairs of functions (f, g) such that the permutation base of f is fixed and has cycle structure $(1^{\lambda_1}, \ldots, k^{\lambda_k})$. Then the total number of commuting pairs of functions from $[n]$ to $[n]$ is

$$\sum_{k=1}^{n} \binom{n}{k} \sum_{\substack{1\lambda_1+\cdots+k\lambda_k \\ \lambda_i\geqslant 0}} \frac{k!}{\lambda_1!\cdots\lambda_k! 1^{\lambda_1}\cdots k^{\lambda_k}} C(\lambda_1, \ldots, \lambda_k).$$

As in the proof of Theorem 2.9 above, we group together all terms for which ℓ is the smallest index such that $\lambda_\ell > 0$, where $1 \leqslant \ell \leqslant k$ is fixed. Then the formula in this case is given by

$$\sum_{k=1}^{n} \sum_{\substack{\ell\lambda_\ell+\cdots+k\lambda_k=k \\ \lambda_\ell>0}} \binom{n}{k} \frac{k!}{\lambda_\ell!\cdots\lambda_k!\ell^{\lambda_\ell}\cdots k^{\lambda_k}} C(0,\ldots,0,\lambda_\ell,\ldots,\lambda_k).$$

We show that this sum is divisible by n for all k and ℓ. Since $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$, it suffices to show that $\frac{k!}{\lambda_\ell!\cdots\lambda_k!\ell^{\lambda_\ell}\cdots k^{\lambda_k}} C(0,\ldots,0,\lambda_\ell,\ldots,\lambda_k)$ is divisible by k. Writing

$$\frac{k!}{\lambda_\ell!\cdots\lambda_k!\ell^{\lambda_\ell}\cdots k^{\lambda_k}} = \frac{k\frac{\ell}{}}{\ell\lambda_\ell} \cdot \frac{(k-\ell)!}{(\lambda_\ell-1)!\lambda_{\ell+1}!\cdots\lambda_k!\ell^{\lambda_\ell-1}(\ell+1)^{\lambda_{\ell+1}}\cdots k^{\lambda_k}},$$

and noting that the second factor is integral as in the proof of Theorem 2.9, in order to show that $\frac{k!}{\lambda_\ell!\cdots\lambda_k!\ell^{\lambda_\ell}\cdots k^{\lambda_k}} C(0,\ldots,0,\lambda_\ell,\ldots,\lambda_k)$ is divisible by k, it suffices to show that $C(0,\ldots,0,\lambda_\ell,\ldots,\lambda_k)$ is divided by $\ell\lambda_\ell$.

For a fixed permutation base of the form $(0,\ldots,0,\lambda_\ell,\ldots,\lambda_k)$, let $C_1,\ldots,C_{\lambda_\ell}$ be the cycles of length ℓ. Let P_1 be the set of ordered pairs (f,g) of commuting functions such that f has the given permutation base and $f^{-1}(C_i) = C_i$ for some cycle C_i. We claim that $\ell\lambda_\ell$ divides $|P_1|$. Let us assume that the index i is minimal and let m be the smallest element of C_i. By Lemma 2.4 and the minimality of ℓ, we see that $g(m)$ must be some member of $[n]$ belonging to one of the cycles of length ℓ, which determines the functional values of g for all members of C_i. Thus we may partition the elements of P_1 into $\ell\lambda_\ell$ classes by requiring that $g(m)$ be the same for all members of a given class. Since $f^{-1}(C_i) = C_i$, it follows by symmetry that each class has the same cardinality, proving the claim.

Let P_2 be the set of ordered pairs (f,g) of commuting functions such that $f^{-1}(C_i) \neq C_i$ for all $i = 1,\ldots,\lambda_\ell$. Let $D_1 = \bigcup_{i=1}^{\lambda_\ell} C_i$. For a function f with the given permutation base, let $D_{f,1} = D_1$ and $D_{f,i+1} = f^{-1}(D_{f,i}) - D_{f,i}$ for $i \geqslant 1$. We will define an equivalence relation R on P_2 by requiring that ordered pairs (f_1,g_1) and (f_2,g_2) be related if the following three conditions are satisfied: (i) $D_{f_1,i} = D_{f_2,i}$ for all $i \geqslant 1$, (ii) $f_1 |_{\bigcup_{i\geqslant 3} D_{f_1,i}} = f_2 |_{\bigcup_{i\geqslant 3} D_{f_2,i}}$, and (iii) there exists a permutation $\sigma : [n] \to [n]$ such that $\sigma(x) = x$ for $x \notin D_1$, $\sigma^{-1}f_1\sigma = f_2$, and if $a_1,\ldots,a_\ell \in D_1$, with $a_1 \mapsto a_2 \mapsto \cdots \mapsto a_\ell \mapsto a_1$ forming a cycle, then $\sigma(a_1) \mapsto \sigma(a_2) \mapsto \cdots \mapsto \sigma(a_\ell) \mapsto \sigma(a_1)$ is a cycle.

Now fix an equivalence class $\mathcal{E}$ of R. Let $D_i = D_{f,i}$ for any element (f,g) of $\mathcal{E}$, and put $D = \bigcup_{i\geqslant 1} D_i$. We partition $\mathcal{E}$ by $(f_1,g_1) \sim (f_1',g_1')$ if f_1 and f_1' have the same restriction to D_2. Then there are $\ell^{\lambda_\ell}\lambda_\ell!$ equivalence classes under $\sim$, since given any representative (f,g) of a class of $\sim$, the $\ell^{\lambda_\ell}\lambda_\ell!$ permutations σ satisfying condition (iii) above produce representatives from each of the other equivalence classes by conjugation. Note that $(f,g) \in P_2$ ensures that each σ produces a member belonging to a distinct equivalence class of $\sim$.

Let E_1 and E_2 be two equivalence classes of $\mathcal{E}$ under $\sim$ and suppose (f_2,g_2) is any member of E_2. Put $f = f_2 |_D$. Then there is a unique σ satisfying condition (iii) above such that $\sigma^{-1}f\sigma = f_1$ for any $(f_1,g_1) \in E_1$.

Define $\phi : E_1 \to E_2$ as follows. Given $(f_1,g_1) \in E_1$, let $f_2(x) = f(x)$ for $x \in D$ and $f_2(x) = f_1(x)$ for $x \notin D$. Note that $\sigma^{-1}f_2\sigma = f_1$. Now define $g_2 = \sigma g_1 \sigma^{-1}$. Put $\phi(f_1,g_1) = (f_2,g_2)$. Observe that we have

$$g_2f_2 = \sigma g_1\sigma^{-1}f_2 = \sigma g_1 f_1 \sigma^{-1} = \sigma f_1 g_1 \sigma^{-1} = f_2\sigma g_1\sigma^{-1} = f_2g_2.$$

The mapping ϕ is seen to be a bijection, which implies all of the equivalence classes of $\mathcal{E}$ under $\sim$ contain the same number of elements. Thus the size of each equivalence class of R is divisible by $\ell^{\lambda_\ell}\lambda_\ell!$, which implies that $|P_2|$ is divisible by $\ell\lambda_\ell$. It follows that $C(0,\ldots,0,\lambda_\ell,\ldots,\lambda_k)$ is divisible by $\ell\lambda_\ell$, which completes the proof. □

4 Conclusion

In this paper, we have considered the problem of enumerating ordered pairs (f,g) of commuting functions from $[n]$ to itself. In the case where f belongs to $\mathcal{S}_n$ (or one of several subsets), an explicit formula counting these pairs is provided and a related divisibility result was established. Building upon this case, an enumeration scheme and formula was developed for the problem of counting ordered pairs of commuting functions with no restrictions. As a consequence, we can show that the number of ordered pairs of commuting functions on $[n]$ is always divisible by n. We seek though a more explicit formula and/or generating function for the number of commuting pairs of functions, in particular for the cardinalities of the sets $S(n_1,\ldots,n_r;\lambda_1,\ldots,\lambda_{n_1})$. It would also be interesting to have asymptotic estimates as n grows large for the number of such pairs. Finally, one might consider the problem of enumerating ordered pairs of commuting elements in other finite semigroup settings, such as the multiplication of $n\times n$ matrices having integral entries taken mod m.

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